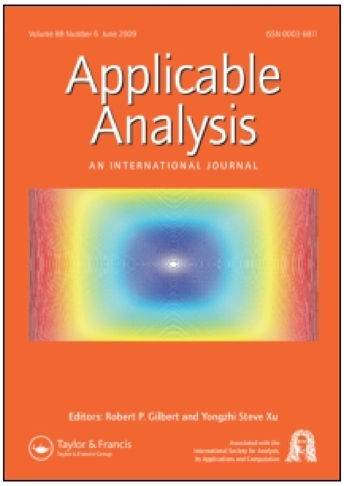

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Perturbation of essential spectra of exterior elliptic problems

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Perturbation of essential spectra of exterior elliptic problems†

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For a second-order symmetric strongly elliptic differential operator on an exterior domain in $\mathbb{R}^n$, it is known from the works of Birman and Solomiak that a change in the boundary condition from the Dirichlet condition to an elliptic Neumann or Robin condition leaves the essential spectrum unchanged, in such a way that the spectrum of the difference between the inverses satisfies a Weyl-type asymptotic formula. We show that one can increase, but not diminish, the essential spectrum by imposition of other Neumann-type nonelliptic boundary conditions. The results are extended to $2m$-order operators, where it is shown that for any selfadjoint realization defined by an elliptic normal boundary condition (other than the Dirichlet condition), one can augment the essential spectrum at will by adding a suitable operator to the mapping from free Dirichlet data to Neumann data. We also show here an extension of the spectral asymptotics formula for the difference between inverses of elliptic problems. The proofs rely on Kreĭn-type formulae for differences between inverses, and cutoff techniques, combined with results on singular Green operators and their spectral asymptotics.

Keywords: exterior domain; essential spectrum; singular Green operator; Schatten class; Krein formula; spectrally negligible cutoffs

AMS Subject Classifications: 35J40; 35P20; 35S15; 47A10

1. Introduction

Let A be a uniformly strongly elliptic differential operator on $\mathbb{R}^n$ $(n \geq 2)$

$$A = -\sum_{j,k=1}^{n} \partial_j a_{jk}(x)\partial_k + a_0(x), \tag{1}$$

with real bounded smooth coefficients with bounded derivatives satisfying $a_{jk} = a_{kj}$ and

$$\sum_{j,k} a_{jk}(x)\xi_j\xi_k \geq c_1|\xi|^2, \ a_0(x) \geq c_2, \quad \text{for } x, \xi \in \mathbb{R}^n, \tag{2}$$

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†Dedicated to Professor Vsevolod Alekseevich Solonnikov on the occasion of his 75th birthday.

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with $c_1, c_2 > 0$. We denote by A_0 the maximal realization in $L_2(\mathbb{R}^n)$; it is selfadjoint positive. Let $\Omega_+ \subset \mathbb{R}^n$ be the exterior of a bounded smooth open set Ω_-, with boundary denoted Σ $(=\partial\Omega_+ = \partial\Omega_-)$, and let A_1, A_2 and A_3 be the selfadjoint lower bounded realizations in $L_2(\Omega_+)$ determined by the Dirichlet condition $(\gamma_0 u \equiv u|_\Sigma = 0)$, the oblique Neumann condition ($\nu_A u = 0$, see (12) below), resp. a Robin condition ($\nu_A u = b(x)\gamma_0 u$) with b real and smooth. The coefficient a_0 is assumed to be taken so large positive that all four operators have positive lower bound.

It is known that the operators A_j have an unbounded essential spectrum, consisting of an interval $[c, \infty[$ if the coefficients converge to a limit for $|x| \to \infty$, and more generally being a subset of $[c, \infty[$ with possible gaps (e.g. when the coefficients are periodic).

Birman showed in [1] a general principle concerning the stability of the essential spectrum:

$$A_0^{-1} - A_j^{-1} \oplus 0_{L_2(\Omega_-)} \in T_{2/n}, \tag{3}$$

$$A_j^{-1} - A_k^{-1} \in T_{2/(n-1)}, \quad \text{for } j,k = 1,2,3, \tag{4}$$

where T_α denotes the class of compact operators whose characteristic values s_l are $O(l^{-\alpha})$ for $l \to \infty$. (When $\Omega_1 \cup \Omega_2$ is a disjoint union of open sets, and P_i acts in $L_2(\Omega_i)$, we denote by $P_1 \oplus P_2$ the operator in $L_2(\Omega_1 \cup \Omega_2)$ that acts like P_i on $L_2(\Omega_i)$, naturally injected in $L_2(\Omega_1 \cup \Omega_2)$.) In particular, all four operators have the same essential spectrum $\sigma_{\mathrm{ess}}(A_0)$; this extends a result of Povzner, as referred to in [1]. (Birman's paper also allowed unbounded coefficients and limited smoothness, but we shall not follow up on those aspects here.)

The result (4) was refined by Birman and Solomiak in [2], where a Weyl-type spectral asymptotics formula was obtained ($s_l l^{2/(n-1)}$ converges to a limit for $l \to \infty$). In Grubb [3], similar spectral asymptotics formulae were shown by methods of pseudodifferential boundary problems, and refinements with a spectral resolvent parameter were studied in [4]. Spectral estimates of resolvent differences have been taken up again in recent works of Alpay and Behrndt [5], Gesztesy and Malamud [6].

This article extends the results to higher order operators, but aims in particular for a slightly different question, namely of how much one can perturb the essential spectrum of A_3 by replacing the Robin condition by a more general *Neumann-type* boundary condition (not necessarily elliptic)

$$\nu_A u = C\gamma_0 u. \tag{5}$$

Let $\widetilde{A}$ denote the realization of A on Ω_+ determined by (5), i.e. with domain

$$D(\widetilde{A}) = \{u \in L_2(\Omega_+) \mid Au \in L_2(\Omega_+),\ \nu_A u = C\gamma_0 u\}. \tag{6}$$

The outcome is as follows:

(1) For any nonzero $a \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(A_0)$, C can be chosen as a pseudodifferential operator of order 1 such that $\widetilde{A}$ is selfadjoint with

$$\sigma_{\mathrm{ess}}(\widetilde{A}) = \sigma_{\mathrm{ess}}(A_0) \cup \{a\}. \tag{7}$$

More generally, when T_0 is an invertible selfadjoint operator in a separable infinite-dimensional Hilbert space Z_0, one can choose an operator C such that $\widetilde{A}$ is selfadjoint and

$$\sigma_{\rm ess}(\widetilde{A}) = \sigma_{\rm ess}(A_0) \cup \sigma_{\rm ess}(T_0). \tag{8}$$

(2) For any choice of an operator C in (5) defining a selfadjoint invertible realization $\widetilde{A}$, $\sigma_{\rm ess}(A_0)$ remains in the essential spectrum of $\widetilde{A}$.

We also reprove the spectral asymptotics formulae, and extend the results to strongly elliptic operators of order $2m$ for positive integer m.

The question of whether points of $\sigma_{\rm ess}(A_0)$ can be removed by a perturbation of the boundary condition was brought up in a conversation with Marletta, Brown and Wood in Cardiff in May 2008; the author thanks these colleagues for useful discussions.

2. Description of the operators in the second-order case

Let us first recall some well-known facts. The Sobolev space $H^s(\mathbb{R}^n)$ ($s \in \mathbb{R}$) can be provided with the norm $\|u\|_s = \|\mathcal{F}^{-1}(\langle\xi\rangle^s \mathcal{F}u)\|_{L_2(\mathbb{R}^n)}$; here, $\mathcal{F}$ is the Fourier transform and $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. There is a standard construction from this of Sobolev spaces over an open subset and over the boundary manifold. We denote by $A_{\max}$ resp. $A_{\min}$ the operators acting like A with domains

$$D(A_{\max}) = \{u \in L_2(\Omega_+) \mid Au \in L_2(\Omega_+)\}, \quad D(A_{\min}) = H_0^2(\Omega_+),$$

here, $A_{\min}$ is closed symmetric, and $A_{\max} = A_{\min}^*$. The operators $\widetilde{A}$ satisfying $A_{\min} \subset \widetilde{A} \subset A_{\max}$ are called the realizations of A.

The symmetric sesquilinear forms

$$\begin{aligned} s_{\mathbb{R}^n}(u,v) &= \int_{\mathbb{R}^n} \sum_{j,k=1}^n (a_{jk}\partial_k u \partial_j \bar{v} + a_0 u\bar{v})\mathrm{d}x, \\ s(u,v) &= \int_{\Omega_+} \sum_{j,k=1}^n (a_{jk}\partial_k u \partial_j \bar{v} + a_0 u\bar{v})\mathrm{d}x, \end{aligned} \tag{9}$$

are bounded on $H^1(\mathbb{R}^n)$ resp. $H^1(\Omega_+)$ and satisfy

$$s_{\mathbb{R}^n}(u,u) \geq c\|u\|^2_{H^1(\mathbb{R}^n)} \text{ resp. } s(u,u) \geq c\|u\|^2_{H^1(\Omega_+)} \tag{10}$$

there, with $c = \min\{c_1, c_2\}$. Moreover,

$$(Au, v)_{L_2(\Omega_+)} = s(u,v) + (\nu_A u, \gamma_0 v)_{L_2(\Sigma)}, \quad u \in H^2(\Omega_+), v \in H^1(\Omega_+), \tag{11}$$

where

$$\nu_A u = \sum_{j,k} a_{jk}\nu_j \gamma_0 \partial_k u, \tag{12}$$

with $(\nu_1(x), \dots, \nu_n(x))$ denoting the interior unit normal to Ω_+ at $x \in \Sigma$. Hence, the standard variational construction (the Lax–Milgram lemma) applied to the triples $(L_2(\mathbb{R}^n), H^1(\mathbb{R}^n), s_{\mathbb{R}^n})$, $(L_2(\Omega_+), H_0^1(\Omega_+), s)$, resp. $(L_2(\Omega_+), H^1(\Omega_+), s)$ defines the positive selfadjoint operators A_0 in $L_2(\mathbb{R}^n)$, A_1 and A_2 in $L_2(\Omega_+)$ mentioned in the introduction. (The variational construction is known, for example from Lions and Magenes [7], and is also explained in Grubb [8].) In view of elliptic regularity theory and the uniform symbol estimates, the domains are in fact contained in H^2. Moreover, the operators representing the nonhomogeneous boundary value problems (cf e.g. [7])

$$\begin{aligned} \mathcal{A}_1 &= \begin{pmatrix} A \\ \gamma_0 \end{pmatrix} : H^{s+2}(\Omega_+) \to \begin{matrix} H^s(\Omega_+) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix}, \\ \mathcal{A}_2 &= \begin{pmatrix} A \\ \nu_A \end{pmatrix} : H^{s+2}(\Omega_+) \to \begin{matrix} H^s(\Omega_+) \\ \times \\ H^{s+\frac{1}{2}}(\Sigma) \end{matrix}, \end{aligned} \tag{13}$$

where $s > -\frac{3}{2}$ resp. $s > -\frac{1}{2}$, have solution operators, continuous in the opposite direction:

$$\mathcal{A}_1^{-1} = \begin{pmatrix} R_1 & K_1 \end{pmatrix}, \quad \mathcal{A}_2^{-1} = \begin{pmatrix} R_2 & K_2 \end{pmatrix}. \tag{14}$$

In modern terminology,

$$R_1 = Q_+ - K_1\gamma_0 Q_+, \quad R_2 = Q_+ - K_2\nu_A Q_+, \tag{15}$$

where Q is the *pseudodifferential operator* $Q = A_0^{-1}$ on $\mathbb{R}^n$ and $Q_\pm = r^\pm Q e^\pm$ is its truncation to $\Omega_\pm$ (here $e^\pm$ extends to $\mathbb{R}^n$ by 0 on $\Omega_\mp$, $r^\pm$ restricts from $\mathbb{R}^n$ to $\Omega_\pm$). The operators K_1 and K_2 are *Poisson operators* solving the respective boundary value problems with nonzero boundary data, zero data in the interior of Ω_+; their mapping properties extend to the full scale of Sobolev spaces with $s \in \mathbb{R}$. R_1 and R_2 act in $L_2(\Omega_+)$ as the inverses of the realizations A_1 resp. A_2 of A with domains

$$D(A_1) = \{u \in H^2(\Omega_+) \mid \gamma_0 u = 0\}, \quad \text{resp. } D(A_2) = \{u \in H^2(\Omega_+) \mid \nu_A u = 0\}. \tag{16}$$

The operator A_3 representing the Robin condition $\nu_A u = b\gamma_0 u$ is defined similarly from the sesquilinear form

$$s_b(u, v) = s(u, v) + (b\gamma_0 u, \gamma_0 v)_{L_2(\Sigma)} \tag{17}$$

on $H^1(\Omega_+)$ and has similar properties as A_2: its domain is $D(A_3) = \{u \in H^2(\Omega_+) \mid (\nu_A - b\gamma_0)u = 0\}$, and the operator

$$\mathcal{A}_3 = \begin{pmatrix} A \\ \nu_A - b\gamma_0 \end{pmatrix} \text{ has inverse } \begin{pmatrix} R_3 & K_3 \end{pmatrix}, \text{ with } R_3 = Q_+ - K_3(\nu_A - b\gamma_0)Q_+. \tag{18}$$

The above facts have been known for many years, although the emphasis was not always placed on including low values of s. Instead of accounting for this aspect in detail here, we mention that the results are covered by the construction in the book by Grubb [9, Chapter 3], and that the general $2m$-order case will be treated below in Section 5.

We shall now regard the realization defined by (5) from the point of view of general nonlocal boundary value problems. The basic theory was presented in Grubb [10] and was taken up again and further developed in a joint work with

Brown et al. [11]; applications to exterior domains are included in [12]. (An introduction is also given in [8].) The fundamental result is that the closed realizations $\widetilde{A}$ are in a 1–1 correspondence with the closed, densely defined operators $T: V \to W$, where V and W are closed subspaces of Z, the nullspace of $A_{\max}$. Many properties are carried along in this correspondence, for example $\widetilde{A}$ is invertible if and only if T is so, and in the affirmative case one has the Kreĭn-type formula

$$\widetilde{A}^{-1} = A_1^{-1} + \mathrm{i}_V T^{-1} \mathrm{pr}_W, \tag{19}$$

where i_V denotes the injection $V \hookrightarrow H$ and pr_V denotes the orthogonal projection onto V, in $H = L_2(\Omega_+)$. We have here taken the Dirichlet realization A_1 as the reference operator for the correspondence theorem.

Consider in particular a realization $\widetilde{A}$ corresponding to an operator $T: Z \to Z$ (i.e. with $V = W = Z$).

As shown in the mentioned references, $\widetilde{A}$ can be interpreted as representing a boundary condition. To describe that boundary condition, we first recall that (11) implies Green's formula valid for $u, v \in H^2(\Omega_+)$,

$$(Au, v)_{L_2(\Omega_+)} - (u, Av)_{L_2(\Omega_+)} = (\nu_A u, \gamma_0 v)_{L_2(\Sigma)} - (\gamma_0 u, \nu_A v)_{L_2(\Sigma)}. \tag{20}$$

We denote by γ_Z the restriction of γ_0 to Z,

$$\gamma_Z : Z \xrightarrow{\sim} H^{-\frac{1}{2}}(\Sigma), \tag{21}$$

with adjoint $\gamma_Z^* : H^{\frac{1}{2}}(\Sigma) \xrightarrow{\sim} Z$ (recall that for $s \in \mathbb{R}$, $H^{-s}(\Sigma)$ identifies with the antidual (conjugate dual) space $(H^s(\Sigma))^*$ of $H^s(\Sigma)$, with a duality consistent with the scalar product in $L_2(\Sigma)$). Moreover, we set

$$P_{\gamma_0,\nu_A} = \nu_A K_1, \quad \Gamma = \nu_A - P_{\gamma_0,\nu_A}\gamma_0, \text{ also equal to } \nu_A A_1^{-1} A_{\max}, \tag{22}$$

here, P_{γ_0,ν_A} is a first-order elliptic pseudodifferential operator over Σ , and Γ is a (nonlocal) trace operator. There holds a generalized Green's formula for all u, $v \in D(A_{\max})$,

$$(Au, v)_{L_2(\Omega_+)} - (u, Av)_{L_2(\Omega_+)} = (\Gamma u, \gamma_0 v)_{\frac{1}{2},-\frac{1}{2}} - (\gamma_0 u, \Gamma v)_{-\frac{1}{2},\frac{1}{2}}, \tag{23}$$

where $(\cdot,\cdot)_{s,-s}$ indicates the (sesquilinear) duality pairing between $H^s(\Sigma)$ and $H^{-s}(\Sigma)$. The boundary condition that $\widetilde{A}$ represents is then found to be

$$\Gamma u = L\gamma_0 u, \tag{24}$$

where L is the closed, densely defined operator from $H^{-\frac{1}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$ defined from T by

$$L = (\gamma_Z^*)^{-1} T \gamma_Z^{-1}, \quad D(L) = \gamma_0 D(T). \tag{25}$$

Since $\Gamma = \nu_A - P_{\gamma_0,\nu_A}\gamma_0$, the condition (24) can also be written

$$\nu_A u = (L + P_{\gamma_0,\nu_A})\gamma_0 u, \tag{26}$$

so it is of the form (5) with C acting like $L + \mathrm{P}_{\gamma_0,\nu_A}$. To sum up:

Proposition 2.1 *When $\widetilde{A}$ corresponds to $T: Z \to Z$, it equals the realization defined by the Neumann-type boundary condition* (5), *where*

$$C = L + P_{\gamma_0,\nu_A}, \quad L = (\gamma_Z^*)^{-1} T \gamma_Z^{-1}, D(C) = D(L) = \gamma_0 D(T). \tag{27}$$

Assume in the following that $0 \in \varrho(\widetilde{A})$, equivalently T has a bounded, everywhere defined inverse $T^{-1}: Z \to Z$, and L has a bounded everywhere defined inverse $L^{-1}: H^{\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma)$. Then (19) takes the form:

$$\widetilde{A}^{-1} = A_1^{-1} + \mathrm{i}_Z T^{-1} \mathrm{pr}_Z = A_1^{-1} + K_1 L^{-1} K_1^*. \tag{28}$$

Here K_1 is the Poisson operator for the Dirichlet problem (cf (14)), considered as a mapping from $H^{-\frac{1}{2}}(\Sigma)$ to $L_2(\Omega_+)$ (also equal to $\mathrm{i}_Z \gamma_Z^{-1}$); its adjoint K_1^* goes from $L_2(\Omega_+)$ to $H^{\frac{1}{2}}(\Sigma)$. The formula (28) can clearly be used to examine $\widetilde{A}^{-1}$ as a perturbation of A_1^{-1}; we pursue this fact below in our analysis of essential spectra.

Remark 1 We are interested in cases where T has an essential spectrum outside of 0. As a specific example, one can think of

$$T = aI \quad \text{on } Z, \text{ with } a \in \mathbb{R} \setminus \{0\}, \tag{29}$$

its essential spectrum is $\{a\}$, since $\dim Z = \infty$. In this case,

$$L = a(\gamma_Z^*)^{-1} \gamma_Z^{-1} = a\Lambda_{(-1)}, \quad \text{where } \Lambda_{(-1)} : H^{-\frac{1}{2}}(\Sigma) \xrightarrow{\sim} H^{\frac{1}{2}}(\Sigma) \tag{30}$$

is a pseudodifferential operator elliptic of order -1, and invertible. (This is in contrast to those boundary conditions (5) that satisfy the Shapiro–Lopatinskiĭ condition; they have L elliptic of order $+1$.) Since this L is defined on all of $H^{-\frac{1}{2}}(\Sigma)$, which is mapped by P_{γ_0,ν_A} to $H^{-\frac{3}{2}}(\Sigma)$, C maps $D(L)$ into $H^{-\frac{3}{2}}(\Sigma)$; it is only the difference $L = C - P_{\gamma_0,\nu_A}$ that is assured to map into $H^{\frac{1}{2}}(\Sigma)$. The realization $\widetilde{A}$ defined by this choice has $Z \subset D(\widetilde{A})$, so $D(\widetilde{A})$ is not contained in $H^s(\Omega_+)$ for any $s > 0$. It is a variant of Kreĭn's 'soft extension'.

3. Cutoff techniques

For the analysis of the operators on exterior domains we shall need to study cutoffs, by multiplication either by a smooth function or by a 'rough' characteristic function supported at a distance from the boundary. In [3,4], smooth cutoffs were used and the exterior singular Green operators estimated by a commutator argument based on a series of nested cutoff functions. We give here a simpler argument based on rough cutoffs.

Let $\Omega_>$ be a smooth open subset of Ω_+ such that $\overline{\Omega}_- \subset \complement\overline{\Omega}_>$, and denote $\Omega_+ \cap \complement\overline{\Omega}_> = \Omega_<$. So $\Omega_+ = \Omega_> \cup \Omega_< \cup \partial\Omega_>$. We denote by $r^>$ resp. $r^<$ the restriction operators from Ω_+ to $\Omega_>$ resp. $\Omega_<$, and by $e^>$ resp. $e^<$ the extension operators extending a function given on $\Omega_>$ resp. $\Omega_<$ to a function on Ω_+ by zero on the complement in Ω_+.

In the following, we draw on the analysis of singular numbers of compact operators as presented in Gohberg and Kreĭn [13]. The operators lying in the intersection of Schatten classes $\bigcap_{r>0} \mathcal{C}_r$ (also equal to $\bigcap_{r>0} T_r$) will be called *spectrally negligible*.

Proposition 3.1 *Let K_1 be the Poisson operator entering in* (14), *continuous from $H^{s-\frac{1}{2}}(\Sigma)$ to $H^s(\Omega_+)$ for all $s \in \mathbb{R}$, and consider the operators $K_{1,>} = r^> K_1 : H^{-\frac{1}{2}}(\Sigma) \to L_2(\Omega_>)$ and $K_{1,>}^* = (r^> K_1)^* = K_1^* e^> : L_2(\Omega_>) \to H^{\frac{1}{2}}(\Sigma)$. Then $r^> K_1$ in fact maps continuously*

$$r^> K_1 : H^{s-\frac{1}{2}}(\Sigma) \to H^{s'}(\Omega_>), \quad \textit{any } s, s' \in \mathbb{R}. \tag{31}$$

Moreover, the operators $K_{1,>}$ *and* $K_{1,>}^*$ *are compact and spectrally negligible.*

Similar statements hold for $K_{j,>} = r^> K_j : H^{-\frac{3}{2}}(\Sigma) \to L_2(\Omega_>)$ *and* $K_{j,>}^* = K_j^* e^> : L_2(\Omega_>) \to H^{\frac{3}{2}}(\Sigma)$ *for* $j = 2, 3$.

Proof Denote by $\gamma_0^>$ the operator restricting to $\partial\Omega_>$. When $\varphi \in H^{-\frac{1}{2}}(\Sigma)$, it follows by the interior regularity for solutions of the Dirichlet problem for A on Ω_+ that $\gamma_0^> K_1 \varphi \in C^\infty(\partial\Omega_>)$. Then $r^> K_1 \varphi$ is a null-solution of the Dirichlet problem for A on $\Omega_>$ with C^∞-boundary value. This will also hold if $\varphi \in H^{s-\frac{1}{2}}(\Sigma)$, any $s \in \mathbb{R}$. We know from the variational theory and the regularity theory for the Dirichlet problem on $\Omega_>$ that a null-solution with C^∞-boundary value lies in $H^{s'}(\Omega_>)$ for any s'; hence, (31) holds. It follows by duality that

$$K_1^* e^> \; : (H^{s'}(\Omega_>))^* \to H^{-s+\frac{1}{2}}(\Sigma), \quad \text{any } s', s \in \mathbb{R}, \tag{32}$$

here, $(H^{s'}(\Omega_>))^* = H^{-s'}(\Omega_>)$ when $|s'| < \frac{1}{2}$ (generally it equals the space $H_0^{-s'}(\Omega_>)$ of distributions in $H^{-s'}(\mathbb{R}^n)$ supported in $\overline{\Omega}_>$). Taking $s' = 0$, we see that

$$K_1^* e^> r^> K_1 : H^s(\Sigma) \to H^{s''}(\Sigma), \quad \text{for all } s, s'', \tag{33}$$

so since Σ is compact, this operator is compact (from $H^s(\Sigma)$ to $H^{s'}(\Sigma)$, any s, s'), and lies in $\bigcap_{r>0} \mathcal{C}_r$, i.e. is spectrally negligible. Then $K_{1,>}$ is compact from $H^s(\Sigma)$ to $L_2(\Omega_>)$ for any s, in particular for $s = \frac{1}{2}$, so $K_{1,>} K_{1,>}^*$ is compact in $L_2(\Omega_>)$, and hence $K_{1,>}^* : L_2(\Omega_>) \to H^{\frac{1}{2}}(\Sigma)$ is compact. In view of the identity $s_l(K_{1,>}^* K_{1,>}) = s_l(K_{1,>} K_{1,>}^*)$, all l, all four operators are spectrally negligible.

The proofs for $K_{2,>}$ and $K_{3,>}$ follow the same pattern. ∎

COROLLARY 3.2 *Let* $\eta \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ *be such that* $\eta = 1$ *on a neighbourhood of* $\overline{\Omega}_<$. *Then the operators* $K_{j,\eta} = (1-\eta)K_j$ *from* $H^{-\frac{1}{2}}(\Sigma)$ *to* $L_2(\Omega_+)$ *for* $j = 1$, *resp. from* $H^{-\frac{3}{2}}(\Sigma)$ *to* $L_2(\Omega_+)$ *for* $j = 2, 3$, *map continuously*

$$(1-\eta)K_j : H^{s-\frac{1}{2}}(\Sigma) \to H^{s'}(\Omega_+), \quad \textit{any } s, s' \in \mathbb{R}, \tag{34}$$

and are spectrally negligible.

Proof We can write $K_{j,\eta} = (1-\eta)K_j = e^>(1-\eta)K_{j,>}$, where Proposition 3.1 applies to $K_{j,>}$, and $e^>(1-\eta)$ is bounded from $H^{s'}(\Omega_>)$ to $H^{s'}(\Omega_+)$, any s'. ∎

COROLLARY 3.3 *Consider the singular Green operators* $G_j = -K_j T_j Q_+$ *as in* (15), (18) *with*

$$T_1 = \gamma_0, \quad T_2 = \nu_A, \quad T_3 = \nu_A - b\gamma_0. \tag{35}$$

For η *as in Corollary* 3.2, *the operators* $(1-\eta)G_j$ *are spectrally negligible.*

Proof This follows since $T_j Q_+$ is bounded from $L_2(\Omega_+)$ to $H^{\frac{3}{2}}(\Sigma)$ for $j = 1$, and from $L_2(\Omega_+)$ to $H^{\frac{1}{2}}(\Sigma)$ for $j = 2, 3$, and the $(1-\eta)K_j$ map into C^∞ and are spectrally negligible by Corollary 3.2. ∎

Remark 1 The proofs given above rely on the solvability properties of the exterior problems for A. The properties can also be inferred from a general principle shown in [9, Lemma 2.4.8], on cutoffs of Poisson operators, prepared for the definition on admissible manifolds (which include exterior domains). Moreover, the lemma deals with a parameter-dependent pseudodifferential boundary operator calculus, including a spectral parameter μ. In this setting, when we consider the Poisson operator family K_j^λ for $\{A-\lambda, T_j\}$, λ on a ray $\{\lambda=-\mu^2 e^{i\theta}\}$ in $\mathbb{C}\setminus\mathbb{R}_+$, it is of *regularity* $\nu=+\infty$. Lemma 2.4.8 then implies that $(1-\eta)K_j^\lambda$ is of order $-\infty$ and regularity $+\infty$, hence maps $H^{s,\mu}(\Sigma)\to H^{s',\mu}(\Omega_+)$ for all $s,s'\in\mathbb{R}$. Then $(K_j^\lambda)^*(1-\eta)^2K_j^\lambda$ maps $H^{s,\mu}(\Sigma)\to H^{s'',\mu}(\Sigma)$ for all $s,s''\in\mathbb{R}$. (The $H^{s,\mu}$-norms are based on the definition of the norm on $H^{s,\mu}(\mathbb{R}^n)$, namely $\|u\|_{s,\mu}=\|\mathcal{F}^{-1}(\langle(\xi,\mu)\rangle^s\mathcal{F}u)\|_{L_2(\mathbb{R}^n)}$.) From this we can conclude both Corollary 3.2 and the fact that any Schatten norm of $(1-\eta)K_j^\lambda$ is $O(\lambda^{-N})$ (any N) for $\lambda\to\infty$ on the ray, as first shown in [4]. Proposition 3.1 follows from this if we replace η by η_1 supported in $\complement\overline{\Omega}_>$ and equal to 1 on a neighbourhood of Ω_-; then $r^>K_j^\lambda=r^>(1-\eta_1)K_j^\lambda$. Also here we get the rapid decrease in λ of the Schatten norms.

We use the results first to reprove the theorems of Birman [1] and Birman–Solomiak [2] with a slight elaboration, essentially as in [3,4].

Theorem 3.4 *For* $j,k=1,2,3$, *let*

$$P_j=A_0^{-1}-A_j^{-1}\oplus 0_{L_2(\Omega_-)},\quad G_j'=A_0^{-1}-A_j^{-1}\oplus(A_0^{-1})_-,\quad G_{jk}=A_j^{-1}-A_k^{-1}. \tag{36}$$

Then

$$P_j\in T_{2/n},\quad G_j' \text{ and } G_{jk}\in T_{2/(n-1)}. \tag{37}$$

Moreover, there are spectral asymptotics formulae for $l\to\infty$:

$$s_l(P_j)l^{2/n}\to C_0,\quad s_l(G_{jk})l^{2/(n-1)}\to C_{jk}, \tag{38}$$

where the constants are determined from the principal symbols. Here C_0 *is the constant in the spectral asymptotics formula for* $(A_0^{-1})_-$, *namely* $C_0=\lim_{l\to\infty}s_l((A_0^{-1})_-)l^{2/n}$, *defined from the principal symbol of* A_0 *on* Ω_-.

Proof We use the notation in (15) ff. and Corollary 3.3; in particular, $A_0^{-1}=Q$. It is well-known that Q_- is compact, with the asserted spectral asymptotics.

Consider first G_{jk}; in view of (15) it can be written

$$G_{jk}=-K_jT_jQ_++K_kT_kQ_+. \tag{39}$$

Let η be as in Corollary 3.2 and let $\eta'\in C_0^\infty(\mathbb{R}^n,\mathbb{R})$, supported in a smooth bounded set Ω' and with $\eta'=1$ on a neighbourhood of $\operatorname{supp}\eta$. We can rewrite $-G_j=K_jT_jQ_+$ as follows:

$$K_jT_jQ_+=K_jT_j\eta Q_+=\eta'K_jT_j\eta Q_+\eta'+\eta'K_jT_j\eta Q_+(1-\eta')+(1-\eta')K_jT_j\eta Q_+. \tag{40}$$

Here the first term is a singular Green operator on $\Omega'\cap\Omega_+$ to which the calculus for bounded domains can be applied, and the two other terms are spectrally negligible. In fact, $(1-\eta')K_j$ is so by Corollary 3.2, and for $\eta Q(1-\eta')$ we can use that it maps $H^s(\mathbb{R}^n)$ continuously into $H^{s'}(\Omega')$ for all s and s' since $\operatorname{supp}\eta\cap\operatorname{supp}(1-\eta')=\emptyset$ so that the operator is of order $-\infty$. Then since Ω' is bounded, the operator is spectrally negligible, and so are its compositions with bounded operators.

The same arguments apply to $K_kT_kQ_+$, so we find that

$$G_{jk}=\eta'(-K_jT_j+K_kT_k)\eta Q_+\eta'+\mathcal{R}, \tag{41}$$

where $\mathcal{R}$ is spectrally negligible and the first term is a singular Green operator in $\Omega' \cap \Omega_+$. To the first term we apply [3, Theorem 4.10], which shows that this term is in $T_{2/(n-1)}$ and satisfies a spectral asymptotics formula as in (38); these facts are preserved when the spectrally negligible term $\mathcal{R}$ is added on. This shows the assertions for the G_{jk}.

The treatments of $K_j T_j Q_+$ in [3] (with misprints) and [4] are a bit more complicated in their use of commutators and nested sequences of cutoff functions.

Next, consider G'_j. Here, since $Q_+ \oplus 0 = e^+ r^+ Q e^+ r^+$ and $0 \oplus Q_- = e^- r^- Q e^- r^-$,

$$\begin{aligned} G'_j &= A_0^{-1} - A_j^{-1} \oplus (A_0^{-1})_- = Q - (Q_+ - K_j T_j Q_+) \oplus Q_- \\ &= Q - Q_+ \oplus Q_- + K_j T_j Q_+ \oplus 0 = e^+ r^+ Q e^- r^- + e^- r^- Q e^+ r^+ + K_j T_j Q_+ \oplus 0. \end{aligned}$$

For $\tilde{G} = e^+ r^+ Q e^- r^- + e^- r^- Q e^+ r^+$ we proceed as in [3, Theorem 5.1]: consider

$$\tilde{G}^2 = e^+ r^+ Q e^- r^- Q e^+ r^+ + e^- r^- Q e^+ r^+ Q e^- r^-. \tag{42}$$

The second term acts like $0 \oplus L_{\Omega_-}(Q, Q)$, where $L_{\Omega_-}(Q, Q) = Q_-^2 - Q_- Q_-$ is the 'leftover operator' for the composition of Q_- with Q_-; it is a singular Green operator and has a spectral asymptotics formula with exponent $4/(n-1)$, by [3, Theorem 4.10]. (It was in the quoted paper that the analysis of leftover operators in terms of $e^+ r^+ Q e^- r^-$ and $e^- r^- Q e^+ r^+$ was first introduced.)

The first term in (42) identifies similarly with a leftover operator on Ω_+, hence a singular Green operator, but since Ω_+ is unbounded, we need more argumentation to show that it is a compact operator with the desired spectral asymptotics. With η and η' as above, we can write:

$$\begin{aligned} L_{\Omega_+}(Q, Q) &= L_{\Omega_+}(Q\eta, \eta Q) \\ &= L_{\Omega_+}(\eta' Q\eta, \eta Q\eta') + L_{\Omega_+}((\eta' Q\eta, \eta Q(1-\eta')) + L_{\Omega_+}((1-\eta')Q\eta, \eta Q). \end{aligned} \tag{43}$$

Here $\eta Q(1-\eta')$ is spectrally negligible as noted above, and its adjoint $(1-\eta')Q\eta$ is likewise spectrally negligible. So $L_{\Omega_+}(Q, Q)$ is the sum of a spectrally negligible part and $L_{\Omega_+}(\eta' Q\eta, \eta Q\eta')$, a singular Green operator in $\Omega' \cap \Omega_+$.

Thus $\tilde{G}^2 = L_{\Omega_+}(\eta' Q\eta, \eta Q\eta') \oplus L_{\Omega_-}(Q, Q)$ plus spectrally negligible terms, so it follows from [3, Theroem 4.10] that $\tilde{G}^2$ has a spectral asymptotics behaviour $s_l(\tilde{G}^2) l^{4/(n-1)} \to C$, and then $\tilde{G}$ satisfies $s_l(\tilde{G}) l^{2/(n-1)} \to C^{\frac{1}{2}}$.

We still have to include the term $K_j T_j Q_+ \oplus 0$, but the nontrivial part was already treated further above, and is seen to have a similar spectral asymptotics behaviour. Adding all contributions and using the rules for s-numbers, we find that $G'_j \in T_{2/(n-1)}$.

For P_j, we simply use that

$$P_j = G'_j + 0_{L_2(\Omega_+)} \oplus Q_-, \tag{44}$$

where perturbation formulae as in [3] show that the spectral asymptotics formula for Q_- dominates the behaviour. One could moreover give remainder estimates (as done in [3]). ■

Remark 2 The estimates also hold when b for A_3 is replaced by a first-order differential operator B such that the realization is elliptic and invertible. Related results are found for $G_{jk}^{(N)} = A_j^{-N} - A_k^{-N}$, which is a singular Green operator on Ω_+ of the form of a sum of Poisson operators composed with trace operators; this leads to asymptotic estimates for all positive integers N: $s_l(G_{jk}^{(N)})l^{2N/(n-1)} \to C_{jk}^{(N)}$ for $l \to \infty$.

4. Perturbations

We shall now investigate the question of perturbations of the essential spectrum.

When $f \in L_2(\Omega_+)$, we also write $r^< f = f_<$, $r^> f = f_>$. Let us rewrite the action of $\widetilde{A}^{-1}$ on $f \in L_2(\Omega)$ in terms of its action on the parts $f_<$ and $f_>$, with matrix notation. When $u = \widetilde{A}^{-1}f$, we have that

$$u = \begin{pmatrix} u_< \\ u_> \end{pmatrix} = \widetilde{A}^{-1}\begin{pmatrix} f_< \\ f_> \end{pmatrix} = \begin{pmatrix} r^< \widetilde{A}^{-1} e^< & r^< \widetilde{A}^{-1} e^> \\ r^> \widetilde{A}^{-1} e^< & r^> \widetilde{A}^{-1} e^> \end{pmatrix}\begin{pmatrix} f_< \\ f_> \end{pmatrix}. \tag{45}$$

Recalling (28), we shall decompose the operators A_1^{-1} and $K_1 L K_1^*$ in a similar way. For A_1^{-1} we have:

$$\begin{aligned} A_1^{-1} &= \begin{pmatrix} r^< A_1^{-1} e^< & r^< A_1^{-1} e^> \\ r^> A_1^{-1} e^< & r^> A_1^{-1} e^> \end{pmatrix} \\ &= \begin{pmatrix} r^< A_1^{-1} e^< & r^< A_1^{-1} e^> \\ r^> A_1^{-1} e^< & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & r^> A_1^{-1} e^> \end{pmatrix}. \end{aligned} \tag{46}$$

The entries in the first matrix are compact in L_2-norm, since $r^< A_1^{-1} e^<$ maps $L_2(\Omega_<)$ into $H^2(\Omega_<)$ and $r^< A_1^{-1} e^>$ maps $L_2(\Omega_>)$ into $H^2(\Omega_<)$, where the injection $H^2(\Omega_<) \hookrightarrow L_2(\Omega_<)$ is compact, and $e^> r^> A_1^{-1} e^< r^<$ is the adjoint of $e^< r^< A_1^{-1} e^> r^>$ in $L_2(\Omega_+)$. Since a compact perturbation leaves the essential spectrum invariant, the second matrix has the same essential spectrum as A_1^{-1}, and we know from Theorem 3.4 that this equals $\sigma_{\mathrm{ess}} A_0^{-1}$. In other words,

$$A_1^{-1} = 0_{L_2(\Omega_<)} \oplus (r^> A_1^{-1} e^>) + S_1, \tag{47}$$

where S_1 is compact in $L_2(\Omega_+)$ and $\sigma_{\mathrm{ess}} A_1^{-1} = \sigma_{\mathrm{ess}} A_0^{-1}$.

Next, we write

$$\begin{aligned} K_1 L^{-1} K_1^* &= \begin{pmatrix} r^< K_1 L^{-1} K_1^* e^< & r^< K_1 L^{-1} K_1^* e^> \\ r^> K_1 L^{-1} K_1^* e^< & r^> K_1 L^{-1} K_1^* e^> \end{pmatrix} \\ &= \begin{pmatrix} r^< K_1 L^{-1} K_1^* e^< & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & r^< K_1 L^{-1} K_1^* e^> \\ r^> K_1 L^{-1} K_1^* e^< & r^> K_1 L^{-1} K_1^* e^> \end{pmatrix}. \end{aligned} \tag{48}$$

In the last matrix, every nonzero element is the composition of a bounded operator with either $r^> K_1$ or $K_1^* e^>$, hence is spectrally negligible in view of Proposition 3.1. So this whole matrix is spectrally negligible. In other words,

$$K_1 L^{-1} K_1^* = (r^< K_1 L^{-1} K_1^* e^<) \oplus 0_{L_2(\Omega_>)} + S_2, \tag{49}$$

where S_2 is spectrally negligible. In particular, $r^< K_1 L^{-1} K_1^* e^< \oplus 0_{L_2(\Omega_>)}$ has the same essential spectrum as $K_1 L^{-1} K_1^*$.

Recall, furthermore, that

$$K_1 L^{-1} K_1^* = \mathrm{i}_Z T^{-1} \mathrm{pr}_Z,$$

where Z is infinite dimensional, hence

$$\sigma_{\mathrm{ess}}(r^< K_1 L^{-1} K_1^* e^<) \cup \{0\} = \sigma_{\mathrm{ess}}(K_1 L^{-1} K_1^*) = \sigma_{\mathrm{ess}} T^{-1} \cup \{0\}. \tag{50}$$

Adding (47) and (49), setting $S = S_1 + S_2$, and observing that $0 \in \sigma_{\mathrm{ess}} A_0^{-1}$, $0 \in \sigma_{\mathrm{ess}} \widetilde{A}^{-1}$ (A_0 and $\widetilde{A}$ are unbounded operators), we conclude with the following theorem.

Theorem 4.1 *Let $\widetilde{A}$ be as in Proposition 2.1, and assume that $0 \in \varrho(\widetilde{A})$. Then $\widetilde{A}^{-1}$ can be written as the sum of a compact operator S in $L_2(\Omega_+)$ and an operator decomposed into a part acting in $L_2(\Omega_<)$ and a part acting in $L_2(\Omega_>)$:*

$$\widetilde{A}^{-1} = (r^< K_1 L^{-1} K_1^* e^<) \oplus (r^> A_1^{-1} e^>) + S. \tag{51}$$

Here,

$$\sigma_{\mathrm{ess}}(r^< K_1 L^{-1} K_1^* e^<) \cup \{0\} = \sigma_{\mathrm{ess}} T^{-1} \cup \{0\}, \quad \sigma_{\mathrm{ess}}(r^> A_1^{-1} e^>) \cup \{0\} = \sigma_{\mathrm{ess}} A_0^{-1}, \tag{52}$$

and hence

$$\sigma_{\mathrm{ess}} \widetilde{A}^{-1} = \sigma_{\mathrm{ess}} T^{-1} \cup \sigma_{\mathrm{ess}} A_0^{-1}. \tag{53}$$

Since the essential spectrum of $\widetilde{A}$ itself is the reciprocal set of the nonzero essential spectrum of $\widetilde{A}^{-1}$, we also have the following corollary.

Corollary 4.2 *When $\widetilde{A}$ is as in Theorem 4.1,*

$$\sigma_{\mathrm{ess}} \widetilde{A} = \sigma_{\mathrm{ess}} A_0 \cup \sigma_{\mathrm{ess}} T. \tag{54}$$

In particular, $\sigma_{\mathrm{ess}} \widetilde{A}$ contains all points of $\sigma_{\mathrm{ess}} T$, and the points in $\sigma_{\mathrm{ess}} A_0$ cannot be removed from $\sigma_{\mathrm{ess}} \widetilde{A}$.

The statements in Section 1 follow: in case (1) we take T as in Remark 1 of Section 2 in order to add a point $\{a\}$; when T acts like aI, C acts like $a\Lambda_{(-1)} + P_{\gamma_0, \nu_A}$. A general choice of a selfadjoint invertible T_0 in a separable infinite-dimensional Hilbert space Z_0 gives rise to a selfadjoint invertible T in the Hilbert space Z with the same essential spectrum, by the use of a unitary operator from Z_0 to Z. The statement in (2) follows since we have covered all possibilities for T in the case of Neumann-type boundary conditions.

5. Higher order cases

Similar results can be shown for higher order elliptic operators. The selfadjoint strongly elliptic even-order case is the natural generalization of the case considered in the preceding sections; here, there is a solvable Dirichlet problem, and a selfadjoint invertible realization defined by another boundary condition can be related to the Dirichlet realization by a Kreĭn-type formula generalizing (28) as in [10,11,14].

Invertible realizations exist in greater generality, though, so to save later repetitions, we consider to begin with a more general class assuring existence of resolvents $(\widetilde{A}-\lambda)^{-1}$ at least when λ is large, lying in a suitable subset of $\mathbb{C}$. We take for A an elliptic operator $A=\sum_{|\alpha|\le 2m} a_\alpha(x)D^\alpha$ of order $2m$, m integer, with complex C^∞ coefficients on $\mathbb{R}^n$ that are bounded with bounded derivatives and with the principal symbol $a^0(x,\xi)=\sum_{|\alpha|=2m} a_\alpha\xi^\alpha$ satisfying (with $c_1>0$)

$$\operatorname{Re} a^0(x,\xi)\ge c_1|\xi|^{2m}, \quad \text{for } x,\xi\in\mathbb{R}^n, \tag{55}$$

uniform strong ellipticity. Here $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_j=-i\partial/\partial x_j$. Denote by A_0 the maximal realization of A in $L_2(\mathbb{R}^n)$; the uniform ellipticity implies that $D(A_0)=H^{2m}(\mathbb{R}^n)$.

A satisfies a Gårding inequality, for which we include a quick proof:

LEMMA 5.1 *There are constants $c_0>0$ and $k\in\mathbb{R}$ such that*

$$\operatorname{Re}(Au,u)\ge c_0\|u\|_m^2-k\|u\|_0^2, \quad \text{for all } u\in H^{2m}(\mathbb{R}^n). \tag{56}$$

Proof Using the calculus of globally estimated pseudodifferential operators as in [15, Section 18.1], and [9], we can write

$$A_1=\Lambda^{-m}A\Lambda^{-m}, \quad \operatorname{Re} A_1=\frac{1}{2}(A_1+A_1^*)=P^*P+B,$$

where $\Lambda^s=\operatorname{Op}(\langle\xi\rangle^s)$, A_1 is of order 0 with principal symbol $a_1^0(x,\xi)$ satisfying

$$\operatorname{Re} a_1^0(x,\xi)\ge c_1'>0,$$

P is of order 0 with principal symbol $p^0=(\operatorname{Re} a_1^0)^{\frac{1}{2}}$, and B is of order -1. Since P is elliptic, it has a parametrix Q of order 0 so that $I-QP$ is of order -1; hence,

$$\begin{aligned}\|v\|_0^2&=\|QPv+(I-QP)v\|_0^2\le C\|Pv\|_0^2+C'\|v\|_{-1}^2\\&=C(P^*Pv,v)+C'\|v\|_{-1}^2\le C\operatorname{Re}(A_1v,v)+C''\|v\|_{-\frac{1}{2}}^2,\end{aligned}$$

for $v\in\mathcal{S}(\mathbb{R}^n)$ (the Schwartz space of rapidly decreasing functions, dense in any $H^s(\mathbb{R}^n)$). It follows that (with $\|u\|_s=\|\Lambda^s u\|_{L_2(\mathbb{R}^n)}$)

$$\begin{aligned}\operatorname{Re}(Au,u)&=\operatorname{Re}(A_1\Lambda^m u,\Lambda^m u)\\&\ge C^{-1}\|u\|_m^2-C^{-1}C''\|u\|_{m-\frac{1}{2}}^2\ge\frac{1}{2}C^{-1}\|u\|_m^2-k\|u\|_0^2,\end{aligned}$$

where we used that $\|u\|_{m-\frac{1}{2}}^2\le\varepsilon\|u\|_m^2+C(\varepsilon)\|u\|_0^2$, any $\varepsilon>0$. ■

Since $|(Au,u)|\le C_1\|u\|_m^2$ and $\|u\|_m\ge\|u\|_0$, we can infer from (56) that

$$\begin{aligned}|\operatorname{Im}(Au,u)|&\le|(Au,u)|\le C_1\|u\|_m^2\le C_1c_0^{-1}(\operatorname{Re}(Au,u)+k\|u\|_0^2)\\\operatorname{Re}(Au,u)&\ge c_0\|u\|_0^2-k\|u\|_0^2=(c_0-k)\|u\|_0^2,\end{aligned}$$

hence, the numerical range of A_0, $\nu(A_0)=\{(A_0u,u)/\|u\|_0^2 \mid u\in D(A_0)\setminus\{0\}\}$, is contained in a sectorial region V,

$$\nu(A_0)\subset V\equiv\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda\ge c_0-k,\ |\operatorname{Im}\lambda|\le c_2(\operatorname{Re}\lambda+k)\}, \tag{57}$$

with $c_2 = C_1 c_0^{-1}$. The numerical range of the adjoint A_0^* is likewise contained in V, and V contains the spectrum of A_0. (The elementary functional analysis used here is explained e.g. in [8, Chapter 12].)

For simplicity we add kI to A, so that we can use the information with $k=0$ in the following, replacing V by

$$V_0 = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq c_0,\ |\operatorname{Im}\lambda| \leq c_2 \operatorname{Re}\lambda\}. \tag{58}$$

The Dirichlet trace γu is in the $2m$-order case defined by

$$\gamma u = \{\gamma_0 u, \dots, \gamma_{m-1} u\},$$

with $\gamma_j u = \gamma_0 (\sum \nu_k D_k)^j u$. For the Dirichlet problems on smooth exterior or interior subsets of $\mathbb{R}^n$, the variational construction gives a realization with numerical range and spectrum likewise contained in V_0. Moreover, there are Sobolev space mapping properties of the solution operator; this is extremely well-known for bounded domains, and for exterior domains it is covered e.g. by Corollary 3.3.3 in [9] (the differential operator $A-\lambda$ is uniformly parameter-elliptic on all rays $\{\lambda = re^{i\theta} \mid r \geq 0\}$ outside V_0, and parameter-ellipticity of the boundary problem holds uniformly for x' in the boundary).

Let us specify the result for Ω_+ and Σ defined as in Section 1. We denote $A_0^{-1} = Q$. Then

$$\mathcal{A}_\gamma = \begin{pmatrix} A \\ \gamma \end{pmatrix} : H^{s+2m}(\Omega_+) \to \begin{matrix} H^s(\Omega_+) \\ \times \\ \prod_{0\leq j<m} H^{s+2m-j-\frac{1}{2}}(\Sigma) \end{matrix} \tag{59}$$

has for $s > -m - \frac{1}{2}$ the solution operator, continuous in the opposite direction,

$$\mathcal{A}_\gamma^{-1} = \begin{pmatrix} R_\gamma & K_\gamma \end{pmatrix}, \quad \text{with } R_\gamma = Q_+ - K_\gamma \gamma Q_+. \tag{60}$$

Here, R_γ is the inverse of the Dirichlet realization A_γ, which acts like A with domain $D(A_\gamma) = H^{2m}(\Omega_+) \cap H_0^m(\Omega_+)$.

The general theory in [10,11] is here interpreted by the use of the Poisson operator $K_\gamma : \prod_{0\leq j<m} H^{-j-\frac{1}{2}}(\Sigma) \to L_2(\Omega_+)$ (and variants with λ-dependence). K_γ acts as an inverse of

$$\gamma_Z : Z \xrightarrow{\sim} \prod_{0\leq j<m} H^{-j-\frac{1}{2}}(\Sigma), \tag{61}$$

Z denoting the $L_2(\Omega_+)$ nullspace of A. The formulae are exactly the same as in [11, Section 3.3].

We shall compare A_γ with the realization $A_{B\varrho}$ of a general *normal* boundary condition, defined as in [11,16, (3.85)]. Let

$$M = \{0, 1, \dots, 2m-1\}, \quad \text{denoting } \varrho u = \{\gamma_j u\}_{j\in M},$$

the Cauchy data. Let J be a subset of M with m elements, and let B be a $J \times M$-matrix of differential operators B_{jk} on Σ of order $j-k$:

$$B = (B_{jk})_{j\in J, k\in M} \text{ with } B_{jk} = 0 \quad \text{for } k > j,\ B_{jj} = I. \tag{62}$$

The boundary condition [11, (3.85)]: $\gamma_j u + \sum_{k<j} B_{jk}\gamma_k u = 0$ for $j \in J$, can then be written

$$B\varrho u = 0,$$

it defines the realization $A_{B\varrho}$ with domain

$$D(A_{B\varrho}) = \{u \in D(A_{\max}) \mid B\varrho u = 0\}.$$

Special examples are the cases where $J = M_0$ or M_1,

$$M_0 = \{0, 1, \dots, m-1\}, \quad M_1 = \{m, m+1, \dots, 2m-1\}, \text{ denoting } \nu u = \{\gamma_j u\}_{j \in M_1},$$

they define Dirichlet-type resp. Neumann-type conditions.

Let us assume that $\{A - \lambda, B\varrho\}$ is uniformly parameter-elliptic for λ on a ray outside V_0; then for large λ on the ray, $\{A - \lambda, B\varrho\}$ is invertible. Take a λ_0 where this invertibility holds (assuming invertibility of $A_0 - \lambda_0$ and $A_\gamma - \lambda_0$ also), and denote in the rest of this section $A - \lambda_0$ by A; then, we are in the situation where

$$\mathcal{A}_{B\varrho} = \begin{pmatrix} A \\ B\varrho \end{pmatrix} : H^{s+2m}(\Omega_+) \to \begin{matrix} H^s(\Omega_+) \\ \times \\ \prod_{j\in J} H^{s+2m-j-\frac{1}{2}}(\Sigma) \end{matrix} \tag{63}$$

for $s > -\frac{1}{2}$ has the solution operator, continuous in the opposite direction,

$$\mathcal{A}_{B\varrho}^{-1} = \begin{pmatrix} R_{B\varrho} & K_{B\varrho} \end{pmatrix}, \quad \text{with } R_{B\varrho} = Q_+ - K_{B\varrho} B\varrho Q_+. \tag{64}$$

Here $R_{B\varrho}$ is the inverse of the realization $A_{B\varrho}$, which acts like A with domain $D(A_{B\varrho}) = \{u \in H^{2m}(\Omega_+) \mid B\varrho u = 0\}$.

The difference between A_γ^{-1} and $A_{B\varrho}^{-1}$, and more generally between two solution operators $A_{B\varrho}^{-1}$ and $A_{\tilde{B}\varrho}^{-1}$, can be described spectrally very much like in Section 3. First, there is a generalization of Proposition 3.1 and its corollaries. Define $\Omega_{\gtrless}$, $r^{\gtrless}$ and $e^{\gtrless}$ as in Section 3.

Proposition 5.2

(1)° *The operators* $K_{B\varrho,>} = r^> K_{B\varrho} : \prod_{j\in J} H^{-j-\frac{1}{2}}(\Sigma) \to L_2(\Omega_>)$ *and* $(K_{B\varrho,>})^* = K_{B\varrho}^* e^> : L_2(\Omega_>) \to \prod_{j\in J} H^{j+\frac{1}{2}}(\Sigma)$ *map continuously*

$$\begin{aligned} r^> K_{B\varrho} &: \prod_{j\in J} H^{s-j-\frac{1}{2}}(\Sigma) \to H^{s'}(\Omega_>), \quad \text{any } s, s' \in \mathbb{R}, \\ K_{B\varrho}^* e^> &: (H^{s'}(\Omega_>))^* \to \prod_{j\in J} H^{-s+j+\frac{1}{2}}(\Sigma), \quad \text{any } s', s \in \mathbb{R}, \end{aligned} \tag{65}$$

and are spectrally negligible.

(2)° *When η is a function in $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ that is 1 on a neighbourhood of $\overline{\Omega}_<$, the operators* $(1-\eta)K_{B\varrho} : \prod_{j\in J} H^{-j-\frac{1}{2}}(\Sigma) \to L_2(\Omega_+)$ *and* $K_{B\varrho}^*(1-\eta) : L_2(\Omega_+) \to \prod_{j\in J} H^{j+\frac{1}{2}}(\Sigma)$ *map continuously*

$$\begin{aligned} (1-\eta) K_{B\varrho} &: \prod_{j\in J} H^{s-j-\frac{1}{2}}(\Sigma) \to H^{s'}(\Omega_+), \quad \text{any } s, s' \in \mathbb{R}, \\ K_{B\varrho}^*(1-\eta) &: (H^{s'}(\Omega_+))^* \to \prod_{j\in J} H^{-s+j+\frac{1}{2}}(\Sigma), \quad \text{any } s', s \in \mathbb{R}, \end{aligned} \tag{66}$$

and are spectrally negligible.

Proof Denote by $\gamma^>$ the Dirichlet trace operator for $2m$-order operators on $\Omega_>$. When $\varphi \in \prod_{j\in J} H^{-j-\frac{1}{2}}(\Sigma)$, $K_{B_Q}\varphi$ is C^∞ on Ω_+, hence $\gamma^> K_{B_Q}\varphi \in C^\infty(\partial\Omega_>)$. Then $r^> K_{B_Q}\varphi$ is a null-solution of the Dirichlet problem for A on $\Omega_>$ with C^∞-boundary value. This will also hold if $\varphi \in H^{s-\frac{1}{2}}(\Sigma)$, any $s \in \mathbb{R}$. We now use that the Dirichlet problem on $\Omega_>$ has a solution operator with mapping properties similar to the problem for Ω_+, and the proof is completed in the same way as the proofs of Proposition 3.1 and Corollary 3.2. ■

Remark 1 It may be observed as in Remark 1 of Section 3 that the proofs can also be inferred from [9, Lemma 2.4.8], and this moreover implies a rapid decrease in λ of any Schatten norm.

With Proposition 5.2 it is easy to generalize Theorem 3.4 as follows:

Theorem 5.3 *For B_Q and $\tilde{B}_Q$ as above, defining invertible elliptic realizations, let*

$$P = A_0^{-1} - A_{B_Q}^{-1} \oplus 0_{L_2(\Omega_-)}, \quad G' = A_0^{-1} - A_{B_Q}^{-1} \oplus (A_0^{-1})_-, \quad G'' = A_{B_Q}^{-1} - A_{\tilde{B}_Q}^{-1}. \tag{67}$$

Then

$$P \in T_{2m/n}, \quad G' \text{ and } G'' \in T_{2m/(n-1)}. \tag{68}$$

Moreover, there are spectral asymptotics formulae for $l \to \infty$:

$$s_l(P) l^{2m/n} \to C, \quad s_l(G'') l^{2m/(n-1)} \to C'', \tag{69}$$

where the constants are determined from the principal symbols. Here C is the same constant as for $(A_0^{-1})_-$, namely $C = \lim_{l\to\infty} s_l((A_0^{-1})_-) l^{2m/n}$.

Proof We proceed as in the proof of Theorem 3.4. First,

$$G'' = -K_{B_Q} B_Q Q_+ + K_{\tilde{B}_Q} \tilde{B}_Q Q_+$$

is written as a singular Green operator on $\Omega' \cap \Omega_+$ plus a spectrally negligible term, by a version of (40) applied to both terms. The assertions for G'' then follow from [3, Theorem 4.10]. Next, G' is treated similarly to G'_j in Theorem 3.4, noting that the operators of order 2 have been replaced by operators of order $2m$. Finally, $P = G' + 0_{L_2(\Omega+)} \oplus Q_-$, where the spectral asymptotics behaviour of Q_- dominates the sum, in view of the rules for s-numbers. ■

It is here allowed to take the set J for B_Q different from the corresponding set $\tilde{J}$ for $\tilde{B}_Q$. There are similar results for differences between higher powers $A_{B_Q}^{-N} - A_{\tilde{B}_Q}^{-N}$, as in Remark 2 of Section 3.

A result of the type $A_{B_Q}^{-N} - A_{\tilde{B}_Q}^{-N} \in T_{2mN/(n-1)}$ has been announced by Gesztesy and Malamud in [6], apparently based on a consideration of M-functions.

In all the calculations, A can be taken to be a $(p \times p)$-system, acting on p-vectors. When A is scalar, the boundary conditions with (62) are the most general ones for which parameter-ellipticity can hold (cf [9, Section 1.5]); in the systems case there exist more general normal boundary conditions, as studied in [14]. The above analysis can be extended to include these, mainly at the cost of a more complicated notational apparatus. Pseudodifferential B_{jk} could be allowed as in [9].

Remark 2 For bounded domains, the result for G'' has been known since 1984, since $A^{-1}_{B_Q} - A^{-1}_{\tilde{B}_Q}$ is then itself a singular Green operator of order $-2m$ on a bounded domain, to which [3, Theorem 4.10] applies. For selfadjoint cases, see also [14, Section 8].

6. Spectral perturbations in higher order cases

For the study of perturbations of essential spectra, we restrict the attention to selfadjoint realizations. First of all, this requires that A equals its formal adjoint A' moreover, it restricts the sets J and matrices B that can be allowed. With the notation $N' = \{k \mid 2m - k - 1 \in N\}$, we have as a necessary condition on J is that it should equal its reversed complement:

$$J = K', \quad \text{where } K = M \setminus J. \tag{70}$$

To explain further, we recall some details from [16]. From Green's formula

$$(Au, v) - (u, Av) = (\mathcal{A}\varrho u, \varrho v) = \left(\begin{pmatrix} \mathcal{A}_{M_0M_0} & \mathcal{A}_{M_0M_1} \\ \mathcal{A}_{M_1M_0} & 0 \end{pmatrix} \begin{pmatrix} \gamma u \\ \nu u \end{pmatrix}, \begin{pmatrix} \gamma u \\ \nu u \end{pmatrix} \right),$$

where $\mathcal{A}$ is skew-selfadjoint and invertible, it is seen that when we set

$$\chi u = \mathcal{A}_{M_0M_1}\nu u + \frac{1}{2}\mathcal{A}_{M_0M_0}\gamma u, \tag{71}$$

(taking $\frac{1}{2}$ of the contribution from $\mathcal{A}_{M_0M0}$ along), we get the symmetric formula

$$(Au, v)_{L_2(\Omega_+)} - (u, Av)_{L_2(\Omega_+)} = (\chi u, \gamma v)_{L_2(\Sigma)^m} - (\gamma u, \chi v)_{L_2(\Sigma)^m}, \tag{72}$$

valid for $u, v \in H^{2m}(\Omega_+)$. Here χ is indexed by M_0, $\chi = \{\chi_j\}_{j \in M_0}$ with χ_j of order $2m - j - 1$; it replaces ν in systematic considerations and maps from $H^s(\Omega_+)$ to $\prod_{j \in M_0} H^{s-2m+j+\frac{1}{2}}(\Sigma)$. Green's formula has the extension to $u \in D(A_{\max})$, $v \in H^{2m}(\Omega_+)$:

$$(Au, v)_{L_2(\Omega_+)} - (u, Av)_{L_2(\Omega_+)} = (\chi u, \gamma v)_{\{-2m+j+\frac{1}{2}\},\{2m-j-\frac{1}{2}\}} - (\gamma u, \chi v)_{\{-j-\frac{1}{2}\},\{j+\frac{1}{2}\}},$$

where $(\cdot,\cdot)_{\{-s_j\},\{sj\}}$ denotes the duality between $\prod H^{-s_j}(\Sigma)$ and $\prod H^{s_j}(\Sigma)$. With χ replaced by the 'reduced Neumann trace operator' Γ , one has for $u, v \in D(A_{\max})$:

$$(Au, v)_{L_2(\Omega_+)} - (u, Av)_{L_2(\Omega_+)} = (\Gamma u, \gamma v)_{\{j+\frac{1}{2}\},\{-j-\frac{1}{2}\}} - (\gamma u, \Gamma v)_{\{-j-\frac{1}{2}\},\{j+\frac{1}{2}\}}, \tag{73}$$

here,

$$P_{\gamma,\chi} = \chi K_\gamma, \quad \Gamma = \chi - P_{\gamma,\chi}\gamma = \chi A_\gamma^{-1} A_{\max}. \tag{74}$$

Now when J satisfies (70), the subsets

$$J_0 = J \cap M_0, \quad J_1 = J \cap M_1, \quad K_0 = K \cap M_0, \quad K_1 = K \cap M_1,$$

satisfy

$$K_1' = J_0, \quad J_1' = K_0. \tag{75}$$

We set

$$\gamma_{J_0} = \{\gamma_j\}_{j \in J_0}, \quad \gamma_{K_0} = \{\gamma_j\}_{j \in K_0}, \quad \chi_{J_0} = \{\chi_j\}_{j \in J_0}, \quad \chi_{K_0} = \{\chi_j\}_{j \in K_0}.$$

As shown in [16] and recalled in [11], the boundary condition $B\varrho u = 0$ may then be rewritten in the form, with differential operators F_0, G_1, G_2,

$$\gamma_{J_0} u = F_0 \gamma_{K_0} u, \quad \chi_{K_0} u = G_1 \gamma_{K_0} u + G_2 \chi_{J_0} u, \tag{76}$$

when we take (75) into account. Here the first condition $\gamma_{J_0} u = F_0 \gamma_{K_0} u$ can be viewed as the 'Dirichlet part', purely concerned with γu, whereas the second condition $\chi_{K_0} u = G_1 \gamma_{K_0} u + G_2 \chi_{J_0} u$ can be viewed as the 'Neumann-type part', where part of the Neumann data $\chi_{K_0} u$ is given as a function of the other data. Note that G_1 links the free Dirichlet data $\gamma_{K_0} u$ to Neumann data and has entries of positive order, and G_2 has entries of order $< m$.

The boundary condition for the adjoint realization is then

$$\gamma_{J_0} u = -G_2^* \gamma_{K_0} u, \quad \chi_{K_0} u = G_1^* \gamma_{K_0} u - F_0^* \chi_{J_0} u. \tag{77}$$

If $J = M_0$, the condition $B\varrho u = 0$ reduces to the Dirichlet condition $\gamma u = 0$. To get a different condition we must take $J \neq M_0$; this means that $K_0 \neq \emptyset$.

We assume in the following that $\{A - \lambda, B\varrho\}$ is uniformly parameter-elliptic on a ray outside V_0 as in the preceding section, so that $D(A_{B\varrho}) \subset H^{2m}(\Omega_+)$. Then

$$G_2^* = -F_0, \quad G_1^* = G_1 \tag{78}$$

are necessary and sufficient for selfadjointness of $A_{B\varrho}$. Equation (78) is assumed from now on.

The operator $A_{B\varrho}$ corresponds to a selfadjoint operator $T: V \to V$ by the general theory, where V is the $L_2(\Omega_+)$-closure of $\mathrm{pr}_\gamma\, D(A_{B\varrho})$ (here $\mathrm{pr}_\gamma = I - A_\gamma^{-1} A_{\max}$). V is mapped by γ onto the closure X of $\gamma D(A_{B\varrho})$ in $\prod_{k \in M_0} H^{-k-\frac{1}{2}}(\Sigma)$. Here X is the graph of F_0, so it is homeomorphic to $\prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Sigma)$, by the mappings

$$\Phi = \begin{pmatrix} I_{K_0 K_0} \\ F_0 \end{pmatrix}, \quad \mathrm{pr}_1 = \begin{pmatrix} I & 0 \end{pmatrix}, \tag{79}$$

$$\Phi : \prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Sigma) \xrightarrow{\sim} X, \quad \mathrm{pr}_1 : X \xrightarrow{\sim} \prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Sigma), \tag{80}$$

as shown in [16] and recalled in [11]. Here $V = K_\gamma X = K_\gamma \Phi \prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Sigma)$.

The restriction of γ to a mapping from V to X is denoted γ_V, so we have:

$$\gamma_V : V \xrightarrow{\sim} X, \quad \mathrm{pr}_1 \gamma_V : V \xrightarrow{\sim} \prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Sigma), \quad \gamma_V^{-1} \Phi : \prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Sigma) \xrightarrow{\sim} V.$$

With these definitions, (76) may be written (using (78))

$$\gamma u = \Phi \gamma_{K_0} u, \quad \Phi^* \chi u = G_1 \gamma_{K_0} u. \tag{81}$$

The operator T in V is carried over to an operator

$$L = (\gamma_V^*)^{-1} T \gamma_V^{-1} : X \to X^*, \tag{82}$$

which is further translated to an operator

$$L_1 = \Phi^* L \Phi : \prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Sigma) \to \prod_{k \in K_0} H^{k+\frac{1}{2}}(\Sigma). \tag{83}$$

We now recall from [16] how the form of L_1 is determined (this detail was not repeated in [11]). Consider the condition defining the correspondence between $A_{B\varrho}$ and T (cf [10,11,16]):

$$(Au, z) = (T \operatorname{pr}_\zeta u, z) \quad \text{for all } u \in D(A_{B\varrho}),\ z \in V. \tag{84}$$

Here the right-hand side is rewritten as

$$(T \operatorname{pr}_\zeta u, z) = (L\gamma u, \gamma z)_{\{k+\frac12\},\{-k-\frac12\}} = (L_1 \gamma_{K_0} u, \gamma_{K_0} z)_{\{k+\frac12\},\{-k-\frac12\}},$$

whereas the left-hand side takes the form, in view of (73) and (81):

$$\begin{aligned}(Au, z) &= (\Gamma u, \gamma z) = (\chi u - P_{\gamma,\chi}\gamma u, \Phi\gamma_{K_0} z)_{\{k+\frac12\},\{-k-\frac12\}}\\ &= (\Phi^*\chi u - \Phi^* P_{\gamma,\chi}\Phi\gamma_{K_0} u, \gamma_{K_0} z)_{\{k+\frac12\},\{-k-\frac12\}}\\ &= ((G_1 - \Phi^* P_{\gamma,\chi}\Phi)\gamma_{K_0} u, \gamma_{K_0} z)_{\{k+\frac12\},\{-k-\frac12\}}.\end{aligned}$$

Since $\gamma_{K_0} z$ runs in a dense subset of $\prod_{k\in K_0} H^{-k-\frac12}(\Sigma)$, (84) implies

$$L_1\gamma_{K_0} u = (G_1 - \Phi^* P_{\gamma,\chi}\Phi)\gamma_{K_0} u, \tag{85}$$

so L_1 acts like $G_1 - \Phi^* P_{\gamma,\chi}\Phi$. The boundary condition may then be rewritten as

$$\gamma u = \Phi\gamma_{K_0} u, \quad \Phi^*\chi u = (L_1 + \Phi^* P_{\gamma,\chi}\Phi)\gamma_{K_0} u. \tag{86}$$

Since $\{A, B\varrho\}$ is elliptic, L_1 is an elliptic selfadjoint mixed-order pseudodifferential operator; its domain is $D(L_1) = \prod_{k\in K_0} H^{2m-k-\frac12}(\Sigma)$.

When $A_{B\varrho}$ is invertible, so are T, L and L_1, and [10, Theorem II.1.4] implies

$$A_{B\varrho}^{-1} = A_\gamma^{-1} + \mathrm{i}_V T^{-1}\operatorname{pr}_V = A_\gamma^{-1} + K_\gamma\Phi L_1^{-1}\Phi^* K_\gamma^*. \tag{87}$$

(It is used here that $\mathrm{i}_V\gamma_V^{-1}\Phi = K_\gamma\Phi$.)

All this is just the implementation of the known results to operators defined for the unbounded set Ω_+. But now we are in a position to consider interesting perturbations.

We replace $T: V \to V$ for $A_{B\varrho}$ by an operator $\widetilde{T}: V \to V$, selfadjoint invertible with a nonempty essential spectrum, and want to see how this effects the realization. As above, $\widetilde{T}$ carries over to

$$\widetilde{L}_1 = \Phi^*(\gamma_V^*)^{-1}\widetilde{T}\gamma_V^{-1}\Phi : \prod_{k\in K_0} H^{-k-\frac12}(\Sigma) \to \prod_{k\in K_0} H^{k+\frac12}(\Sigma), \tag{88}$$

with $D(\widetilde{L}_1) = \operatorname{pr}_1\gamma D(\widetilde{T})$, and the boundary condition now takes the form

$$\begin{aligned}&\gamma u = \Phi\gamma_{K_0} u, \quad \Phi^*\chi u = \widetilde{G}_1\gamma_{K_0} u, \quad \gamma_{K_0} u \in D(\widetilde{L}_1),\\ &\qquad \text{where } \widetilde{G}_1 = \widetilde{L}_1 + \Phi^* P_{\gamma,\chi}\Phi = G_1 + \widetilde{L}_1 - L_1.\end{aligned} \tag{89}$$

Here

$$\widetilde{A}^{-1} = A_\gamma^{-1} + \mathrm{i}_V\widetilde{T}^{-1}\operatorname{pr}_V = A_\gamma^{-1} + K_\gamma\Phi\widetilde{L}_1^{-1}\Phi^* K_\gamma^*. \tag{90}$$

THEOREM 6.1 *Consider the realization $A_{B\varrho}$ of A in $L_2(\Omega_+)$ defined by a normal boundary condition $B\varrho u = 0$ (cf (62)) with $J \neq M_0$, and assume that ellipticity and* *selfadjointness holds, cf* (76)–(78). A_{B_Q} *corresponds to an operator* $T: V \to V$, *where* $V = K_\gamma X = K_\gamma \Phi \prod_{k\in K_0} H^{-k-\frac{1}{2}}(\Sigma)$, *cf also* (82), (83), (85).

Let $\widetilde{T}$ *be a selfadjoint invertible operator in* V *with nonempty essential spectrum, and let* $\widetilde{A}$ *be the realization of* A *corresponding to* $\widetilde{T}: V \to V$, *i.e. where the boundary condition* (76), *equivalently written* (81), *is replaced by* (89). *Then*

$$\sigma_{\rm ess}\widetilde{A} = \sigma_{\rm ess}A_0 \cup \sigma_{\rm ess}\widetilde{T}. \tag{91}$$

Proof The proof goes in exactly the same way as the proof of Theorem 4.1 and Corollary 4.2. We cut Ω_+ in a bounded part $\Omega_<$ and an exterior part $\Omega_>$, and use (90) and Proposition 5.2 with $B_Q = \gamma$ to see that $\widetilde{A}$ can be written as

$$\widetilde{A}^{-1} = \left(r^< K_\gamma \Phi \widetilde{L}_1^{-1} \Phi^* K_\gamma^* e^<\right) \oplus (r^> A_\gamma^{-1} e^>) + S, \tag{92}$$

where S is compact, the operator $r^> A_\gamma^{-1} e^>$ in $L_2(\Omega_>)$ has the same essential spectrum as A_γ^{-1}, and the operator $r^< K_\gamma \Phi \widetilde{L}_1^{-1} \Phi^* K_\gamma^* e^<$ in $L_2(\Omega_<)$ has the same essential spectrum as $K_\gamma \Phi \widetilde{L}_1^{-1} \Phi^* K_\gamma^* = {\rm i}_V \widetilde{T}^{-1} {\rm pr}_V$ outside 0. ■

Briefly expressed, the theorem states that any normal boundary condition (apart from the Dirichlet condition) defining a selfadjoint invertible elliptic realization, can be perturbed by addition of a suitable operator to G_1 (the map from the free Dirichlet data to Neumann data) to provide a selfadjoint invertible realization with a prescribed augmentation of the essential spectrum.

Example 6.2 Let $A = \Delta^2 + 1$. Clearly, A satisfies the positivity and selfadjointness requirements, and it has Green's formula (72) with

$$\gamma = \{\gamma_0, \gamma_1\}, \quad \chi = \{\chi_0, \chi_1\} = \{-\gamma_1\Delta, \gamma_0\Delta\},$$

as in [11, Example 3.14]. The Dirichlet operator

$$\mathcal{A}_\gamma = \begin{pmatrix} \Delta^2 + 1 \\ \gamma \end{pmatrix} : H^{s+4}(\Omega_+) \to \begin{matrix} H^s(\Omega_+) \\ \times \\ H^{s+\frac{7}{2}}(\Sigma) \times H^{s+\frac{5}{2}}(\Sigma) \end{matrix}, \tag{93}$$

where $s > -\frac{5}{2}$, has an inverse $(R_\gamma \;\; K_\gamma)$ continuous in the opposite direction. Let us take (as in [11, Example 3.14]) $J = \{0, 2\} \subset M = \{0, 1, 2, 3\}$; it satisfies (70), and $J_0 = \{0\}$, $K_0 = \{1\}$. With this choice, the boundary condition (76) is of the form

$$\gamma_0 u = 0, \quad \gamma_0 \Delta u = G_1 \gamma_1 u. \tag{94}$$

(F_0 and G_2 vanish, being differential operators of negative order.) G_1 is of order 1. Selfadjointness of A_{B_Q} requires $G_1^* = G_1$, and if this holds and the problem is elliptic, then A_{B_Q} is selfadjoint with domain $D(A_{B_Q}) = \{u \in H^4(\Omega_+) \mid (94) \text{ holds.}\}$ Continuing under this assumption, we find that

$$X = \{0\} \times H^{-\frac{3}{2}}(\Sigma), \text{ naturally identified with } H^{-\frac{3}{2}}(\Sigma),$$

and L_1 is the first-order pseudodifferential operator

$$L_1 = G_1 - \mathrm{pr}_2 P_{\gamma,\chi} \mathrm{i}_2 : H^{-\frac{3}{2}}(\Sigma) \to H^{\frac{3}{2}}(\Sigma), \tag{95}$$

with $D(L_1) = H^{\frac{5}{2}}(\Sigma)$ in view of the ellipticity. There is a corresponding operator $T: V \to V$ where $V = K_\gamma(\{0\} \times H^{-\frac{3}{2}}(\Sigma))$. Invertibility holds e.g. when L_1 has a positive lower bound.

Replacing $T: V \to V$ by $\widetilde{T}: V \to V$, selfadjoint and invertible with a nonempty essential spectrum, corresponds to replacing G_1 by

$$\widetilde{G}_1 = G_1 + \widetilde{L}_1 - L_1, \quad \widetilde{L}_1 = \mathrm{pr}_2 (\gamma_V^*)^{-1} \widetilde{T} \gamma_V^{-1} \mathrm{i}_2. \tag{96}$$

The corresponding realization $\widetilde{A}$ is defined by the boundary condition

$$\gamma_0 u = 0, \quad \gamma_0 \Delta u = \widetilde{G}_1 \gamma_1 u, \quad \gamma_1 u \in D(\widetilde{L}_1), \tag{97}$$

and satisfies $\sigma_{\mathrm{ess}} \widetilde{A} = \sigma_{\mathrm{ess}} A_0 \cup \sigma_{\mathrm{ess}} \widetilde{T}$.

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