Eigenvalue asymptotics for nonsmooth singular Green operators

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Plan

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1. Singular Green operators in the smooth case

- Ω bounded open $\subset \mathbb{R}^n$ with $C^\infty$-boundary $\partial \Omega = \Sigma$.
- A strongly elliptic on $\Omega$, $C^\infty$-coefficients,
  $$Au = -\sum_{j,k=1}^n \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^n a_j \partial_j u + a_0 u,$$
  with
  $$\text{Re} \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq c_0 |\xi|^2 \text{ for } x \in \overline{\Omega}, \xi \in \mathbb{R}^n; \quad c_0 > 0.$$
- $\partial^n_j u|_{\Sigma} = \gamma_j u$, $j \in \mathbb{N}_0$. $\nu u = \sum n_j \gamma_0(a_{jk}\partial_k u) \quad (= \gamma_1 u \text{ when } A = -\Delta)$,
  $\vec{n} = (n_1, \ldots, n_n)$ the normal to $\Sigma$.
- The Dirichlet realization $A_{\gamma}$ acts like $A$ with $D(A_{\gamma}) = \{u \in H^2(\Omega) \mid \gamma_0 u = 0\}$,
- Define a Neumann-type realization $A_{\nu,C}$ with $D(A_{\nu,C}) = \{u \in H^2(\Omega) \mid \nu u = C\gamma_0 u\}; C$ a first-order diff.op. on $\Sigma$.

If both are invertible, then $A_{\nu,C}^{-1} - A_{\gamma}^{-1}$ is a singular Green operator.
When $B$ is compact in a Hilbert space $H$, set $s_j(B) = \mu_j(B^*B)^{\frac{1}{2}}$.

It is well-known (starting with Weyl 1912, . . . ) that each of the operators $A_\gamma$ and $A_{\nu,C}$ has a spectral asymptotic behavior

$$s_j(A_\gamma^{-1}) \text{ and } s_j(A_{\nu,C}^{-1}) \sim c_A j^{-2/n} \text{ for } j \to \infty, \quad (1)$$

with a constant $c_A$ determined from $A$. Remainders improved to $O(j^{-3/n})$ or more exact formulas (Hörmander ’69, Ivrii ’80s, . . . ).

It is also well-known (Birman and Solomyak, Grubb in ’70s and ’80s) that

$$s_j(A_{\nu,C}^{-1} - A_\gamma^{-1}) \sim c j^{-2/(n-1)} \text{ for } j \to \infty. \quad (2)$$

Again the remainder can be refined using Hörmander, Ivrii results.

The “weak Schatten class” $\mathcal{S}_{p,\infty}(H)$ consists of those $B$ such that $s_j(B)$ is $O(j^{-1/p})$ for $j \to \infty$; with quasi-norm $N_p(B) \equiv \sup_j s_j(B)j^{1/p}$.

These are just upper estimates.

Here $A_\gamma^{-1}$ and $A_{\nu,C}^{-1}$ are in $\mathcal{S}_{n/2,\infty}$, and $A_{\nu,C}^{-1} - A_\gamma^{-1} \in \mathcal{S}_{(n-1)/2,\infty}$.
The dimension $n-1$ comes in because the resolvent difference has its essential effect in the neighborhood of the boundary $\partial \Omega$. More generally, the singular Green operators defined by Boutet de Monvel '71 in a calculus of pseudodifferential boundary operators ($\psi$dbo's) satisfy, by G '84:

When $G$ is a singular Green operator on $\Omega$ of order $-t < 0$ (and class zero), then

$$s_j(G) \sim c_G j^{-t/(n-1)} \text{ for } j \to \infty. \quad (3)$$

**Question:** Extend asymptotic estimates like (2) and (3) to operators with nonsmooth $x$-dependence.

Upper estimates for (2) are known from Birman '62, when the coefficients are in $C^0 \cap W^{1,\infty}$ and $\partial \Omega$ is $C^2$.

The $\psi$do calculus deals with matrices:

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : H^{s+d}(\Omega)^N \times H^s(\Sigma)^M \to H^{s+d}(\Sigma)^M \times H^s(\Sigma)^M',$$

- $P$ is a pseudodifferential operator ($\psi$do) on $\mathbb{R}^n$ of order $d$, and $P_+ = r^+ Pe^+$ is its truncation to $\Omega$ ($r^+$ restricts to $\Omega$ and $e^+$ extends by zero).
- $T$ is a trace operator from $\Omega$ to $\Sigma$ of order $d - \frac{1}{2}$, $K$ is a Poisson operator from $\Sigma$ to $\Omega$ of order $d + \frac{1}{2}$, $S$ is a $\psi$do on $\Sigma$ of order $d$.
- $G$ is a singular Green operator of order $d$, e.g. of type $KT$.

$P$ and $G$ are defined in local coordinates by Fourier integrals

$$(Pu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} p(x, \xi) \hat{u}(\xi) \, d\xi,$$

$$(Gu)(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{-ix' \cdot \xi'} \int_0^\infty \tilde{g}(x', x_n, y_n, \xi') \hat{u}(\xi', y_n) \, dy_n d\xi',$$
(when $G$ is of class 0). Here $\hat{u} = \mathcal{F}u$, $\hat{u}(\xi', y_n) = \mathcal{F}_{y' \to \xi'} u(y', y_n)$, $y' = (y_1, \ldots, y_{n-1})$.

For the $2'$ order elliptic operator $A$ we have the examples:

- $Q = A^{-1}$ is the $\psi$do inverse of $A$ on $\mathbb{R}^n$, $Q_+$ its truncation to $\Omega$,
- $A_{\gamma}^{-1} = Q_+ + G_{\gamma}$, where $G_{\gamma}$ is the s.g.o. $-K_{\gamma} \gamma_0 Q_+$; here $K_{\gamma}$ is the Poisson solution operator for the Dirichlet problem.
- $A_{\nu, C}^{-1} - A_{\gamma}^{-1} = K_{\gamma} L^{-1}(K'_{\gamma})^*$, a Krein resolvent formula. Here $L = C - P_{\gamma, \nu}$, where $P_{\gamma, \nu}$ is the Dirichlet-to-Neumann operator $\nu K_{\gamma}$, a $\psi$do on $\Sigma$.

The $\psi$dbo calculus was generalized to symbols with $C^\tau$- Hölder continuity in $x$ ($\tau > 0$) by Abels ’05. The Krein resolvent formula was extended to nonsmooth cases by Abels-G-Wood ’12, when the coefficients of $A$ are in $W^1_q$ (for some $q > n$) and the domain has a $B^{3/2}_{p,2}$-boundary; this contains $C^{3/2+\varepsilon}$-domains for all $\varepsilon > 0$.

Some tools for spectral estimates:
Lemma A. If $\Xi$ is bounded smooth $m$-dimensional, and $B \in \mathcal{L}(L_2(\Xi), H^t(\Xi))$ with $t > 0$, then $B \in \mathcal{S}_{m/t, \infty}$; indeed,

$$N_{m/t}(B) \equiv \sup_j s_j(B) j^{t/m} \leq C \|B\|_{\mathcal{L}(L_2, H^t)}.$$

Lemma B. 1° Let $B = B_0 + R$, then for $j \to \infty$,

$$\lim s_j(B_0) j^{1/p} = C_0, \quad \lim s_j(R) j^{1/p} = 0 \implies \lim s_j(B) j^{1/p} = C_0.$$

2° Let $B = B_M + B'_M$ for $M \in \mathbb{N}_0$, then $\lim s_j(B_M) j^{1/p} = C_M$, $\lim_M C_M = C_0$ and $\lim_M N_p(B'_M) = 0$ imply $\lim s_j(B) j^{1/p} = C_0$.

Lemma C. When $P$ is a classical $\psi$do system of order $-t < 0$, cut down to $\Xi$, with principal symbol $p^0(x, \xi)$, then

$$\lim_j s_j(P) j^{t/m} = c(p^0)^{t/m},$$

where

$$c(p^0) = \frac{1}{m(2\pi)^m} \int_{\Xi} \int |\xi|=1 \text{tr}((p^0* p^0)^{m/2t}) \, d\omega \, dx. \quad (4)$$
The pseudodifferential calculus for symbols $p(x, \xi)$ with full estimates in $\xi$ but only $C^{\tau}$-smoothness in $x$ ($\tau > 0$) was developed by Kumano-go and Nagase ’78, J. Marschall ’87 and M. Taylor ’91. Here when $P_i = \text{OP}(p_i(x, \xi))$ of order $d_i$, we only have

$$P_i : H^{s+d_i}(\mathbb{R}^m) \rightarrow H^s(\mathbb{R}^m) \text{ for } |s| < \tau,$$

$$P_1 P_2 - \text{OP}(p_1 p_2) : H^{s+d_1+d_2-\theta}(\mathbb{R}^m) \rightarrow H^s(\mathbb{R}^m) \text{ for } s, s+d_1 \in ]-\tau+\theta, \tau[.$$

Marschall shows that operator norms depend on $N$ symbol estimates ($N$ linked with the dimension). Then Lemma C extends:

**Theorem 1.** If $P$ is a classical $C^{\tau}$-smooth $\psi$do system of order $-t < 0$, defined on a compact $m$-dimensional $C^\infty$-manifold $\Xi$ without boundary, then

$$s_j(P) j^{t/m} \rightarrow C(p^0), \text{ for } j \rightarrow \infty.$$

**Proof:** Approximate $P$ by $C^\infty$-smooth operators $P_k$ obeying Lemma C. Now $\|P - P_k\|_{\mathcal{L}(H^{-t}, H^0)} \rightarrow 0$, so $P_k \rightarrow P$ in $\mathfrak{S}_m t, \infty$ by Lemma A. Apply Lemma B to the decompositions $P = P_k + (P - P_k)$. 

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**Eigenvalue asymptotics for nonsmooth singular Green opera...**
We now address the question for nonsmooth singular Green operators on smooth bounded domains $\Omega \subset \mathbb{R}^n$, in particular resolvent differences, where the boundary dimension $n - 1$ should come in.

The nonsmooth $\psi$dbo calculus (Abels ’05) has similar difficulties as the $\psi$do calculus: Sobolev mapping properties are valid only for $s$ in a small interval, in particular this holds for remainders in composition rules $A_1 A_2 - \text{OP}(a_1 \circ a_2)$.

For spectral estimates the calculus must be sharpened to operator norms depending on specific \textit{finite} sets of symbol seminorms, as in Marschall’s $\psi$do treatment.

However, there is an additional difficulty in the application of spectral theory to nonsmooth singular Green operators:
If $G = G_0 + R$ on $\Omega$, where $G_0$ of order $-t$ has the expected asymptotic behavior

$$s_j(G_0) \sim C(G_0) j^{-t/(n-1)},$$

and $R$ is of lower order, bounded from $H^{-t-\theta}(\Omega)$ to $H^0(\Omega)$ for some $\theta > 0$, then Lemma A for operators on $\Omega$ only gives that

$$s_j(R) \leq C j^{-(t+\theta)/n}.$$

But $j^{-(t+\theta)/n}$ decreases faster than $j^{-t/(n-1)}$ only when

$$\theta > t/(n - 1),$$

and we usually do not have such large values of $\theta$ available. So the remainders arising in compositions and approximations are a major problem. One has to involve the boundary more directly.
Consider a $C^\tau$-smooth s.g.o. of order $-t$ and class 0 on $\mathbb{R}^n_+$,

$$(Gu)(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} \int_0^\infty \tilde{g}(x', x_n, y_n, \xi') \hat{u}(\xi', y_n) \, dy_n d\xi'.$$

**Theorem 2.** Let $G$ be selfadjoint $\geq 0$ on $\mathbb{R}^n_+$, and let $\psi(x) = \psi_0(x')\psi_n(x_n)$ with $C_0^\infty$ functions equal to 1 near 0. Then

$$\mu_j(\psi G\psi)^{t/(n-1)} \rightarrow c(\psi_0^2 g^0)^{t/(n-1)} \text{ for } j \rightarrow \infty;$$

$$c(\psi_0^2 g^0) = \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_\Sigma \int_{|\xi|=1} \text{tr}((\psi_0^2 g^0(x', \xi', D_n))^{(n-1)/t}) \, d\omega dx'. \quad (5)$$

The proof uses that the symbol-kernel $\tilde{g}$ has a rapidly convergent double expansion in Laguerre functions $\varphi_m(x_n, \sigma)$,

$$\tilde{g}(x', x_n, y_n, \xi') = \sum_{l, m \in \mathbb{N}_0} c_{lm}(x', \xi') \varphi_l(x_n, \langle \xi' \rangle) \varphi_m(y_n, \langle \xi' \rangle).$$

Here $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$, the $c_{lm}$ are $\psi$do symbols of order $-t$, and the $\varphi_m(x_n, \sigma)$ are of the form $\text{pol}_m(x_n) e^{-x_n\sigma}$, with polynomials of degree $m$ in $x_n$ with coefficients depending on $\sigma$.

The $\varphi_m(x_n, \sigma)$, $m \in \mathbb{N}_0$, are an orthonormal basis of $L_2(\mathbb{R}_+)$. 

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Let $\Phi_m$ denote the Poisson operator with symbol-kernel $\varphi_m(x_n, \langle \xi' \rangle)$, then we can write, with $C^\tau$-smooth $\psi$do's $C_{lm}$ of order $-t$ on $\mathbb{R}^{n-1}$,

$$G = \sum_{l,m\in \mathbb{N}_0} \Phi_l C_{lm} \Phi_m^*.$$ 

The idea of proof is then (we leave out cut-off functions): Write

$$G = G_M + G_M^\dagger,$$

where

$$G_M = \sum_{l,m<M} \Phi_l C_{lm} \Phi_m^*,$$

$$G_M^\dagger = \sum_{l \text{ or } m \geq M} \Phi_l C_{lm} \Phi_m^*.$$ 

We can use the rapid decrease of the $C_{lm}$, combined with the control of $\mathcal{S}_{p,\infty}$-quasinorms in terms of finite sets of symbol seminorms, to show that $G_M^\dagger \to 0$ in $\mathcal{S}_{(n-1)/t,\infty}$ for $M \to \infty$. Next,

$$G_M = (\Phi_0 \cdots \Phi_{M-1}) \begin{pmatrix} C_{00} & \cdots & C_{0,M-1} \\ \vdots & & \vdots \\ C_{M-1,0} & \cdots & C_{M-1,M-1} \end{pmatrix} \begin{pmatrix} \Phi_0^* \\ \vdots \\ \Phi_{M-1}^* \end{pmatrix} = \mathcal{K}_M C_M \mathcal{K}_M^*.$$
Hence the $j$-th eigenvalue satisfies

$$
\mu_j(G_M) = \mu_j(K_M C_M K_M^*) = \mu_j(C_M K_M^* K_M) = \mu_j(C_M),
$$

since $K_M^* K_M = I_M$ in view of the orthonormality of the Laguerre system.

Here $C_M$ is an $M \times M$-matrix-formed $C^\tau$-smooth $\psi$do of order $-t$ on $\mathbb{R}^{n-1}$, to which Theorem 1 applies to give a spectral asymptotic estimate.

Now Lemma B is applied to the decomposition $G = G_M + G_M^\dagger$ for $M \to \infty$, to complete the proof.

There is an extension of the theorem to selfadjoint $C^\tau$-smooth s.g.o.s on bounded open smooth sets $\Omega \subset \mathbb{R}^n$. 
Finally consider the Krein resolvent formula

\[ A^{-1}_{\nu, C} - A^{-1}_{\gamma} = K_{\gamma}L^{-1}(K'_{\gamma})^* \equiv G_{C}, \]

for \( A \) with \( W_{1}^{1} \)-coefficients; take \( \Omega \) smooth to begin with. We want to find spectral asymptotics of \( G_{C} \); recall that \( K_{\gamma} \) is the Dirichlet Poisson operator, and \( L = C - P_{\gamma, \nu} \).

In the selfadjoint case, \( G_{C} \) the sum of a s.g.o. of order \(-2\) (as treated above) and a lower-order term. However, perturbation methods fail, since the lower-order term is linked with dimension \( n \).

Instead we shall use that \( G_{C} \) is here already in a product form passing via the boundary, and we can even allow nonselfadjointness.

In the original boundary problems for \( A = -\sum_{j,k=1}^{n} \partial^j a_{jk} \partial^k + \sum_{j=1}^{n} a_{j} \partial^j + a_0 \), we approximate the coefficients by \( C^{\infty} \)-functions \( a_{\epsilon}^{jk}, a_{\epsilon}^{j} \) (by convolution with an approximate identity), and we likewise approximate \( C \) by smoothed out versions \( C^{\epsilon} \).
Following the construction of $A_\gamma^{-1}$, $K_\gamma$ and $L^{-1}$ in Abels-G-Wood '12, we can show that for $\varepsilon \to 0$,

\[
\|K_\gamma^\varepsilon - K_\gamma\|_{\mathcal{L}(H^{s-\frac{1}{2}}(\Sigma), H^s(\Omega))} \to 0, \text{ each } s \in [0, 2],
\]

\[
\|K'_\gamma^\varepsilon - K'_\gamma\|_{\mathcal{L}(H^{s-\frac{1}{2}}(\Sigma), H^s(\Omega))} \to 0, \text{ each } s \in [0, 2],
\]

(6)

\[
\|(L^\varepsilon)^{-1} - L^{-1}\|_{\mathcal{L}(H^{\frac{1}{2}}(\Sigma), H^\frac{3}{2}(\Sigma))} \to 0.
\]

It follows by use of Lemma A that

\[
G_C^\varepsilon - G_C = K_\gamma^\varepsilon (L^\varepsilon)^{-1} (K'_\gamma)^* - K_\gamma L^{-1}(K'_\gamma)^* \to 0 \text{ in } \mathcal{S}(n-1)/2, \infty,
\]

for $\varepsilon \to 0$.

Then, since the result is known in the smooth case, we conclude by use of Lemma B:
Theorem 3. For the resolvent difference \( G_C = K_\gamma L^{-1}(K'_\gamma)^* \) defined from the Dirichlet realization and a Neumann-type realization of a strongly elliptic operator \( A \) with \( W^1_q \)-smooth coefficients, \( q > n \), on a bounded smooth set \( \Omega \),

\[
 s_j(G_C)j^{2/(n-1)} \to c(g_C^0)^{2/(n-1)} \text{ for } j \to \infty,
\]

where \( c(g_C^0) \) is defined similarly to (5).

Earlier, Birman ’62 had upper estimates when coefficients are in \( C^0 \cap W^{1,\infty} \). The constant satisfies:

Theorem 4. With \( l^0(x', \xi') \) denoting the principal symbol of \( L \) and \( \lambda^\pm(x', \xi') \) denoting the root in \( \mathbb{C}_\pm \) of the principal symbol \( a^0(x', 0, \xi', \xi_n) \) of \( A \) (as a polynomial in \( \xi_n \), in local coordinates)

\[
 c(g_C^0) = \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{\Sigma} \int_{|\xi'|=1} |4(l^0)^2 \text{Im } \lambda^+ \text{Im } \lambda^-|^{-(n-1)/4} d\omega dx'. \quad (7)
\]

There is a recent extension to nonsmooth sets \( \Omega \):
Assume that $\Omega$ has a $B_{p,2}^{3/2}$-boundary; here $p, q > 2, p \leq \infty$, with

$$1 - \frac{n}{q} \geq \frac{1}{2} - \frac{n-1}{p} \equiv \tau_0 > 0.$$ 

Note that $C^{3/2+\varepsilon} \subset B_{p,2}^{3/2} \subset C^{1+\tau_0}$ for all $\varepsilon > 0$. Birman assumed a $C^2$-boundary to get upper estimates (in $\mathcal{S}(n-1)/2, \infty$).

**Theorem 5.** The asymptotic estimate for $G_C$ extends to this case.

**Proof ingredients:** One can construct a sequence of $C^{1+\tau_0}$-diffeomorphisms $\lambda_l : \overline{\Omega} \to \overline{\Omega}_l$ where the $\overline{\Omega}_l$ are $C^\infty$-domains, such that $\lambda_l \to \text{Id}$ on a neighborhood of $\overline{\Omega}$, for $l \to \infty$.

The boundary value problems carried over to $\overline{\Omega}_l$ define Krein terms $G_{C_l}$, to which the preceding considerations apply.

Moreover, with $\varrho_l(x)$ denoting the square root of the Jacobian,

$$\varrho_l = (|\det(\partial \lambda_{l,j}/\partial x_k)_{j,k=1,...,n}|)^{1/2},$$

$G_C$ is unitarily equivalent with $\varrho_l^{-1} G_{C_l} \varrho_l$, for each $l$.

Since $\sup \varrho_l, \sup \varrho_l^{-1} \to 1$ for $l \to \infty$, we can use perturbation arguments to carry the spectral estimates for the $G_{C_l}$ over to $G_C$. 

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