SUPPLEMENT TO
G. GRUBB: "DISTRIBUTIONS AND OPERATORS"

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Additional miscellaneous exercises, *updated January 23, 2012***.**

Exercise 6.39. Denote by $\ell_2^N(\mathbb{N})$ the Hilbert space of complex sequences $\underline{x} = (x_k)_{k\in\mathbb{N}}$ with norm $\|\underline{x}\|_{\ell_2^N(\mathbb{N})} = \big(\sum_{k\in\mathbb{N}} |k^N x_k|^2\big)^{\frac12} < \infty$; the corresponding scalar product is $(\underline{x}, \underline{y})_{\ell_2^N(\mathbb{N})} = \sum_{k\in\mathbb{N}} k^{2N} x_k \bar{y}_k$.
(a) Show that $V = \ell_2^1(\mathbb{N})$ and $H = \ell_2^0(\mathbb{N})$ satisfy the hypotheses around (12.36).
(b) Show that when V^* is considered as in Lemma 12.16, it may be identified with $\ell_2^{-1}(\mathbb{N})$.
(c) Let $a(\underline{x}, \underline{y}) = (\underline{x}, \underline{y})_{\ell_2^1(\mathbb{N})} + 2(\underline{x}, \underline{y})_{\ell_2^0(\mathbb{N})}$, with domain V. Find the associated operator A in H defined by Definition 12.14, and check the properties resulting from Theorem 12.18.

Exercise 6.40. Let I be an interval of $\mathbb{R}$. Show, by construction, that the equation $Du = f$ has a solution $u \in \mathscr{D}'(I)$ for any $f \in \mathscr{D}'(I)$. Describe all solutions for a given f.
(*Hint:* The mapping from φ to ψ defined in the proof of Theorem 4.19 may be helpful.)

Exercise 6.41. Define the sesquilinear form a_1 by

$$a_1(u, v) = \int_0^\infty (u''\bar{v}'' + 2u'\bar{v}' + u\bar{v})\, dx, \quad u, v \in H^2(\mathbb{R}_+),$$

and let a_0 be its restriction to $H_0^2(\mathbb{R}_+)$. Let $H = L_2(\mathbb{R}_+)$, $V_1 = H^2(\mathbb{R}_+)$, $V_0 = H_0^2(\mathbb{R}_+)$.
(a) Show that the triples (H, V_0, a_0) and (H, V_1, a_1) satisfy the conditions for application of the Lax-Milgram theorem (Theorem 12.18).
(b) Denoting the hereby defined operators by A_0 resp. A_1, find how these operators act and what their domains are.
(c) Show that the operators are selfadjoint positive.

Exercise 6.42. Let $Q =]-1, 1[\times]-1, 1[\subset \mathbb{R}^2$, and let $u(x, y)$ be the function on $\mathbb{R}^2$ defined by

$$u(x, y) = \begin{cases} x + y & \text{for } (x, y) \in Q, \\ 0 & \text{for } (x, y) \notin Q. \end{cases}$$

(a) Find the Fourier transform of u.

(*Hint.* One can first determine the Fourier transform of the function 1_Q and then use rules of calculus.)

(b) Find the Fourier transforms of $D_x u$ and $D_y u$.

(c) Determine whether $u \in H^0(\mathbb{R}^2)$, and whether $u \in H^1(\mathbb{R}^2)$.

Exercise 6.43. Let $I =]-1,1[$, and let $\mathscr{B}$ denote the space of functions $\varphi \in C^\infty(\overline{I})$ satisfying $\varphi(0) = 0$. Show that $\mathscr{B}$ is dense in $L_2(I)$, but not in $H^1(I)$.
(*Hint.* Recall that convergence in $H^1(I)$ implies convergence in $C^0(\overline{I})$.)

Exercise 6.44. **(a)** Show that all distributions in $\mathscr{S}'(\mathbb{R}^n)$ are of finite order.

(b) Let $f(t)$ be the function on $\mathbb{R}$ defined as $f(x) = 1 - |x|$ for $x \in [-1,1]$ and $f(x) = 0$ outside $[-1,1]$. Let Λ be the functional on $\mathscr{D}(\mathbb{R})$ defined by

$$\langle \Lambda, \varphi \rangle = \sum_{j \in \mathbb{N}_0} \langle f(x-3j), \partial^j \varphi \rangle, \quad \varphi \in \mathscr{D}(\mathbb{R}).$$

Show that $\Lambda \in \mathscr{D}'(\mathbb{R})$.

(c) Is $\Lambda \in \mathscr{S}'(\mathbb{R})$?

Exercise 6.45. Consider the differential operator

$$A = -\partial_1^2 - 2\partial_2^2 + 3\partial_1 + 4\partial_2$$

on $\Omega = B(0,1) \subset \mathbb{R}^2$.

(a) Show that A is elliptic.

(b) Let $H = L_2(\Omega)$ and $V = H_0^1(\Omega)$. Find a sesquilinear form $a(u,v)$ with domain V, such that the triple $\{H, V, a\}$ satisfies the hypotheses of the Lax-Milgram Theorem Th. 12.18, and the associated operator A_0 extends $A|_{C_0^\infty(\Omega)}$.

(c) Find a lower bound for A_0, striving for a large value (one can obtain 3/2).

(d) Find constants c_1 and c_2 such that the spectrum of A_0 is contained in the set

$$\{z \in \mathbb{C} \mid |\operatorname{Im} z| \le c_1(\operatorname{Re} z + c_2)\}.$$

(*Hints.* Show that $(\partial_j u, u)$ is purely imaginary when $u \in V$. The Poincaré inequality can also be of use.)

Exercise 6.46. Let $a \in \mathbb{R}$, and let N be a positive integer.

(a) Show that the Fourier transform of δ_a is the function $e^{-ia\xi}$, and find the Fourier transforms of ${\delta_a}'$ and $(x-a)\delta_a$.
[We recall that for $a \in \mathbb{R}$, δ_a denotes the distribution defined by $\langle \delta_a, \varphi \rangle = \varphi(a)$.]

(b) Find all solutions $u \in \mathscr{S}'(\mathbb{R})$ to the equation

$$x^N u = 0.$$

(c) Find all solutions $u \in \mathscr{S}'(\mathbb{R})$ to the equation

$$x^N u = \delta_a.$$

Exercise 6.47. Show that the function $(x+iy)^{-1}$ on $\mathbb{R}^2$ is in $L_{1,\mathrm{loc}}(\mathbb{R}^2)$ and satisfies

$$(\partial_x + i\partial_y)\frac{1}{x+iy} = -2\pi\delta.$$

(*Hint.* Application of the left-hand side to a testfunction reduces, by integration by parts, to a sum of integrals over $B(0,\varepsilon)$ and $\partial B(0,\varepsilon)$, that can be evaluated for $\varepsilon \to 0$.)

(*Comment.* This allows giving a very explicit description of the solution of Exercise 5.8.)