II. PSEUDODIFFERENTIAL METHODS

§7. Pseudodifferential operators on open sets

7.1 Symbols and operators, mapping properties.

The Fourier transform is an important tool in the theory of PDE because of its very convenient property of *replacing differentiation by multiplication* by a polynomial:

$$\mathcal{F}(D^\alpha u) = \xi^\alpha \hat{u},$$

and the fact that $(2\pi)^{-n/2}\mathcal{F}$ defines a unitary operator in $L_2(\mathbb{R}^n)$ with a similar inverse $(2\pi)^{-n/2}\overline{\mathcal{F}}$. We have exploited this for example in the definition of Sobolev spaces of all orders

$$H^s(\mathbb{R}^n) = \{\, u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle\xi\rangle^s \hat{u} \in L_2(\mathbb{R}^n)\,\},$$

used in Chapter 6 to discuss the regularity of the distribution solutions of elliptic equations. For constant coefficient elliptic operators the Fourier transform is easy to use, for example in the simple case of the operator $I - \Delta$ that has the solution operator

$$\operatorname{Op}\Big(\frac{1}{1+|\xi|^2}\Big) = \mathcal{F}^{-1}\frac{1}{1+|\xi|^2}\mathcal{F}.$$

When we more generally define the operator $\operatorname{Op}(p(\xi))$ with symbol $p(\xi)$ by the formula

$$\operatorname{Op}(p(\xi))u = \mathcal{F}^{-1}(p(\xi)\mathcal{F}u),$$

we have a straightforward composition rule

$$\operatorname{Op}(p(\xi))\operatorname{Op}(q(\xi)) = \operatorname{Op}(p(\xi)q(\xi)), \tag{7.1}$$

where composition of operators is turned into multiplication of symbols.

However, these simple mechanisms hold only for x-independent ("constant coefficient") operators. As soon as one has to deal with differential operators with variable coefficients, the situation becomes much more complicated.

Pseudodifferential operators (ψdo's) were introduced as a tool to handle this, and to give a common framework for partial differential operators and their solution integral operators. A symbol p is now taken to depend (smoothly) on x also, and we define $P \equiv \operatorname{Op}\big(p(x,\xi)\big) \equiv p(x,D)$ by

$$\begin{aligned} \operatorname{Op}(p(x,\xi))u &= \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}\, đ\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p(x,\xi)u(y)\, dy đ\xi. \end{aligned} \tag{7.2}$$

We here (and in the following) use the notation

$$đ\xi = (2\pi)^{-n} d\xi, \tag{7.2a}$$

which was first introduced in the Russian literature on the subject. In the second line, the expression for the Fourier transform of u has been inserted. This formula will later be generalized, allowing p to depend on y also, see (7.14). Note that $(Pu)(x) = \{\mathcal{F}^{-1}_{\xi\to x}[p(z,\xi)(\mathcal{F}u)(\xi)]\}_{z=x}$.

With this notation, a differential operator $A = \sum_{|\alpha|\le d} a_\alpha(x)D^\alpha$ with C^∞ coefficients $a_\alpha(x)$ on $\mathbb{R}^n$ can be written as

$$A = \sum_{|\alpha|\le d} a_\alpha(x)D^\alpha = \sum_{|\alpha|\le d} a_\alpha(x)\mathcal{F}^{-1}\xi^\alpha\mathcal{F} = \operatorname{Op}(a(x,\xi)),$$
$$\text{where } a(x,\xi) = \sum_{|\alpha|\le d} a_\alpha(x)\xi^\alpha, \text{ the symbol of } A.$$

For operators as in (7.2) we do not have a simple product rule like (7.1). But it is important here that for “reasonable choices” of symbols, one can show something that is approximately as good:

$$\operatorname{Op}(p(x,\xi))\operatorname{Op}(q(x,\xi)) = \operatorname{Op}(p(x,\xi)q(x,\xi)) + \mathcal{R}, \tag{7.3}$$

where $\mathcal{R}$ is an operator that is “of lower order” than $\operatorname{Op}(pq)$.

We shall now describe a couple of the reasonable choices, namely the space S^d of so-called *classical* (or polyhomogeneous) symbols of order d (as systematically presented by Kohn and Nirenberg in [KN65]), and along with it the space $S^d_{1,0}$ (of Hörmander [H67]). We shall go rapidly through the main points in the classical theory without explaining everything in depth; detailed introductions are found e.g. in Seeley [S69], Hörmander [H71], [H85], Taylor [T81], Treves [T80].

In the next two definitions, n and n' are positive integers, Σ is an open subset of $\mathbb{R}^{n'}$ whose points are denoted X, and $d \in \mathbb{R}$. As usual, $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$.

Definition 7.1. *The space $S^d_{1,0}(\Sigma, \mathbb{R}^n)$ of symbols of degree d and type $1, 0$ is defined as the set of functions $p(X, \xi) \in C^\infty(\Sigma \times \mathbb{R}^n)$ such that for all indices $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^{n'}$ and any compact set $K \subset \Sigma$, there is a constant $c_{\alpha,\beta,K}$ such that*

$$|D_X^\beta D_\xi^\alpha p(X, \xi)| \le c_{\alpha,\beta,K} \langle \xi \rangle^{d-|\alpha|}. \tag{7.4}$$

When $p \in S^{m_0}_{1,0}(\Sigma, \mathbb{R}^n)$ and there exists a sequence of symbols p_{m_j}, $j \in \mathbb{N}_0$, with $p_{m_j} \in S^{m_j}_{1,0}(\Sigma, \mathbb{R}^n)$, $m_j \searrow -\infty$, such that $p - \sum_{j<M} p_{m_j} \in S^{m_M}_{1,0}(\Sigma, \mathbb{R}^n)$ for all M, we say that p has the asymptotic expansion $\sum_{j \in N_0} p_{m_j}$, in short, $p \sim \sum_j p_{m_j}$ in $S^{m_0}_{1,0}(\Sigma, \mathbb{R}^n)$.

Actually, [H67] also introduces some more general spaces $S^d_{\varrho,\delta}$, where $0 \le \delta \le 1$, $0 \le \varrho \le 1$, and the estimates (7.4) are replaced by estimates

$$|D_X^\beta D_\xi^\alpha p(X, \xi)| \le c_{\alpha,\beta,K} \langle \xi \rangle^{d-\varrho|\alpha|+\delta|\beta|}.$$

Many of the results we discuss in the following are also valid for these spaces, when $0 \le 1 - \varrho \le \delta < \varrho \le 1$. We shall not pursue the study of symbol spaces of type ϱ, δ here.

A prominent example of a function in $S^d_{1,0}(\Sigma, \mathbb{R}^n)$ is a function $p_d(X, \xi) \in C^\infty(\Sigma \times \mathbb{R}^n)$, which is *positively homogeneous of degree d in ξ for $|\xi| \ge 1$*, i.e. satifies

$$p_d(X, t\xi) = t^d p_d(X, \xi) \text{ for } |\xi| \ge 1,\ t \ge 1. \tag{7.5}$$

For such a function we have:

$$|p_d(X, \xi)| = |\xi|^d |p_d(X, \xi/|\xi|)| \le c(X) \langle \xi \rangle^d \text{ for } |\xi| \ge 1,$$

and its α'th derivative in ξ is homogeneous of degree $d - |\alpha|$, hence bounded by $c(X)\langle \xi \rangle^{d-|\alpha|}$, for $|\xi| \ge 1$. (For the latter homogeneity, note that $\partial_{\xi_j} p_d(X, \xi) = \partial_{\xi_j}(t^{-d} p_d(X, t\xi)) = t^{-d+1}(\partial_{\xi_j} p_d)(X, t\xi)$.)

Definition 7.2. *The space $S^d(\Sigma, \mathbb{R}^n)$ of* **polyhomogeneous symbols of degree** *d is defined as the set of symbols $p(X, \xi) \in S^d_{1,0}(\Sigma, \mathbb{R}^n)$ for which there exists a sequence of functions $p_{d-l}(X, \xi) \in C^\infty(\Sigma \times \mathbb{R}^n)$ for $l \in \mathbb{N}_0$, satisfying* (i) *and* (ii)*:*

(i) *Each p_{d-l} is positively homogeneous of degree $d - l$ in ξ for $|\xi| \ge 1$,*

(ii) *p has the asymptotic expansion*

$$p(X, \xi) \sim \sum_{l \in \mathbb{N}_0} p_{d-l}(X, \xi) \text{ in } S^d_{1,0}(\Sigma, \mathbb{R}^n); \tag{7.6}$$

in other words, for any compact set $K \subset \Sigma$, any multiindices $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^{n'}$ and any $M \in \mathbb{N}_0$, there is a constant $c_{\alpha,\beta,M,K}$ such that for all $(X,\xi) \in K \times \mathbb{R}^n$,

$$|D_X^\beta D_\xi^\alpha [p(X,\xi) - \sum_{0 \le l < M} p_{d-l}(X,\xi)] | \le c_{\alpha,\beta,M,K} \langle \xi \rangle^{d-|\alpha|-M}. \tag{7.7}$$

These symbols are also often called *classical* in the literature. Some authors call them *one-step polyhomogeneous* to underline that the degree of homogeneity fall in steps of length 1 (noninteger or varying steps could be needed in other contexts).

The leading term $p_d(X,\xi)$ is called *the principal symbol,* also denoted $p^0(X,\xi)$; the term $p_{d-l}(X,\xi)$ is called the symbol (or term) of degree $d-l$; and the series $\sum_{l=M}^\infty p_{d-l}(X,\xi)$ is called the symbol of degree $\le d-M$ (of p). (For ψdo's we use the word *degree* interchangeably with *order*, where the latter reflects their continuity properties, see Theorem 7.3 below.) From a given symbol $p(X,\xi) \in S^d(\Sigma, \mathbb{R}^n)$ one can determine the terms of degree $d-l$ successively by the formulas

$$p_{d-l}(X,\xi) = \lim_{t\to\infty} (t^{-d+l}[p(X,t\xi) - \sum_{j<l} p_{d-j}(X,t\xi)]), \text{ for } |\xi| \ge 1. \tag{7.8}$$

In view of the estimates (7.7), this convergence is uniform, locally in X and in ξ.

Observe that the series $\sum_{l\in\mathbb{N}_0} p_{d-l}$ is by no means assumed to be convergent; it is an *asymptotic series*, and its connection with p is described in a precise way in (7.7). It is important to know that there holds the following "reconstruction lemma" for general $S^d_{1,0}(\Sigma,\mathbb{R}^n)$ symbols:

Lemma 7.2a. *For any sequence of symbols $p_{m_j}(X,\xi)$ in $S^{m_j}_{1,0}(\Sigma,\mathbb{R}^n)$, $m_j \searrow -\infty$, there exists a function $p(X,\xi)$ such that $p \sim \sum_j p_{m_j}$ in $S^{m_0}_{1,0}(\Sigma,\mathbb{R}^n)$.*

For the proof, one takes

$$p(X,\xi) = \sum_{j\in\mathbb{N}_0} p_{m_j}(X,\xi)(1-\chi(\varepsilon_j\xi)), \tag{7.9}$$

where χ is our usual cut-off function, and ε_j goes to zero sufficiently rapidly for $j\to\infty$. Details are given in [S69] and [H71], [H85], see e.g. [H85, Prop. 18.1.3]. There is also a proof in [S91, Lemma 2.2]. The construction is a generalization of an old construction by Borel of a C^∞ function with arbitrarily given Taylor coefficients at a point.

A simple but important example of a symbol in $S^d(\mathbb{R}^n,\mathbb{R}^n)$ with $d=1$ is the function $\langle\xi\rangle=(1+|\xi|^2)^{\frac12}$, which for $|\xi|>1$ is the sum of a convergent series

$$\langle\xi\rangle=|\xi|(1+\tfrac12|\xi|^{-2}-\tfrac18|\xi|^{-4}+\dots+\tbinom{\frac12}{j}|\xi|^{-2j}+\cdots), \tag{7.10}$$

where $\binom{s}{j}=s(s-1)\cdots(s-j+1)/j!$. Then $\langle\xi\rangle$ has the asymptotic expansion

$$\langle\xi\rangle\sim\eta(\xi)|\xi|+\tfrac12\eta(\xi)|\xi|^{-1}-\tfrac18\eta(\xi)|\xi|^{-3}+\dots+\tbinom{\frac12}{j}\eta(\xi)|\xi|^{1-2j}+\cdots, \tag{7.10a}$$

where $\eta(\xi)=1-\chi(2\xi)$ was inserted to make the terms smooth near 0.

The space $S^d_{1,0}(\Sigma,\mathbb{R}^n)$ is a Fréchet space with the seminorms defined as the least constants entering in (7.4) for each choice of α,β and K (the K can be replaced by an exhausting sequence $K_j\to\Sigma$). Similarly, the space $S^d(\Sigma,\mathbb{R}^n)$ is a Fréchet space with the seminorms defined as the least constants entering in (7.7) for each choice of α,β,M and K. Clearly,

$$\begin{aligned}S^d_{1,0}(\Sigma,\mathbb{R}^n)&\subset S^{d'}_{1,0}(\Sigma,\mathbb{R}^n)\text{ when }d'>d,\\ S^d(\Sigma,\mathbb{R}^n)&\subset S^{d'}(\Sigma,\mathbb{R}^n)\text{ when }d'-d\in\mathbb{N}_0.\end{aligned}$$

We can define

$$\begin{aligned}S^\infty_{1,0}(\Sigma,\mathbb{R}^n)&=\bigcup_{d\in\mathbb{R}}S^d_{1,0}(\Sigma,\mathbb{R}^n),\quad S^\infty(\Sigma,\mathbb{R}^n)=\bigcup_{d\in\mathbb{R}}S^d(\Sigma,\mathbb{R}^n),\\ S^{-\infty}_{1,0}(\Sigma,\mathbb{R}^n)&=\bigcap_{d\in\mathbb{R}}S^d_{1,0}(\Sigma,\mathbb{R}^n)=\bigcap_{d\in\mathbb{R}}S^d(\Sigma,\mathbb{R}^n)=S^{-\infty}(\Sigma,\mathbb{R}^n).\end{aligned}\tag{7.11}$$

The symbols can be $(N'\times N)$-matrix formed; then the symbol space will be indicated by $S^d(\Sigma,\mathbb{R}^n)\otimes\mathcal{L}(\mathbb{C}^N,\mathbb{C}^{N'})$ or just $S^d(\Sigma,\mathbb{R}^n)$. The norm in (7.4) or (7.7) then stands for a matrix norm (some convenient norm on $\mathcal{L}(\mathbb{C}^N,\mathbb{C}^{N'})$, chosen once and for all).

Condition (ii) can be replaced by the following equivalent condition (ii′):

(ii′) For any indices $\alpha\in\mathbb{N}_0^n,\beta\in\mathbb{N}_0^{n'}$ and $M\in\mathbb{N}_0$, there is a continuous function $c(X)$ on Σ (depending on α,β and M but not on ξ) so that

$$|D_X^\beta D_\xi^\alpha[p(X,\xi)-\sum_{l<M}p_{d-l}(X,\xi)]|\le c(X)\langle\xi\rangle^{d-|\alpha|-M}; \tag{7.12}$$

a formulation we shall often use. Similarly, we can reformulate the estimates (7.4) in the form

$$|D_X^\beta D_\xi^\alpha p(X,\xi)|\le c(X)\langle\xi\rangle^{d-|\alpha|}. \tag{7.13}$$

In the case where $\Sigma = \mathbb{R}^{n'}$, one can instead work with more restrictive symbol classes where the estimates in (7.4), (7.7) or (7.12) (local in X) are replaced by global estimates on $\mathbb{R}^{n'}$ (with constants independent of K or X); this is done in [H85, Sect. 18.1] and in [S91]. The basic calculations such as proofs of composition rules are a little bit harder in that case than in the case we consider here, but the global calculus has the advantage that the rules can be made exact, without remainder terms. One can get the local calculus from the global calculus by use of cut-off functions.

Besides the need to construct a symbol with a given asymptotic series, we shall also sometimes need to *rearrange* an asymptotic series. For example, let $p \sim \sum_{l\in\mathbb{N}_0} p_{d-l}$ in $S^d_{1,0}$, where $p_{d-l} \in S^{d-l}_{1,0}$, and assume that each p_{d-l} has an asymptotic expansion $p_{d-l} \sim \sum_{k\in\mathbb{N}_0} p_{d-l,k}$ in $S^{d-l}_{1,0}$ with $p_{d-l,k} \in S^{d-l-k}_{1,0}$. Then we also have that

$$p \sim \sum_{l\in\mathbb{N}_0} q_{d-l}, \text{ where } q_{d-l} = \sum_{j+k=l} p_{d-j,k}.$$

In fact, $q_{d-l} = \sum_{j+k=l} p_{d-j,k}$ is a finite sum of terms in $S^{d-l}_{1,0}$ for each l, and $p - \sum_{l<M} q_{d-l}$ is the sum of $p - \sum_{l<M} p_{d-l} \in S^{d-M}_{1,0}$ and finitely many "tails" $p_{d-j} - \sum_{k<M-j} p_{d-j,k} \in S^{d-M}_{1,0}$. This is useful e.g. if p is given as a series of polyhomogeneous symbols of decreasing orders, and we want to rearrange the terms, collecting those that have the same degree of homogeneity.

We now specialize Σ somewhat. When $n = n'$, i.e. Σ is an open subset of $\mathbb{R}^n$ (with points x) and $p(x,\xi) \in S^d_{1,0}(\Sigma, \mathbb{R}^n)$, then $p(x,\xi)$ defines a pseudodifferential operator $P \equiv \operatorname{Op}\big(p(x,\xi)\big) \equiv p(x,D)$ by the formula (7.2), considered e.g. for $u \in C_0^\infty(\Sigma)$ or $u \in \mathcal{S}(\mathbb{R}^n)$.

Another interesting case is when $\Sigma = \Omega_1 \times \Omega_2$, Ω_1 and Ω_2 open $\subset \mathbb{R}^n$ (so $n' = 2n$), the points in Σ denoted (x,y). Here a symbol $p(x,y,\xi)$ defines an operator P, also denoted $\operatorname{Op}\big(p(x,y,\xi)\big)$, by the formula

$$(Pu)(x) = \int e^{i(x-y)\cdot\xi} p(x,y,\xi) u(y)\, dy đ\xi, \tag{7.14}$$

for $u \in C_0^\infty(\Omega_2)$. This generalizes the second line in (7.2). (The functions $p(x,y,\xi)$ are in some texts called amplitude functions, see Remark 7.5a below.)

The integration over ξ is defined in the sense of *oscillatory integrals* (cf. [H71], [H85], [S91]). A brief explanation goes as follows: When $d < -n$, the integrand in (7.14) is in L_1 since it is $O(\langle\xi\rangle^d)$, so the integral has the usual meaning. Otherwise, insert a convergence factor $\chi(\varepsilon\xi)$, and let $\varepsilon \to 0$

(note that then $\chi(\varepsilon x) \to 1$ pointwise). The limit exists and can be found as follows: Inserting

$$e^{-iy\cdot\xi} = (1+|\xi|^2)^{-N}(1-\Delta_y)^N e^{-iy\cdot\xi} \tag{7.14a}$$

(with N so large that $d - 2N < -n$) and integrating by parts with respect to y, we see that

$$\begin{aligned}(Pu)(x) &= \lim_{\varepsilon\to 0}\int \chi(\varepsilon\xi)e^{i(x-y)\cdot\xi}p(x,y,\xi)u(y)\,dy đ\xi \\ &= \lim_{\varepsilon\to 0}\int \chi(\varepsilon\xi)\langle\xi\rangle^{-2N}[(1-\Delta_y)^N e^{i(x-y)\cdot\xi}]p(x,y,\xi)u(y)\,dy đ\xi \\ &= \lim_{\varepsilon\to 0}\int \chi(\varepsilon\xi)e^{i(x-y)\cdot\xi}\langle\xi\rangle^{-2N}(1-\Delta_y)^N[p(x,y,\xi)u(y)]\,dy đ\xi \\ &= \int e^{i(x-y)\cdot\xi}\langle\xi\rangle^{-2N}(1-\Delta_y)^N[p(x,y,\xi)u(y)]\,dy đ\xi,\end{aligned} \tag{7.15}$$

where the limit exists since the integrand in the third line equals $\chi(\varepsilon\xi)$ times an L_1-function, and $\chi(\varepsilon\xi) \to 1$ boundedly (equals 1 for $|\xi| \le 1/\varepsilon$). The last expression defines a continuous function of x; it is independent of N since N was arbitrarily chosen ($> \frac{1}{2}(d+n)$). Thus we can use the last expression as the definition of Pu (for any large N). For $2N > n+d+m$, it allows differentiation with respect to x of order up to m, carried through the integral sign; since m can be taken arbitrarily large, we can conclude that $Pu \in C^\infty(\Omega_1)$ (when $u \in C_0^\infty(\Omega_2)$). One can also verify that P is continuous from $C_0^\infty(\Omega_2)$ to $C^\infty(\Omega_1)$ on the basis of these formulas.

In the following we shall use (without further argumentation) that the occurring integrals all have a sense as oscillatory integrals. Oscillatory integrals have many of the properties of usual integrals, allowing change of variables, change of order of integration (Fubini theorems), etc.

When Ω_1 and $\Omega_2 \subset \mathbb{R}^n$, the notation $S^d(\Omega_1,\mathbb{R}^n)$ and $S^d(\Omega_1\times\Omega_2,\mathbb{R}^n)$ is often abbreviated to $S^d(\Omega_1)$ resp. $S^d(\Omega_1\times\Omega_2)$, and the space of operators defined from these symbols is denoted $\operatorname{Op} S^d(\Omega_1)$ resp. $\operatorname{Op} S^d(\Omega_1\times\Omega_2)$ (with similar notation for $S^d_{1,0}$).

The pseudodifferential operators have the continuity property with respect to Sobolev spaces:

$$\begin{aligned}&P\colon H^s_{\mathrm{comp}}(\Omega_2) \to H^{s-d}_{\mathrm{loc}}(\Omega_1) \quad \text{continuously, when}\\ &P = \operatorname{Op}\big(p(x,y,\xi)\big)\,,\ p \in S^d_{1,0}(\Omega_1\times\Omega_2,\mathbb{R}^n).\end{aligned} \tag{7.16}$$

This follows from the following theorem, when we use that for φ and ψ in $C_0^\infty(\Omega_1)$ resp. $C_0^\infty(\Omega_2)$,

$$\varphi P(\psi u) = \operatorname{Op}\big(\varphi(x)p(x,y,\xi)\psi(y)\big)u.$$

Theorem 7.3. *Let $p(x,y,\xi)\in S^d_{1,0}(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, vanishing for $|x|>a$ and for $|y|\ge a$, for some $a>0$. For each $s\in\mathbb{R}$ there is a constant C, depending only on n,d,a and s; such that the norm of $P=\operatorname{Op}\big(p(x,y,\xi)\big)$ as an operator from $H^s(\mathbb{R}^n)$ to $H^{s-d}(\mathbb{R}^n)$ satisfies*

$$\begin{aligned}\|P\|_{s,s-d}&\equiv\sup\{\,\|Pu\|_{s-d}\mid u\in\mathcal{S}(\mathbb{R}^n),\|u\|_s=1\,\}\\ &\le C\sup\big\{\,|\langle\xi\rangle^{-d}D^\beta_{x,y}p|\mid x,y,\xi\in\mathbb{R}^n,\ |\beta|\le 2(\max\{|d-s|,|s|\}+n+2)\,\big\}.\end{aligned}\tag{7.17}$$

Proof. Let $u,v\in\mathcal{S}(\mathbb{R}^n)$. By Fourier transformation, we find:

$$\begin{aligned}|(Pu,v)|&=|\int_{\mathbb{R}^{3n}}e^{i(x-y)\cdot\xi}p(x,y,\xi)u(y)\overline{v}(x)\,dy\,đ\xi dx|\\ &=|\int_{\mathbb{R}^{5n}}e^{ix\cdot(\xi-\theta)-iy\cdot(\xi-\eta)}p(x,y,\xi)\hat{u}(\eta)\overline{\hat{v}}(\theta)\,đ\xi đ\eta đ\theta dxdy|\\ &\le\int_{\mathbb{R}^{3n}}|p(\widehat{\theta-\xi},\widehat{\xi-\eta},\xi)\hat{u}(\eta)\overline{\hat{v}}(\theta)|\,đ\xi đ\eta đ\theta,\end{aligned}\tag{7.17a}$$

where $p(\widehat{\theta},\widehat{\eta},\xi)=\mathcal{F}_{x\to\theta}\mathcal{F}_{y\to\eta}p(x,y,\xi)$. Now for any $N\in\mathbb{N}_0$,

$$\begin{aligned}&\langle\xi\rangle^{-d}|\langle\theta\rangle^{2N}\langle\eta\rangle^{2N}p(\widehat{\theta},\widehat{\eta},\xi)|\\ &\qquad\le\langle\xi\rangle^{-d}\|(1-\Delta_x)^N(1-\Delta_y)^Np(x,y,\xi)\|_{L_{1,x,y}(B(0,a)\times B(0,a))}\\ &\qquad\le C_{a,N}\sup\big\{\,|\langle\xi\rangle^{-d}D^\beta_{x,y}p|\mid x,y,\xi\in\mathbb{R}^n,|\beta|\le 4N\,\big\}\\ &\qquad\equiv M,\end{aligned}$$

so the symbol p satisfies

$$|p(\widehat{\theta},\widehat{\eta},\xi)|\le M\langle\xi\rangle^d\langle\theta\rangle^{-2N}\langle\eta\rangle^{-2N}.\tag{7.18}$$

By the Peetre inequality (6.13), we have that

$$\langle\xi\rangle^d=\langle\xi\rangle^s\langle\xi\rangle^{d-s}\le C'_{s,d}\langle\eta\rangle^s\langle\xi-\eta\rangle^{|s|}\langle\theta\rangle^{d-s}\langle\xi-\theta\rangle^{|d-s|}.$$

Then we find from (7.17a), by applying the Schwarz inequality (and absorbing universal constants in c and c'):

$$\begin{aligned}|(Pu,v)|&\le M\int\langle\xi\rangle^d\langle\xi-\eta\rangle^{-2N}\langle\xi-\theta\rangle^{-2N}|\hat{u}(\eta)\hat{v}(\theta)|\,đ\xi đ\eta đ\theta\\ &\le cM\int\langle\xi-\eta\rangle^{|s|-2N}\langle\xi-\theta\rangle^{|d-s|-2N}\langle\eta\rangle^s\langle\theta\rangle^{d-s}|\hat{u}(\eta)\hat{v}(\theta)|\,đ\xi đ\eta đ\theta\\ &\le c'M\|\langle\xi-\eta\rangle^{\frac12|s|-N}\langle\xi-\theta\rangle^{\frac12|d-s|-N}\langle\eta\rangle^s\hat{u}(\eta)\|_{L^2_{\xi,\eta,\theta}}\cdot\\ &\qquad\cdot\|\langle\xi-\eta\rangle^{\frac12|s|-N}\langle\xi-\theta\rangle^{\frac12|d-s|-N}\langle\theta\rangle^{d-s}\hat{v}(\theta)\|_{L^2_{\xi,\eta,\theta}}.\end{aligned}\tag{7.19}$$

Using a change of variables, we calculate e.g.:

$$\begin{aligned}\|\langle\xi-\eta\rangle^{\frac12|s|-N}\langle\xi-\theta\rangle^{\frac12|d-s|-N}\langle\eta\rangle^{s}\hat u(\eta)\|^2_{L^2_{\xi,\eta,\theta}}\\ &=\int\langle\varrho\rangle^{|s|-2N}\langle\sigma\rangle^{|d-s|-2N}\langle\eta\rangle^{2s}|\hat u(\eta)|^2\,đ\varrho đ\sigma đ\eta\\ &=c''\int\langle\eta\rangle^{2s}|\hat u(\eta)|^2\,đ\eta=c''\|u\|_s^2,\end{aligned}$$

when $2N>\max\{|s|,|d-s|\}+n+1$. It follows that

$$|(Pu,v)|\le CM\|u\|_s\|v\|_{d-s},$$

with C depending only on a,N,s,d,n. Then

$$\|P\|_{s,s-d}=\sup\{\,|(Pu,v)|\mid \|u\|_s=1,\|v\|_{d-s}=1\,\}$$

satisfies (7.17). $\square$

Approximating the elements in $H^s_{\mathrm{comp}}(\Omega_2)$ by $C_0^\infty(\Omega_2)$ functions, we extend P by continuity to a mapping from $H^s_{\mathrm{comp}}(\Omega_2)$ to $H^{s-d}_{\mathrm{loc}}(\Omega_1)$ (it is continuous, since it is clearly continuous from $H^s_{K_j}$ to $H^{s-d}_{\mathrm{loc}}(\Omega_1)$ for each K_j compact $\subset\Omega_2$). Since s can be arbitrary in $\mathbb{R}$, an application of the Sobolev theorem confirms that P maps $C_0^\infty(\Omega_2)$ continuously into $C^\infty(\Omega_1)$. Note also that since each element of $\mathcal{E}'(\Omega_2)$ lies in $H^t_{K_j}$ for some t, some K_j compact $\subset\Omega_2$ (cf. Theorem 6.15), P maps $\mathcal{E}'(\Omega_2)$ into $\mathcal{D}'(\Omega_1)$ (continuously).

One advantage of including the formulation (7.14) is that the formal adjoint $P^\times$ of P is easy to describe; where $P^\times$ stands for the operator from $C_0^\infty(\Omega_1)$ to $\mathcal{D}'(\Omega_2)$ for which

$$\langle P^\times u,\overline{v}\rangle_{\Omega_2}=\langle u,\overline{Pv}\rangle_{\Omega_1}\text{ for }u\in C_0^\infty(\Omega_1),v\in C_0^\infty(\Omega_2).\tag{7.20}$$

We here find that when $P=\mathrm{Op}\big(p(x,y,\xi)\big)$, $P^\times$ is simply $\mathrm{Op}\big(p_1(x,y,\xi)\big)$, where $p_1(x,y,\xi)$ is defined as the conjugate transpose of $p(y,x,\xi)$:

$$p_1(x,y,\xi)={}^t\overline{p}(y,x,\xi)\equiv p(y,x,\xi)^*;\tag{7.21}$$

this is seen by writing out the integrals (7.20) and interchanging integrations (all is justified in the sense of oscillatory integrals). In particular, if $P=\mathrm{Op}\big(p(x,\xi)\big)$, then $P^\times=\mathrm{Op}\big(p(y,\xi)^*\big)$. (The transposition is only relevant when p is matrix formed.) Note that in fact $P^\times$ maps $C_0^\infty(\Omega_1)$ into $C^\infty(\Omega_2)$, since it is a ψdo.

We can relate this to the extension of P to distributions: The operator

$$P^\times : C_0^\infty(\Omega_1) \to C^\infty(\Omega_2)$$

has an adjoint (not just a formal adjoint)

$$(P^\times)^* : \mathcal{E}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$$

that coincides with P on the set $C_0^\infty(\Omega_2)$. Since $C_0^\infty(\Omega_2)$ is dense in $\mathcal{E}'(\Omega_2)$ (a distribution $u \in \mathcal{E}'(\Omega_2)$ is the limit of $h_j * u$ lying in $C_0^\infty(\Omega_2)$ for j sufficiently large, cf. Lemma 3.19), there can be at most one extension of P to a continuous operator from $\mathcal{E}'(\Omega_2)$ to $\mathcal{D}'(\Omega_1)$. Thus $(P^\times)^*$ acts in the same way as $P : \mathcal{E}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$, and as the other extensions of P we have defined. We use the notation P for all of them, since they are consistent with each other.

It is customary in the pseudodifferential theory to use the notation P^* for any true or formal adjoint of any of the versions of P. We shall follow this custom, as long as it does not conflict with the strict rules for adjoints of Hilbert space operators recalled in Chapter 12. (The notation P' can then be used more liberally for other purposes.)

7.2 Negligible operators.

Another advantage of working with the formulation (7.14) and not just (7.2) is that in this way, the so-called *negligible pseudodifferential operators* are included in the theory in a natural way:

When $p(x,y,\xi) \in S_{1,0}^{-\infty}(\Omega_1 \times \Omega_2, \mathbb{R}^n)$, then

$$\begin{aligned} \operatorname{Op}(p)u(x) &= \int_{\Omega_2} K_p(x,y)u(y)\,dy, \text{ with} \\ K_p(x,y) &= \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p(x,y,\xi)\,đ\xi = \mathcal{F}^{-1}_{\xi\to z} p(x,y,\xi)|_{z=x-y}, \end{aligned} \tag{7.22}$$

via an interpretation in terms of oscillatory integrals. Since this p is in $\mathcal{S}(\mathbb{R}^n)$ as a function of ξ, $\mathcal{F}^{-1}_{\xi\to z} p$ is in $\mathcal{S}(\mathbb{R}^n)$ as a function of z. Taking the smooth dependence of p on x and y into account, one finds that $K_p(x,y) \in C^\infty(\Omega_1 \times \Omega_2)$. So in fact $\operatorname{Op}(p)$ is an integral operator with C^∞ kernel; we call such operators *negligible*. Conversely, if $\mathcal{R}$ is an integral operator from Ω_2 to Ω_1 with kernel $K(x,y) \in C^\infty(\Omega_1 \times \Omega_2)$, then there is a symbol $r(x,y,\xi) \in S_{1,0}^{-\infty}(\Omega_1 \times \Omega_2, \mathbb{R}^n)$ such that $\mathcal{R} = \operatorname{Op}\big(r(x,y,\xi)\big)$, namely

$$r(x,y,\xi) = c e^{i(y-x)\cdot\xi} K(x,y)\chi(\xi), \tag{7.24}$$

where the constant c equals $\big(\int \chi(\xi)đ\xi\big)^{-1}$. (Integral operators with C^∞ kernels are in some other texts called smoothing operators, or regularizing operators.)

The reason that we need to include the negligible operators in the calculus is that there is a certain vagueness in the definition. For example, for the polyhomogeneous symbols there is primarily a freedom of choice in how each term $p_{d-l}(x,\xi)$ (or $p_{d-l}(x,y,\xi)$) is extended as a C^∞ function in the set $|\xi| \le 1$ where it is not assumed to be homogeneous; and secondly, p is only associated with the series $\sum_{l\in\mathbb{N}_0} p_{d-l}$ in an asymptotic sense (cf. Definition 7.2), which also leaves a free choice of the value of p, as long as the estimates are respected. These choices are free precisely modulo symbols of order $-\infty$. *Moreover*, we shall find that when the *composition* of two operators $P' = \operatorname{Op}\big(p'(x,\xi)\big)$ and $P'' = \operatorname{Op}\big(p''(x,\xi)\big)$ with symbols in $S^\infty_{1,0}(\Omega,\mathbb{R}^n)$ is defined, the resulting operator $P = P'P''$ need not be of the exact form $\operatorname{Op}\big(p(x,\xi)\big)$, but does have the form

$$P = \operatorname{Op}\big(p(x,\xi)\big) + \mathcal{R},$$

for some negligible operator $\mathcal{R}$ (in $\operatorname{Op} S^{-\infty}(\Omega\times\Omega)$), see below. (As mentioned earlier, one can get a more exact calculus on $\mathbb{R}^n$ by working with the more restrictive class of globally estimated symbols introduced in [H85, 18.1], see also e.g. Saint Raymond [S91].)

When p and p' are symbols in $S^\infty_{1,0}(\Sigma,\mathbb{R}^n)$ with $p - p' \in S^{-\infty}(\Sigma,\mathbb{R}^n)$, we say that $p \sim p'$. When P and P' are linear operators from $C_0^\infty(\Omega_2)$ to $C^\infty(\Omega_1)$ with $P - P'$ a negligible ψdo, we say that $P \sim P'$.

From now on, we restrict the attention to cases where $\Omega_1 = \Omega_2 = \Omega$.

We shall now discuss the question of whether a symbol is determined from a given ψdo. There are the following facts: On one hand, the (x,y)-dependent symbols $p(x,y,\xi)$ are very far from uniquely determined from $\operatorname{Op}(p(x,y,\xi))$; for example, the symbol (where $a(x) \in C_0^\infty(\Omega) \setminus \{\,0\,\}$)

$$p(x,y,\xi) = \xi_j a(y) - a(x)\xi_j - D_{x_j} a(x) \tag{7.25}$$

is an element of $S^1(\Omega\times\Omega) \setminus S^0(\Omega\times\Omega)$, for which $\operatorname{Op}(p)$ is the zero operator. On the other hand, it can be shown that an x-dependent symbol $p(x,\xi)$ *is* uniquely determined from $P = \operatorname{Op}(p)$, modulo $S^{-\infty}(\Omega)$, in a sense that we shall explain below. First we need to introduce a restricted class of operators:

Definition 7.3a. *A ψdo P will be said to be properly supported in Ω when both P and P^* have the property: For each compact $K \subset \Omega$ there is a compact $K' \subset \Omega$ such that distributions supported in K are mapped into distributions supported in K'.*

When P is properly supported, P and P^* map $C_0^\infty(\Omega)$ into itself, and hence P extends to a mapping from $\mathcal{D}'(\Omega)$ to itself, as the adjoint of P^* on

$C_0^\infty(\Omega)$. Moreover, P maps $C^\infty(\Omega)$ to itself (since P^* maps $\mathcal{E}'(\Omega)$ to itself), and it maps $H^s_{\mathrm{comp}}(\Omega)$ to $H^{s-d}_{\mathrm{comp}}(\Omega)$ and $H^s_{\mathrm{loc}}(\Omega)$ to $H^{s-d}_{\mathrm{loc}}(\Omega)$ for all s when of order d. — Note that *differential* operators are always properly supported.

Consider a properly supported ψdo P in Ω. If $u \in C^\infty(\Omega)$ and we want to evaluate Pu at $x \in \Omega$, we can replace Pu by ϱPu, where $\varrho = 1$ at x and is C^∞, supported in a compact set $K \subset \Omega$. Then if K' is chosen for P^* according to Definition 7.3a, and $\psi = 1$ on K', $\psi \in C_0^\infty(\Omega)$, we have for any $\varphi \in C_0^\infty(\Omega)$ with $\operatorname{supp} \varphi \subset K$:

$$\langle Pu, \bar\varphi \rangle = \langle u, \overline{P^*\varphi} \rangle = \langle u, \psi \overline{P^*\varphi} \rangle = \langle P(\psi u), \bar\varphi \rangle.$$

So $Pu = P\psi u$ on K°, and $(\varrho Pu)(x) = (\varrho P\psi u)(x)$. This allows us to give a meaning to $e^{-ix\cdot\xi} P(e^{i(\cdot)\cdot\xi})$, namely as

$$e^{-ix\cdot\xi} P(e^{i(\cdot)\cdot\xi}) = e^{-ix\cdot\xi} \varrho(x) P(\psi(y) e^{iy\cdot\xi}),$$

for any pair of ϱ and ψ chosen as just described. With a certain abuse of notation, this function of x and ξ is often just denoted $e^{-ix\cdot\xi} P(e^{ix\cdot\xi})$.

Now we claim that if $P = \mathrm{Op}(p(x,\xi))$ and is properly supported, then p is determined from P by

$$p(x,\xi) = e^{-ix\cdot\xi} P(e^{ix\cdot\xi}), \tag{7.26}$$

for $x \in \Omega$. For then, by reading the integrals as forwards or backwards Fourier transforms,

$$\begin{aligned} \mathrm{Op}(p)(e^{i(\cdot)\cdot\xi}) &= \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\eta} p(x,\eta) e^{iy\cdot\xi}\, dy đ\eta \\ &= \int_{\mathbb{R}^n} p(x, \widetilde{x-y}) e^{iy\cdot\xi}\, dy \\ &= e^{ix\cdot\xi} \int_{\mathbb{R}^n} p(x, \widetilde{x-y}) e^{-i(x-y)\cdot\xi}\, dy = e^{ix\cdot\xi} p(x,\xi), \end{aligned}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$. We have here used the notation $p(x, \widetilde{z})$ for the inverse Fourier transform with respect to the last variable, $\mathcal{F}^{-1}_{\eta\to z} p(x,\eta)$. This shows that $p(x,\xi)$ is uniquely determined from $\mathrm{Op}(p)$.

On the other hand, if p is defined from P by (7.26), then when $u \in C_0^\infty(\Omega)$, one can justify the calculation

$$Pu = P\Big(\int_{\mathbb{R}^n} e^{i(\cdot)\cdot\xi} \hat u(\xi)\, đ\xi\Big) = \int_{\mathbb{R}^n} P(e^{i(\cdot)\cdot\xi}) \hat u(\xi)\, đ\xi = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi) \hat u(\xi)\, đ\xi,$$

by inserting the Fourier transform of u and writing the integral as a limit of Riemann sums, such that the linearity and continuity of P allows us to pull it through the integration; this shows that $P = \mathrm{Op}(p(x,\xi))$.

All this implies:

Lemma 7.3b. *When P is properly supported in Ω, there is a unique symbol $p(x,\xi)\in S^\infty(\Omega)$ such that $P=\operatorname{Op}\big(p(x,\xi)\big)$, namely the one determined by (7.26).*

As we shall see below in Theorem 7.5, an operator $P=\operatorname{Op}(p(x,\xi))$ can always be written as a sum $P=P'+\mathcal{R}$, where $P'=\operatorname{Op}(p'(x,y,\xi))$ is properly supported and $\mathcal{R}$ is negligible. By the preceding remarks there is then also a symbol $p''(x,\xi)$ (by (7.26)) so that $P=\operatorname{Op}\big(p''(x,\xi)\big)+\mathcal{R}$, and then $\mathcal{R}=\operatorname{Op}(r(x,\xi))$ with $r(x,\xi)=p(x,\xi)-p''(x,\xi)$. Moreover, one can show that when $r(x,\xi)$ defines a negligible operator, then necessarily $r(x,\xi)\in S^{-\infty}(\Omega)$ (for example by use of Remark 7.8 below). We conclude:

Proposition 7.3c. *The symbol $p(x,\xi)$ in a representation*

$$P=\operatorname{Op}\big(p(x,\xi)\big)+\mathcal{R}\,, \tag{7.27}$$

with $\operatorname{Op}(p(x,\xi))$ properly supported and $\mathcal{R}$ negligible, is determined from P uniquely modulo $S^{-\infty}(\Omega)$.

It remains to establish Theorem 7.5. For an open set $\Omega\subset\mathbb{R}^n$, denote $\operatorname{diag}(\Omega\times\Omega)=\{\,(x,y)\in\Omega\times\Omega\mid x=y\,\}$.

Lemma 7.4. *Let $p(x,y,\xi)\in S^d_{1,0}(\Omega\times\Omega,\mathbb{R}^n)$. When $\varphi(x,y)\in C^\infty(\Omega\times\Omega)$ with $\operatorname{supp}\varphi\subset(\Omega\times\Omega)\setminus\operatorname{diag}(\Omega\times\Omega)$, then $\operatorname{Op}\big(\varphi(x,y)p(x,y,\xi)\big)$ is negligible.*

Proof. Since $\varphi(x,y)$ vanishes on a neighborhood of the diagonal $\operatorname{diag}(\Omega\times\Omega)$, we may write it, for any $N\in\mathbb{N}_0$, as

$$\varphi(x,y)=|y-x|^{2N}\varphi_N(x,y), \tag{7.28}$$

where also the $\varphi_N(x,y)$ are in $C^\infty(\Omega\times\Omega)$ with support in $(\Omega\times\Omega)\setminus\operatorname{diag}(\Omega\times\Omega)$. Then an integration by parts (in the oscillatory integrals) gives

$$\begin{aligned}
&\operatorname{Op}\big(\varphi(x,y)p(x,y,\xi)\big)u\\
&\quad=\int e^{i(x-y)\cdot\xi}|y-x|^{2N}\varphi_N(x,y)p(x,y,\xi)u(y)\,dy đ\xi\\
&\quad=\int\big[(-\Delta_\xi)^N e^{i(x-y)\cdot\xi}\big]\varphi_N(x,y)p(x,y,\xi)u(y)\,dy đ\xi\\
&\quad=\int e^{i(x-y)\cdot\xi}\varphi_N(x,y)(-\Delta_\xi)^N p(x,y,\xi)u(y)\,dy đ\xi\\
&\quad=\operatorname{Op}\big(\varphi_N(x,y)(-\Delta_\xi)^N p(x,y,\xi)\big)u,
\end{aligned} \tag{7.29}$$

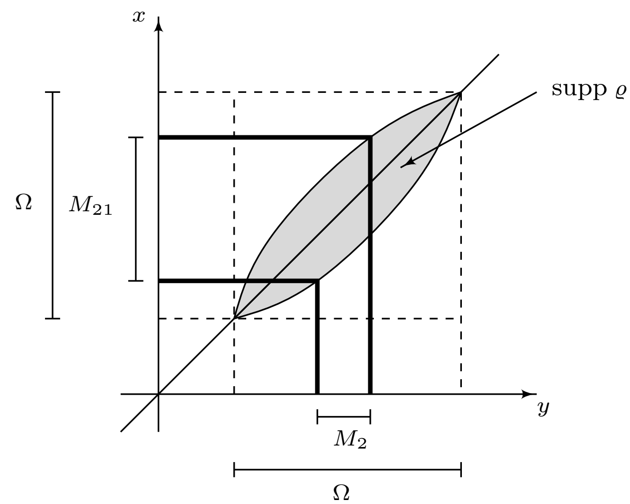
where the symbol is in $S^{d-2N}_{1,0}(\Omega\times\Omega,\mathbb{R}^n)$. Calculating the kernel of this operator as in (7.23), we get a function of (x,y) with more continuous derivatives the larger N is taken. Since the original expression is independent of N, we conclude that $\operatorname{Op}(\varphi p)$ is an integral operator with kernel in $C^\infty(\Omega\times\Omega)$, i.e., is a negligible ψdo. $\square$

Theorem 7.5. *Any $P = \mathrm{Op}(p(x,y,\xi))$ with $p \in S^d_{1,0}(\Omega \times \Omega)$ can be written as the sum of a properly supported operator P' and a negligible operator $\mathcal{R}$.*

Proof. The basic idea is to obtain the situation of Lemma 7.4 with $\varphi(x,y) = 1 - \varrho(x,y)$, where ϱ has the following property: Whenever M_1 and M_2 are compact $\subset \Omega$, then the sets

$$M_{12} = \{\, y \in \Omega \mid \exists x \in M_1 \text{ with } (x,y) \in \operatorname{supp} \varrho \,\}$$
$$M_{21} = \{\, x \in \Omega \mid \exists y \in M_2 \text{ with } (x,y) \in \operatorname{supp} \varrho \,\}$$

are compact. We then say that $\varrho(x,y)$ is *properly supported.*



Once we have such a function, we can take

$$p(x,y,\xi) = \varrho(x,y)p(x,y,\xi) + (1-\varrho(x,y))p(x,y,\xi); \tag{7.30}$$

here the first term defines a properly supported operator $P = \mathrm{Op}(\varrho p)$ and the second term defines, by Lemma 7.4, a negligible operator $\mathcal{R} = \mathrm{Op}\big((1-\varrho)p\big) = \mathrm{Op}\big(\varphi p\big)$. Then the statement in the theorem is obtained.

To construct the function ϱ, one can use a partition of unity $1 = \sum_{j\in\mathbb{N}_0} \psi_j$ for Ω as in Theorem 2.16. Take

$$J = \{(j,k) \in \mathbb{N}_0^2 \mid \operatorname{supp} \psi_j \cap \operatorname{supp} \psi_k = \emptyset\}, \quad J' = \mathbb{N}_0^2 \setminus J,$$
$$\varphi(x,y) = \sum_{(j,k)\in J} \psi_j(x)\psi_k(y), \quad \varrho(x,y) = \sum_{(j,k)\in J'} \psi_j(x)\psi_k(y).$$

In the proof that φ and ϱ are as asserted it is used again and again that any compact subset of Ω meets only finitely many of the supports of the ψ_j:

To see that ϱ is properly supported, let M_2 be a compact subset of Ω. Then there is a finite set $I_2 \subset \mathbb{N}_0$ such that $\operatorname{supp} \psi_k \cap M_2 = \emptyset$ for $k \notin I_2$, and hence

$$\varrho(x,y) = \sum_{(j,k)\in J', k\in I_2} \psi_j(x)\psi_k(y) \text{ for } y \in M_2.$$

By definition of J', the indices j that enter here are at most those for which $\operatorname{supp} \psi_j \cap M_2' \neq \emptyset$, where M_2' is the compact set $M_2' = \bigcup_{k\in I_2} \operatorname{supp} \psi_k$ in Ω. There are only finitely many such j; let I_1 denote the set of these j. Then $\varrho(x,y)$ vanishes for $x \notin M_{21} = \bigcup_{j\in I_1} \operatorname{supp} \psi_j$, when $y \in M_2$. — There is a similar proof with the roles of x and y exchanged.

To see that φ vanishes on a neighborhood of the diagonal, let $x_0 \in \Omega$, and let $B \subset \Omega$ be a closed ball around x_0. There is a finite set $I_0 \subset \mathbb{N}_0$ such that $\operatorname{supp} \psi_k \cap B = \emptyset$ for $k \notin I_0$, so

$$\varphi(x,y) = \sum_{(j,k)\in J, j\in I_0, k\in I_0} \psi_j(x)\psi_k(y) \text{ for } (x,y) \in B \times B.$$

This is a finite sum, and we examine each term. Consider $\varphi_j(x)\varphi_k(y)$. The supports of φ_j and φ_k have a positive distance r_{jk}, by definition of J, so $\psi_j(x)\varphi_k(y) = 0$ for $|x-y| < r_{jk}$. Take as r the smallest occurring r_{jk}, then φ vanishes on $\{(x,y) \in B \times B \mid |x-y| < r\}$, a neighborhood of (x_0,x_0). $\square$

A further consequence of Lemma 7.4 is the "pseudolocal" property of pseudodifferential operators. For a $u \in \mathcal{D}'(\Omega)$, define

$$\Omega_\infty(u) = \bigcup \{\, \omega \text{ open } \subset \Omega \mid u|_\omega \in C^\infty(\omega) \,\}; \tag{7.30a}$$

it is the largest open subset of Ω where u coincides with a C^∞ function. Define the singular support of u as the complement

$$\operatorname{sing\,supp} u = \Omega \setminus \Omega_\infty(u), \tag{7.30b}$$

it is clearly a closed subset of $\operatorname{supp} u$. Differential operators preserve supports

$$\operatorname{supp} Pu \subset \operatorname{supp} u, \text{ when } P \text{ is a differential operator;} \tag{7.31}$$

in short: They are *local*. Ψdo's do not in general have the property in (7.31), but Lemma 7.4 implies that they are *pseudolocal:*

Proposition 7.3d. *A ψdo P preserves singular supports:*

$$\operatorname{sing\,supp} Pu \subset \operatorname{sing\,supp} u. \tag{7.32}$$

Proof. Let $u \in \mathcal{E}'(\Omega)$ and write $u = u_\varepsilon + v_\varepsilon$ where $\operatorname{supp} u_\varepsilon \subset \operatorname{sing\,supp} u + B(0,\varepsilon)$, and $v_\varepsilon \in C_0^\infty(\Omega)$. Using the decomposition (7.30) with ϱ supported in $\operatorname{diag}(\Omega \times \Omega) + B(0,\varepsilon)$, $\varrho = 1$ on a neighborhood of diag $(\Omega \times \Omega)$, we find

$$\begin{aligned} Pu &= \operatorname{Op}(\varrho p)(u_\varepsilon + v_\varepsilon) + \operatorname{Op}\big((1-\varrho)p\big)u \\ &= \operatorname{Op}(\varrho p)u_\varepsilon + f_\varepsilon, \end{aligned}$$

where $f_\varepsilon \in C^\infty(\Omega)$ and $\operatorname{supp} \operatorname{Op}(\varrho p)u_\varepsilon \subset \operatorname{sing\,supp} u + B(0, 2\varepsilon)$. Since ε can be taken arbitrarily small, this implies (7.32). $\square$

Remark 7.5a. Symbols of the form $p(x,y,\xi)$ are sometimes called amplitude functions (to distinguish them from the sharper notion of symbols $p(x,\xi)$), since they are far from uniquely determined by the operator. We shall stay with the more vague terminology where everything is called a symbol, but distinguish by speaking of symbols "in x-form" (symbols $p(x,\xi)$), "in y-form" (symbols $p(y,\xi)$), "in (x,y)-form" (symbols $p(x,y,\xi)$). Moreover, we can speak of for example symbols "in (y',x_n)-form" (symbols $p(y',x_n,\xi)$), etc.

In preparation for general composition rules, we observe:

Lemma 7.6. *When $P = \operatorname{Op}(p(x,\xi))$ is properly supported and $\mathcal{R}$ is negligible, then $P\mathcal{R}$ and $\mathcal{R}P$ are negligible.*

Sketch of proof. Let $K(x,y)$ be the kernel of $\mathcal{R}$. Then for $u \in C_0^\infty(\Omega)$,

$$\begin{aligned} P\mathcal{R}u &= \int_{\Omega\times\Omega\times\mathbb{R}^n} e^{i(x-z)\cdot\xi} p(x,\xi)K(z,y)u(y)\,dydz đ\xi \\ &= \int_\Omega K'(x,y)u(y)\,dy, \text{ with} \\ K'(x,y) &= \int_{\Omega\times\mathbb{R}^n} e^{i(x-z)\cdot\xi} p(x,\xi)K(z,y)\,dz đ\xi \end{aligned}$$

(oscillatory integrals). For each y, $K'(x,y)$ is $\operatorname{Op}(p)$ applied to the C^∞ function $K(\cdot,y)$, hence is C^∞ in x. Moreover, since $K(\cdot,y)$ depends smoothly on y, so does $K'(\cdot,y)$. So $K'(x,y)$ is C^∞ in (x,y); this shows the statement for $P\mathcal{R}$. For $\mathcal{R}P$ one can use that $(\mathcal{R}P)^* = P^*\mathcal{R}^*$ is of the type already treated. $\square$

7.3 Composition rules.

Two pseudodifferential operators can be composed for instance when one of them is properly supported, or when the ranges and domains fit together in some other way. The "rules of calculus" are summarized in the following theorem (where $\overline{D}$ stands for $+i\partial$).

Theorem 7.7. *In the following, Ω is an open subset of $\mathbb{R}^n$, and d and $d' \in \mathbb{R}$.*

1° *Let $p(x,y,\xi) \in S^d_{1,0}(\Omega \times \Omega)$. Then*

$$\begin{aligned} &\operatorname{Op}\big(p(x,y,\xi)\big) \sim \operatorname{Op}\big(p_1(x,\xi)\big) \sim \operatorname{Op}\big(p_2(y,\xi)\big)\,, \text{ where} \\ &\quad p_1(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha p(x,y,\xi)\,|_{y=x}\,, \text{ and} \qquad (7.33) \\ &\quad p_2(y,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!}\partial_x^\alpha \overline{D}_\xi^\alpha p(x,y,\xi)\,|_{x=y}, \qquad (7.34) \end{aligned}$$

as symbols in $S^d_{1,0}(\Omega)$.

2° *When $p(x,\xi) \in S^d_{1,0}(\Omega)$, then*

$$\begin{aligned} &\operatorname{Op}\big(p(x,\xi)\big)^* \sim \operatorname{Op}\big(p_3(x,\xi)\big)\,, \text{ where} \\ &\quad p_3(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!}\partial_x^\alpha D_\xi^\alpha p(x,\xi)^* \quad \text{in } S^d_{1,0}(\Omega). \qquad (7.35) \end{aligned}$$

3° *When $p(x,\xi) \in S^d_{1,0}(\Omega)$ and $p'(x,\xi) \in S^{d'}_{1,0}(\Omega)$, with $\operatorname{Op}(p)$ or $\operatorname{Op}(p')$ properly supported, then*

$$\begin{aligned} &\operatorname{Op}\big(p(x,\xi)\big)\operatorname{Op}\big(p'(x,\xi)\big) \sim \operatorname{Op}\big(p''(x,\xi)\big)\,, \text{ where} \\ &p''(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!} D_\xi^\alpha p(x,\xi)\partial_x^\alpha p'(x,\xi) \quad \text{in } S^{d+d'}_{1,0}(\Omega). \qquad (7.36) \end{aligned}$$

In each of the rules, polyhomogeneous symbols give rise to polyhomogeneous symbols (when rearranged in series of terms according to degree of homogeneity).

Proof. The principal step is the proof of 1°. Inserting the Taylor expansion of order N in y of $p(x,y,\xi)$ at $y = x$, we find

$$\begin{aligned} \operatorname{Op}(p)u &= \int e^{i(x-y)\cdot\xi} p(x,y,\xi)u(y)\,dy đ\xi \\ &= \int e^{i(x-y)\cdot\xi} \sum_{|\alpha|<N} \tfrac{1}{\alpha!}(y-x)^\alpha \partial_y^\alpha p(x,x,\xi)u(y)\,dy đ\xi \\ + \int e^{i(x-y)\cdot\xi} &\sum_{|\alpha|=N} \tfrac{N(y-x)^\alpha}{\alpha!} \int_0^1 (1-h)^{N-1}\partial_y^\alpha p(x,x+(y-x)h,\xi)u(y)\,dh dy đ\xi. \end{aligned}$$

The first integral gives the terms for $|\alpha| < N$ in the series (7.33), by an integration by parts like in (7.29). Also in the second integral, an integration by parts transforms the factor $(y-x)^\alpha$ to a derivation D_ξ^α on p; then for sufficiently large N the integral equals

$$\int K_N(x,y)u(y)dy,$$

with continuous kernel

$$K_N(x,y) = \sum_{|\alpha|=N} c_\alpha \int_{\mathbb{R}^n} e^{i(x-y)\xi} \int_0^1 (1-h)^{N-1} \partial_y^\alpha D_\xi^\alpha p\big(x, x+(y-x)h, \xi\big)\, dh đ\xi.$$

More precisely, this integral defines a continuous function when $N > d+n$ because $\partial_y^\alpha D_\xi^\alpha p$ is integrable in ξ then; and K_N has continuous derivatives in (x,y) up to order k, when $N > d+n+k$. Let $p_1(x,\xi)$ be a symbol satisfying (7.33), then $\operatorname{Op}(p_1 - \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha p(x,y,\xi) \,|_{y=x})$ has a continuous kernel $K_{1,N}(x,y)$ with similar properties for large N. Altogether, $\operatorname{Op}(p)$ differs from $\operatorname{Op}(p_1)$ by an operator with a kernel $K_N - K_{1,N}$, which is C^k when $N > d+n+k$. Since $\operatorname{Op}\big(p(x,y,\xi)\big) - \operatorname{Op}\big(p_1(x,\xi)\big)$ is independent of N, its kernel is independent of N; then since we can take N arbitrarily large, it must be C^∞. Hence $\operatorname{Op}\big(p(x,y,\xi)\big) - \operatorname{Op}\big(p_1(x,\xi)\big)$ is negligible. This shows (7.33).

(7.34) is obtained by considering the adjoints (cf. (7.21)):

$$\operatorname{Op}\big(p(x,y,\xi)\big)^* = \operatorname{Op}\big({}^t\overline{p}(y,x,\xi)\big) \sim \operatorname{Op}\big(p_2'(x,\xi)\big),$$

where

$$p_2'(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!}\partial_w^\alpha D_\xi^\alpha\, {}^t\overline{p}(w,z,\xi) \,|_{z=w=x},$$

by (7.33). Here $\operatorname{Op}\big(p_2'(x,\xi)\big)^* = \operatorname{Op}\big(p_2(y,\xi)\big)$, where

$$p_2(y,\xi) = {}^t\overline{p}_2'(y,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!}\partial_x^\alpha \overline{D}_\xi^\alpha p(x,y,\xi) \,|_{x=y} \,.$$

This completes the proof of 1°; and 2° is obtained as a simple corollary, where we apply 1° to the symbol ${}^t\overline{p}(y,\xi)$.

Point 3° can likewise be obtained as a corollary, when we use 1° twice. Assume first that both $\operatorname{Op}(p)$ and $\operatorname{Op}(p')$ are properly supported. By 1°, we can replace $\operatorname{Op}\big(p'(x,\xi)\big)$ by an operator in y-form, cf. (7.34):

$$\operatorname{Op}\big(p'(x,\xi)\big) \sim \operatorname{Op}\big(p_2(y,\xi)\big), \text{ where}$$
$$p_2(y,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!}\partial_y^\alpha \overline{D}_\xi^\alpha p'(y,\xi).$$

So $P' = \mathrm{Op}(p_2(y,\xi)) + \mathcal{R}$ with $\mathcal{R}$ negligible, and then $PP' \sim \mathrm{Op}(p_1(x,\xi))\,\mathrm{Op}(p_2(y,\xi))$ in view of Lemma 7.6. Now we find

$$\begin{aligned}\mathrm{Op}\big(p(x,\xi)\big)\,&\mathrm{Op}\big(p_2(y,\xi)\big)u(x)\\ &= \int e^{i(x-y)\cdot\xi} p(x,\xi) e^{i(y-z)\cdot\theta} p_2(z,\theta)u(z)\, dz đ\theta dy đ\xi\\ &= \int e^{i(x-z)\cdot\xi} p(x,\xi)p_2(z,\xi)u(z)\, dz đ\xi \qquad (7.38)\\ &= \mathrm{Op}\big(p(x,\xi)p_2(y,\xi)\big)u(x),\end{aligned}$$

since the integrations in θ and y represent a backwards and a forwards Fourier transform:

$$\int e^{i(y-z)\cdot\theta} p_2(z,\theta)\, đ\theta = \mathcal{F}^{-1}_{\theta\to y-z} p_2(z,\theta) = p_2(z,\widetilde{y-z}),$$

and then (with $e^{i(x-y)\cdot\xi} = e^{i(x-z)\cdot\xi}e^{-i(y-z)\cdot\xi}$)

$$\int e^{-i(y-z)\cdot\xi} p_2(z,\widetilde{y-z})\, dy = p_2(z,\xi).$$

The resulting symbol of the composed operator is a simple product $p(x,\xi)p_2(y,\xi)$! One checks by use of the Leibniz formula, that it is a symbol of order $d+d'$.

Next, we use 1° to reduce $\mathrm{Op}\big(p(x,\xi)p_2(y,\xi)\big)$ to x-form. This gives $PP' \sim \mathrm{Op}\big(p''(x,\xi)\big)$, where

$$p''(x,\xi) \sim \sum_{\beta\in\mathbb{N}_0^n} \tfrac{1}{\beta!} D_\xi^\beta \big[p(x,\xi)\partial_x^\beta \sum_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!}\partial_x^\alpha \overline{D}_\xi^\alpha p'(x,\xi)\big].$$

This can be further reduced by use of the "backwards Leibniz formula" shown in Lemma 7.7a below. Doing a rearrangement as mentioned after (7.13), and appying (7.37), we find

$$\begin{aligned}p''(x,\xi) &\sim \sum_{\beta\in\mathbb{N}_0^n}\sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!\beta!} D_\xi^\beta\big[p\overline{D}_\xi^\alpha \partial_x^{\alpha+\beta} p'\big]\\ &\sim \sum_{\theta\in\mathbb{N}_0^n} \frac{1}{\theta!} \sum_{\alpha+\beta=\theta} \frac{\theta!}{\alpha!\beta!} D_\xi^\beta\big[p\overline{D}_\xi^\alpha \partial_x^\theta p'\big] = \sum_{\theta\in\mathbb{N}_0^n} \frac{1}{\theta!}(D_\xi^\theta p)\partial_x^\theta p',\end{aligned}$$

which gives the formula (7.36).

In all the formulas one makes rearrangements, when one starts with a polyhomogeneous symbol and wants to show that the resulting symbol is likewise polyhomogeneous.

Point 3° could alternatively have been shown directly by inserting a Taylor expansion for p in the ξ-variable, integrating by parts and using inequalities like in the proof of Theorem 7.3.

Finally, if only one of the operators $\operatorname{Op}(p)$ and $\operatorname{Op}(p')$, say $\operatorname{Op}(p')$, is properly supported, we reduce to the first situation by replacing the other operator $\operatorname{Op}(p(x,\xi))$ by $\operatorname{Op}(p_1(x,\xi)) + \mathcal{R}'$, where $\operatorname{Op}(p_1)$ is properly supported and $\mathcal{R}'$ is negligible; then $\operatorname{Op}(p)\operatorname{Op}(p') \sim \operatorname{Op}(p_1)\operatorname{Op}(p')$ in view of Lemma 7.6. □

Lemma 7.7a. (The backwards Leibniz formula.) *For $u, v \in C^\infty(\Omega)$,*

$$(D^\theta u)v = \sum_{\alpha,\beta\in\mathbb{N}_0,\alpha+\beta=\theta} \frac{\theta!}{\alpha!\beta!} D^\beta\big(u\overline{D}^\alpha v\big). \tag{7.37}$$

Proof. This is deduced from the usual Leibniz formula by noting that

$$\begin{aligned}\langle (D^\theta u)v, \varphi\rangle &= (-1)^{|\theta|}\langle u, D^\theta(v\varphi)\rangle = (-1)^{|\theta|}\langle u, \sum_{\alpha+\beta=\theta}\frac{\theta!}{\alpha!\beta!}D^\alpha v D^\beta\varphi\rangle \\ &= \sum_{\alpha+\beta=\theta}(-1)^{|\theta|-|\beta|}\frac{\theta!}{\alpha!\beta!}\langle D^\beta(uD^\alpha v), \varphi\rangle,\end{aligned}$$

where $|\theta| - |\beta| = |\alpha|$. □

Formula (7.37) extends of course to cases where u or v is in $\mathcal{D}'$.

Definition 7.7b. *The symbol (defined modulo $S^{-\infty}(\Omega)$) in the right-hand side of* (7.36) *is denoted $p(x,\xi)\circ p'(x,\xi)$ and is called the Leibniz product of p and p'.*

The symbol in the right-hand side of (7.35) *is denoted $p^{\circ *}(x,\xi)$.*

The rule for $p\circ p'$ is a generalization of the usual (Leibniz) rule for composition of differential operators with variable coefficients. The notation $p \,\#\, p'$ is also often used.

Note that (7.36) shows

$$\begin{aligned} p(x,\xi)\circ p'(x,\xi) &\sim p(x,\xi)p'(x,\xi) + r(x,\xi) \text{ with} \\ r(x,\xi) &\sim \sum_{|\alpha|\ge 1}\tfrac{1}{\alpha!}D_\xi^\alpha p(x,\xi)\partial_x^\alpha p'(x,\xi),\end{aligned} \tag{7.39}$$

where r is of order $d+d'-1$. Thus (7.3) has been obtained for these symbol classes with $\operatorname{Op}(pq)$ of order $d+d'$ and $\mathcal{R}$ of order $d+d'-1$.

In calculations concerned with elliptic problems, this information is sometimes sufficient; one does not need the detailed information on the structure of $r(x,\xi)$. But there are also applications where the terms in r are important, first of all the term of order $d+d'-1$, $\sum_{j=1}^n D_{\xi_j}p\,\partial_{x_j}p'$. For example, the commutator of $\operatorname{Op}(p)$ and $\operatorname{Op}(p')$ (for scalar operators) has the symbol

$$p\circ p' - p'\circ p \sim -i\sum_{j=1}^n \big(\partial_{\xi_j}p\,\partial_{x_j}p' - \partial_{x_j}p\,\partial_{\xi_j}p'\big) + r' \tag{7.39a}$$

with r' of order $d+d'-2$. The sum over j is called the Poisson bracket of p and p'; it plays a role in many considerations.

Remark 7.8. We here *sketch* a certain spectral property. Let $p(x,\xi)\in S^0(\Omega,\mathbb{R}^n)$ and let (x_0,ξ_0) be a point with $x_0\in\Omega$ and $|\xi_0|=1$; by translation and dilation we can obtain that $x_0=0$ and $B(0,2)\subset\Omega$. The sequence

$$u_k(x) = k^{n/2}\chi(kx)e^{ik^2x\cdot\xi_0},\quad k\in\mathbb{N}_0, \tag{7.39b}$$

has the following properties:

$$\begin{aligned} &\text{(i)} && \|u_k\|_0 = \|\chi\|_0\ (\neq 0) \text{ for all } k,\\ &\text{(ii)} && (u_k,v)\to 0 \text{ for } k\to\infty,\ \text{all } v\in L_2(\mathbb{R}^n),\\ &\text{(iii)} && \|\chi\operatorname{Op}(p)u_k - p^0(0,\xi_0)\cdot u_k\|_0\to 0 \text{ for } k\to\infty. \end{aligned} \tag{7.39c}$$

Here (i) and (ii) imply that u_k has no convergent subsequence in $L_2(\mathbb{R}^n)$. Now if $\operatorname{Op}(p)$ is continuous from $H^0_{\text{comp}}(\Omega)$ to $H^1_{\text{loc}}(\Omega)$, then $p^0(0,\xi_0)$ must equal zero, for the compactness of the injection $H^1(B(0,2))\hookrightarrow H^0(B(0,2))$ (cf. Section 8.2) then shows that $\chi\operatorname{Op}(p)u_k$ has a convergent subsequence in $H^0(B(0,2))$, and then, unless $p(0,\xi_0)=0$, (iii) will imply that u_k has a convergent subsequence in L_2 in contradition to (i)–(ii). Applying this argument to every point (x_0,ξ_0) with $|\xi_0|=1$, one finds that if $\operatorname{Op}(p)$ is of order 0 *and* maps $H^0_{\text{comp}}(\Omega)$ continuously into $H^1_{\text{loc}}(\Omega)$, its principal symbol must equal 0. (The proof is found in [H67] and is sometimes called Hörmander's variant of Gohberg's lemma, referring to a version given by Gohberg in [G60].)

The properties (i)–(iii) mean that u_k is a *singular sequence* for the operator P_1-a with $P_1=\chi\operatorname{Op}(p)$ and $a=p^0(0,\xi_0)$; this implies that a belongs to the *essential spectrum* of P_1, namely the set

$$\operatorname{ess\,spec}(P_1) = \bigcap\{\operatorname{spec}(P_1+K)\mid K \text{ compact in } L_2(\Omega)\}.$$

Since the operator norm is $\geq$ the spectral radius, the operator norm of P_1 (and of any P_1+K) must be $\geq |a|$. It follows that the operator norm in $L_2(\Omega)$ of ψP for any $\psi\in C_0^\infty$ is $\geq \sup_{x\in\Omega,|\xi|=1}|\psi(x)p^0(x,\xi)|$. (So if we know that the norm of ψP is $\le C$ for all $|\psi| \le 1$, then also $\sup|p^0| \le C$; a remark that can be very useful.)

By compositions with properly supported versions of $\operatorname{Op}(\langle\xi\rangle^t)$ (for suitable t), it is seen more generally that if $P = \operatorname{Op}\big(p(x,\xi)\big)$ is of order d and maps $H^s_{\mathrm{comp}}(\Omega)$ into $H^{s-d+1}_{\mathrm{loc}}(\Omega)$, then its principal symbol equals zero. In particular, if $P = \operatorname{Op}\big(r(x,\xi)\big)$ where $r \in S^{+\infty}(\Omega,\mathbb{R}^n)$, and maps $\mathcal{E}'(\Omega)$ into $C^\infty(\Omega)$, then all the homogeneous terms in each asymptotic series for $r(x,\xi)$ are zero, i.e. $r(x,\xi) \sim 0$ (hence is in $S^{-\infty}(\Omega,\mathbb{R}^n)$). This gives a proof that a symbol in x-form is determined from the operator it defines, uniquely modulo $S^{-\infty}(\Omega,\mathbb{R}^n)$.

7.4 Elliptic pseudodifferential operators.

One of the most important applications of Theorem 7.7 is the construction of a *parametrix* (an almost-inverse) to an *elliptic* operator. We shall now define ellipticity, and here we include matrix-formed symbols and operators.

The space of $(N' \times N)$-matrices of symbols in $S^d(\Sigma,\mathbb{R}^n)$ (resp. $S^d_{1,0}(\Sigma,\mathbb{R}^n)$) is denoted $S^d(\Sigma,\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N,\mathbb{C}^{N'})$ (resp. $S^d_{1,0}(\Sigma,\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N,\mathbb{C}^{N'})$) since complex $(N' \times N)$-matrices can be identified with linear maps from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$ (i.e., elements of $\mathcal{L}(\mathbb{C}^N,\mathbb{C}^{N'})$). The symbols in these classes of course define $(N' \times N)$-matrices of operators (notation: when $p \in S^d_{1,0}(\Omega\times\Omega,\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N,\mathbb{C}^{N'})$ then $P = \operatorname{Op}(p)$ sends $C_0^\infty(\Omega)^N$ into $C^\infty(\Omega)^{N'}$). Ellipticity is primarily defined for square matrices (the case $N = N'$), but has a natural extension to general matrices.

Definition 7.9. 1° *Let $p \in S^d(\Omega,\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)$. Then p, and $P = \operatorname{Op}(p)$, and any ψdo P' with $P' \sim P$, is said to be* **elliptic** *of order d, when the principal symbol $p_d(x,\xi)$ is invertible for all $x \in \Omega$ and all $|\xi| \ge 1$.*

2° *Let $p \in S^d(\Omega,\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N,\mathbb{C}^{N'})$. Then p (and $\operatorname{Op}(p)$ and any $P' \sim \operatorname{Op}(p)$) is said to be* **injectively elliptic** *of order d, resp.* **surjectively elliptic** *of order d, when $p_d(x,\xi)$ is injective, resp. surjective, from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$, for all $x \in \Omega$ and $|\xi| \ge 1$. (In particular, $N' \ge N$ resp. $N' \le N$.)*

Note that since p_d is homogeneous of degree d for $|\xi| \ge 1$, it is only necessary to check the invertibility for $|\xi| = 1$. The definition (and its usefulness) extends to the classes $S^d_{\varrho,\delta}$, for the symbols that have a principal part in a suitable sense, cf. e.g. [H67], [T80] and [T81].

Surjectively elliptic systems are sometimes called underdetermined elliptic or right elliptic systems, and injectively elliptic systems are sometimes called overdetermined elliptic or left elliptic systems.

When $P = \operatorname{Op}(p)$ and $Q = \operatorname{Op}(q)$ are pseudodifferential operators on Ω, we say that Q is a *right parametrix* for P if PQ can be defined and

$$PQ \sim I, \tag{7.40}$$

and q is a *right parametrix symbol* for p if

$$p \circ q \sim 1 \text{ (read as the identity, for matrix formed operators)},$$

in the sense of the equivalences and the composition rules introduced above. Similarly, Q is a *left parametrix*, resp. q is a *left parametrix symbol* for p, when

$$QP \sim I \text{ resp. } q \circ p \sim 1.$$

When Q is both a right and a left parametrix, it is called a two-sided parametrix or simply a parametrix. (When P is of order d and Q is a one-sided parametrix of order $-d$ then it is two-sided if $N' = N$, as we shall see below.)

Theorem 7.10. 1° *Let $p(x,\xi) \in S^d(\Omega,\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)$. Then p has a parametrix symbol $q(x,\xi)$ belonging to $S^{-d}(\Omega,\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)$ if and only if p is elliptic.*

2° *Let $p(x,\xi) \in S^d(\Omega,\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^N,\mathbb{C}^{N'})$. Then p has a right (left) parametrix symbol $q(x,\xi)$ belonging to $S^{-d}(\Omega,\mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^{N'},\mathbb{C}^N)$ if and only if p is surjectively (resp. injectively) elliptic.*

Proof. 1° (The case of square matrices.) Assume that p is elliptic. Let $q_{-d}(x,\xi)$ be a C^∞ function on $\Omega\times\mathbb{R}^n$ ($(N\times N)$-matrix formed), that coincides with $p_d(x,\xi)^{-1}$ for $|\xi| \geq 1$ (one can extend p_d to be homogeneous for all $\xi \neq 0$ and take $q_{-d}(x,\xi) = \big[1-\chi(2\xi)\big]p_d(x,\xi)^{-1}$). By Theorem 7.7 3° (cf. (7.39)),

$$p(x,\xi) \circ q_{-d}(x,\xi) \sim 1 - r(x,\xi), \tag{7.41a}$$

for some $r(x,\xi) \in S^{-1}(\Omega)$. For each M, let

$$r^{\circ M}(x,\xi) \sim r(x,\xi) \circ r(x,\xi) \circ \cdots \circ r(x,\xi) \text{ (M factors)},$$

it lies in $S^{-M}(\Omega)$. Now

$$\begin{aligned} &p(x,\xi) \circ q_{-d}(x,\xi) \circ \big(1 + r(x,\xi) + r^{\circ 2}(x,\xi) + \cdots + r^{\circ M}(x,\xi)\big) \\ &\quad \sim \big(1 - r(x,\xi)\big) \circ \big(1 + r(x,\xi) + r^{\circ 2}(x,\xi) + \cdots + r^{\circ M}(x,\xi)\big) \\ &\quad \sim 1 - r^{\circ(M+1)}(x,\xi). \end{aligned} \tag{7.41}$$

Here each term $r^{\circ M}(x,\xi)$ has an asymptotic development in homogeneous terms of degree $-M-j$, $j = 0,1,2,\dots$,

$$r^{\circ M}(x,\xi) \sim r^{\circ M}_{-M}(x,\xi) + r^{\circ M}_{-M-1}(x,\xi) + \cdots\,,$$

and there exists a symbol $r'(x,\xi)\in S^{-1}(\Omega)$ with

$$r'(x,\xi)\sim\sum_{M\ge 1} r^{\circ M}(x,\xi)\ ,\quad \text{defined as } \sum_{M\ge 1}\Big(\sum_{1\le j\le M} r^{\circ j}_{-M}\Big)$$

(here we use rearrangement; the point is that there are only M terms of each degree $-M$). Finally,

$$p\circ q_{-d}\circ(1+r')\sim 1, \tag{7.42}$$

which is seen as follows: For each M, there is a symbol $r'_{(M+1)}$ such that

$$r'_{(M+1)}(x,\xi)\sim\sum_{k\ge M+1} r^{\circ k}(x,\xi);$$

it is in $S^{-M-1}(\Omega)$. Then (cf. also (7.41))

$$p\circ q_{-d}\circ(1+r')\sim p\circ q_{-d}\circ(1+\dots+r^{\circ M})+p\circ q_{-d}\circ r'_{(M+1)}\sim 1+r''_{(M+1)},$$

where $r''_{(M+1)}=-r^{\circ(M+1)}+p\circ q_{-d}\circ r'_{(M+1)}$ is in $S^{-M-1}(\Omega)$. Since this holds for any M, $p\circ q_{-d}\circ(1+r')-1$ is in $S^{-\infty}(\Omega)$. In other words, (7.42) holds.

This gives a right parametrix of p, namely

$$q\sim q_{-d}\circ(1+r')\in S^{-d}(\Omega).$$

Similarly, there exists a left parametrix q' for p. Finally, $q'\sim q$, since

$$q'-q\sim q'\circ(p\circ q)-(q'\circ p)\circ q\sim 0,$$

so q itself is also a left parametrix. We have then shown that when p is elliptic of order d, it has a parametrix symbol q of order $-d$, and any left/right parametrix symbol is also a right/left parametrix symbol.

Conversely, if $q\in S^{-d}(\Omega)$ is such that $p\circ q\sim 1$, then the principal symbols satisfy

$$p_d(x,\xi)q_{-d}(x,\xi)=1\ \text{ for } |\xi|\ge 1$$

in view of Remark 7.8, so p_d is elliptic.

2° We now turn to the case where p is not necessarily a square matrix; assume for instance that $N\ge N'$. Here $p_d(x,\xi)$ is, when p is surjectively elliptic of order d, a matrix defining a surjective operator from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$ for each (x,ξ) with $|\xi|\ge 1$; and hence

$$\tilde p(x,\xi)=p_d(x,\xi)p_d(x,\xi)^*\colon \mathbb{C}^{N'}\to\mathbb{C}^{N'}$$

is bijective. Let $\tilde{q}(x,\xi) = \tilde{p}(x,\xi)^{-1}$ for $|\xi| \geq 1$, extended to a C^∞ function for $|\xi| \leq 1$; and note that $\tilde{q} \in S^{-2d}(\Omega, \mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^{N'}, \mathbb{C}^{N'})$. Now, by Theorem 7.7 3°,

$$p(x,\xi) \circ p_d(x,\xi)^* \circ \tilde{q}(x,\xi) \sim 1 - r(x,\xi),$$

where $r(x,\xi) \in S^{-1}(\Omega, \mathbb{R}^n) \otimes \mathcal{L}(\mathbb{C}^{N'}, \mathbb{C}^{N'})$. We can then proceed exactly as under 1°, and construct the complete right parametrix symbol $q(x,\xi)$ as

$$q(x,\xi) \sim p_d(x,\xi)^* \circ \tilde{q}(x,\xi) \circ \big(1 + \sum_{M \geq 1} r^{\circ M}(x,\xi)\big). \tag{7.43}$$

(One could instead have taken $\tilde{p}' \sim p \circ {p_d}^*$, observed that it has principal symbol $p_d {p_d}^*$ (in S^{2d}) in view of the composition formula (7.36), thus is elliptic, and applied 1° to this symbol. This gives a parametrix symbol $\tilde{q}'$ such that $p \circ {p_d}^* \circ \tilde{q}' \sim 1$, and then ${p_d}^* \circ \tilde{q}'$ is a right parametrix symbol for p.)

The construction of a left parametrix symbol in case $N \leq N'$ is analogous. The necessity of ellipticity is seen as under 1°. □

The above proof is *constructive*; it shows that p has the parametrix symbol

$$\begin{gathered} q(x,\xi) \sim q_{-d}(x,\xi) \circ (1 + \sum_{M \geq 1} (1 - p(x,\xi) \circ q_{-d}(x,\xi))^{\circ M}) \\ \text{with } q_{-d}(x,\xi) = p_d(x,\xi)^{-1} \text{ for } |\xi| \geq 1, \end{gathered} \tag{7.44}$$

in the square matrix case, and (7.43) in case $N \geq N'$ (with a related left parametrix in case $N \leq N'$). Sometimes one is interested in the precise structure of the lower order terms, and they can be calculated from the above formulas.

Corollary 7.11. 1° *When P is a square matrix formed ψdo on Ω that is elliptic of order d, then it has a properly supported parametrix Q that is an elliptic ψdo of order $-d$. The parametrix Q is unique up to a negligible term.*

2° *When P is a surjectively elliptic ψdo of order d on Ω, then it has a properly supported right parametrix Q, that is an injectively elliptic ψdo of order $-d$. When P is an injectively elliptic ψdo of order d on Ω, then it has a properly supported left parametrix Q, that is a surjectively elliptic ψdo of order $-d$.*

Proof. That P is surjectively/injectively elliptic of order d means that $P = \operatorname{Op}\big(p(x,\xi)\big) + \mathcal{R}$, where $\mathcal{R}$ is negligible and $p \in S^d(\Omega) \otimes \mathcal{L}(\mathbb{C}^N, \mathbb{C}^{N'})$ with p_d surjective/injective for $|\xi| \geq 1$. Let $q(x,\xi)$ be the parametrix symbol constructed according to Theorem 7.10. For Q we can then take any properly supported operator $Q = \operatorname{Op}\big(q(x,\xi)\big) + \mathcal{R}'$ with $\mathcal{R}'$ negligible.

In the square matrix formed case, we have that when Q is a properly supported right parametrix, and Q' is a properly supported left parametrix, then since $PQ = I - \mathcal{R}_1$ and $Q'P = I - \mathcal{R}_2$,

$$Q' - Q = Q'(PQ + \mathcal{R}_1) - (Q'P + \mathcal{R}_2)Q = Q'\mathcal{R}_1 - \mathcal{R}_2 Q$$

is negligible; and Q and Q' are, both of them, two-sided parametrices. □

There are some immediate consequences for the solvability of the equation $Pu = f$, when P is elliptic.

Corollary 7.12. *When P is injectively elliptic of order d on Ω, and properly supported, then any solution $u \in \mathcal{D}'(\Omega)$ of the equation*

$$Pu = f \quad \text{in } \Omega\,, \quad \text{with } f \in H^s_{\text{loc}}(\Omega), \tag{7.45}$$

satisfies $u \in H^{s+d}_{\text{loc}}(\Omega)$.

Proof. Let Q be a properly supported left parametrix of P, it is of order $-d$. Then $QP = I - \mathcal{R}$, with $\mathcal{R}$ negligible, so u satisfies

$$u = QPu + \mathcal{R}u = Qf + \mathcal{R}u.$$

Here Q maps $H^s_{\text{loc}}(\Omega)$ into $H^{s+d}_{\text{loc}}(\Omega)$ and $\mathcal{R}$ maps $\mathcal{D}'(\Omega)$ into $C^\infty(\Omega)$, so $u \in H^{s+d}_{\text{loc}}(\Omega)$. □

This is a far-reaching generalization of the regularity result in Theorem 6.25.

Without assuming that P is properly supported, we get the same conclusion for $u \in \mathcal{E}'(\Omega)$.

Corollary 7.12 is a *regularity* result, that is interesting also for uniqueness questions: If one knows that there is uniqueness of "smooth" solutions, then injective ellipticity gives uniqueness of any kind of solution. – Another consequence of injective ellipticity is that there is always *uniqueness modulo C^∞ solutions*: If u and u' satisfy $Pu = Pu' = f$, then (with notations as above)

$$u - u' = (QP + \mathcal{R})(u - u') = \mathcal{R}(u - u') \in C^\infty(\Omega). \tag{7.46}$$

Note also that there is an *estimate* for $u \in H^s_{\text{comp}}(\Omega)$ with support in $K \subset \Omega$:

$$\|u\|_s = \|QPu + \mathcal{R}u\|_s \le C(\|Pu\|_{s-d} + \|u\|_{s-1}), \tag{7.47}$$

since $\mathcal{R}$ is of order ≤ -1.

For surjectively elliptic differential operators we can show an *existence* result for data with small support.

Corollary 7.13. *Let P be a surjectively elliptic differential operator of order d in Ω, let $x_0 \in \Omega$ and let $r > 0$ be such that $\overline{B}(x_0, r) \subset \Omega$. Then there exists $r_0 \le r$ so that for any $f \in L_2(B(x_0, r_0))$, there is a $u \in H^d(B(x_0, r_0))$ satisfying $Pu = f$ on $B(x_0, r_0)$.*

Proof. Let Q be a properly supported right parametrix of P. Then $PQ = I - \mathcal{R}$ where $\mathcal{R}$ is negligible, and

$$PQf = f - \mathcal{R}f \quad \text{ for } f \in L_{2,\mathrm{comp}}(\Omega).$$

Let $r_0 \le r$ (to be fixed later), and denote $B(x_0, r_0) = B_0$. Similarly to earlier conventions, functions in $L_2(B_0)$ will be identified with their extension by 0 on $\Omega \setminus B_0$, which belongs to $L_2(\Omega)$ and has support in $\overline{B}_0$. We denote by r_{B_0} the operator that restricts to B_0.

When f is supported in $\overline{B}_0$,

$$r_{B_0} PQf = f - 1_{B_0} \mathcal{R}f \text{ on } B_0.$$

We know that $\mathcal{R}$ has a C^∞ kernel $K(x, y)$, so that

$$(\mathcal{R}f)(x) = \int_{B_0} K(x, y) f(y)\, dy, \text{ when } f \in L_2(B_0);$$

moreover,

$$\begin{aligned} \|r_{B_0}\mathcal{R}f\|^2_{L_2(B_0)} &= \int_{B_0} \Big| \int_{B_0} K(x, y) f(y)\, dy \Big|^2 dx \\ &\le \int_{B_0} \Big(\int_{B_0} |K(x, y)|^2\, dy \Big) \Big(\int_{B_0} |f(y)|^2\, dy \Big) dx \\ &= \int_{B_0 \times B_0} |K(x, y)|^2\, dxdy\, \|f\|^2_{L_2(B_0)}. \end{aligned}$$

When $r_0 \to 0$, $\int_{B_0 \times B_0} |K(x, y)|^2\, dxdy \to 0$, so there is an $r_0 > 0$ such that the integral is $\le \frac{1}{4}$; then

$$\|r_{B_0}\mathcal{R}f\|_{L_2(B_0)} \le \tfrac{1}{2} \|f\|_{L_2(B_0)} \text{ for } f \in L_2(B_0).$$

So the norm of the operator S in $L_2(B_0)$ defined from $r_{B_0}\mathcal{R}$,

$$S \colon f \mapsto r_{B_0}\mathcal{R}f,$$

is $\le \frac{1}{2}$. Then $I - S$ can be inverted by a Neumann series

$$(I - S)^{-1} = I + S + S^2 + \cdots = \sum_{k=0}^{\infty} S^k \qquad (7.48)$$

converging in norm to a bounded operator in $L_2(B_0)$. It follows that when $f \in L_2(B_0)$, we have on B_0 (using that $r_{B_0} P 1_{B_0} v = r_{B_0} P v$ since P is a differential operator):

$$r_{B_0} P 1_{B_0} Q(I-S)^{-1} f = r_{B_0} PQ(I-S)^{-1} f = r_{B_0}(I - 1_{B_0}\mathcal{R})(I-S)^{-1} f = f,$$

which shows that $u = 1_{B_0} Q(I-S)^{-1} f$ solves the equation $Pu = f$ on B_0. Since Q is of order $-d$, $r_{B_0} u$ lies in $H^d(B_0)$. $\square$

More powerful conclusions can be obtained for ψdo's on compact manifolds; they will be taken up in Chapter 8.

7.5 Strongly elliptic operators, the Gårding inequality.

A polyhomogeneous pseudodifferential operator $P = \operatorname{Op}(p(x,\xi))$ of order d on Ω is said to be *strongly elliptic*, when the principal symbol satifies

$$\operatorname{Re} p^0(x,\xi) \geq c_0(x)|\xi|^d, \text{ for } x \in \Omega,\ |\xi| \geq 1, \tag{7.49}$$

with $c_0(x)$ continuous and positive. It is *uniformly strongly elliptic* when $c_0(x)$ has a positive lower bound (this holds of course on compact subsets of Ω). The definition extends to $(N \times N)$-matrix formed operators, when we define $\operatorname{Re} p^0 = \frac{1}{2}(p^0 + p^{0*})$ and read (7.49) in the matrix sense:

$$(\operatorname{Re} p^0(x,\xi)v, v) \geq c_0(x)|\xi|^d |v|^2, \text{ for } x \in \Omega,\ |\xi| \geq 1,\ v \in \mathbb{C}^N. \tag{7.50}$$

When P is uniformly strongly elliptic of order $d > 0$, one can show that $\operatorname{Re} P = \frac{1}{2}(P + P^*)$ has a certain lower semiboundedness property called the *Gårding inequality* (it was shown for differential operators by Gårding in [G53]):

$$\operatorname{Re}(Pu, u) \geq c_1 \|u\|_{d/2}^2 - c_2 \|u\|_0^2 \tag{7.51}$$

holds for $u \in C_0^\infty(\Omega)$, with some $c_1 > 0$, $c_2 \in \mathbb{R}$.

Before giving the proof, we shall establish a useful interpolation property of Sobolev norms.

Theorem 7.14. *Let s and $t \in \mathbb{R}$.*

1° *For any $\theta \in [0,1]$ one has for all $u \in H^{\max\{s,t\}}(\mathbb{R}^n)$:*

$$\|u\|_{\theta s + (1-\theta)t, \wedge} \leq \|u\|_{s,\wedge}^\theta \|u\|_{t,\wedge}^{1-\theta}. \tag{7.52}$$

2° *Let $s < r < t$. For any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that*

$$\|u\|_{r,\wedge} \leq \varepsilon \|u\|_{t,\wedge} + C(\varepsilon)\|u\|_{s,\wedge} \text{ for } u \in H^t(\mathbb{R}^n). \tag{7.53}$$

Proof. 1°. When $\theta = 0$ or 1, the inequality is trivial, so let $\theta \in]0,1[$. Then we use the Hölder inequality with $p = 1/\theta$, $p' = 1/(1-\theta)$:

$$\begin{aligned}\|u\|^2_{\theta s+(1-\theta)t,\wedge} &= \int_{\mathbb{R}^n} \langle\xi\rangle^{2\theta s+2(1-\theta)t}|\hat{u}(\xi)|^2\, d\!\!{}^-\xi \\ &= \int_{\mathbb{R}^n} (\langle\xi\rangle^{2s}|\hat{u}(\xi)|^2)^{1/p}(\langle\xi\rangle^{2t}|\hat{u}(\xi)|^2)^{1/p'}\, d\!\!{}^-\xi \\ &\le \Big(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s}|\hat{u}(\xi)|^2\, d\!\!{}^-\xi\Big)^{1/p}\Big(\int_{\mathbb{R}^n} \langle\xi\rangle^{2t}|\hat{u}(\xi)|^2\, d\!\!{}^-\xi\Big)^{1/p'} \\ &= \|u\|^{2\theta}_{s,\wedge}\|u\|^{2(1-\theta)}_{t,\wedge}.\end{aligned}$$

2°. Taking $\theta = (r-t)/(s-t)$, we have (7.52) with $\theta s + (1-\theta)t = r$. For $\theta \in [0,1]$ and a and $b \ge 0$ there is the general inequality:

$$a^\theta b^{1-\theta} \le \max\{a,b\} \le a+b \tag{7.54}$$

(since e.g. $a^\theta \le b^\theta$ when $0 \le a \le b$). We apply this to $b = \varepsilon\|u\|_{t,\wedge}$, $a = \varepsilon^{(-1+\theta)/\theta}\|u\|_{s,\wedge}$, $\theta = (r-t)/(s-t)$, which gives:

$$\begin{aligned}\|u\|_{r,\wedge} &\le (\varepsilon^{(-1+\theta)/\theta}\|u\|_{s,\wedge})^\theta(\varepsilon\|u\|_{t,\wedge})^{1-\theta} \\ &\le \varepsilon^{(-1+\theta)/\theta}\|u\|_{s,\wedge} + \varepsilon\|u\|_{t,\wedge}. \quad \square\end{aligned}$$

Theorem 7.16. (The Gårding Inequality.) *Let A be a properly supported $(N\times N)$-matrix formed ψdo on Ω_1 of order $d>0$, strongly elliptic on Ω_1. Denote $\frac12 d = d'$. Let Ω be an open subset such that $\overline{\Omega}$ is compact in Ω_1. There exist constants $c_0 > 0$ and $k \in \mathbb{R}$ such that*

$$\operatorname{Re}(Au,u) \ge c_0\|u\|^2_{d'} - k\|u\|^2_0, \text{ when } u \in C_0^\infty(\Omega)^N. \tag{7.55}$$

Proof. The symbol $\operatorname{Re} a^0(x,\xi) = \frac12(a^0(x,\xi) + a^0(x,\xi)^*)$ is the principal symbol of $\operatorname{Re} A = \frac12(A + A^*)$ and is positive definite by assumption. Let

$$p^0(x,\xi) = \sqrt{\operatorname{Re} a^0(x,\xi)} \text{ for } |\xi| \ge 1,$$

extended smoothly to $|\xi| \le 1$, it is elliptic of order d'. (When $N > 1$, one can define the squareroot as

$$\frac{i}{2\pi}\int_{\mathcal{C}} \lambda^{\frac12}(\operatorname{Re} a^0(x,\xi) - \lambda)^{-1}\, d\lambda,$$

where $\mathcal{C}$ is a closed curve in $\mathbb{C} \setminus \overline{\mathbb{R}}_+$ encircling the spectrum of $\operatorname{Re} a^0(x,\xi)$ in the positive direction.)

Let P be a properly supported ψdo on Ω_1 with symbol $p^0(x,\xi)$; its adjoint P^* is likewise properly supported and has principal symbol $p^0(x,\xi)$. Then P^*P has principal symbol $\operatorname{Re} a^0(x,\xi)$, so

$$\operatorname{Re} A = P^*P + S,$$

where S is of order $d-1$. For any $s \in \mathbb{R}$, let Λ_s be a properly supported ψdo on Ω_1 equivalent with $\operatorname{Op}(\langle\xi\rangle^s I_N)$ (I_N denotes the $(N \times N)$-unit matrix). Then $\Lambda_{-d'}\Lambda_{d'} \sim I$, and we can rewrite

$$S = S_1 S_2 + \mathcal{R}_1, \text{ where } S_1 = S\Lambda_{-d'},\ S_2 = \Lambda_{d'},$$

with S_1 of order $d'-1$, S_2 of order d' and $\mathcal{R}_1$ of order $-\infty$, all properly supported.

Since P is elliptic of order d', it has a properly supported parametrix Q of order $-d'$. Because of the properly supportedness there are bounded sets Ω' and Ω'' with $\overline{\Omega} \subset \Omega'$, $\overline{\Omega'} \subset \Omega''$, $\overline{\Omega''} \subset \Omega_1$, such that when $\operatorname{supp} u \subset \Omega$, then $Au, Pu, S_1u, S_1^*u, S_2u$ are supported in Ω' and $QPu, S_1S_2u, \mathcal{R}_1u, \mathcal{R}_2u$ are supported in Ω''.

For $u \in C_0^\infty(\Omega)^N$ we have:

$$\begin{aligned}\operatorname{Re}(Au,u) &= \tfrac12[(Au,u)+(u,Au)] = ((\operatorname{Re} A)u,u)\\ &= (P^*Pu,u) + (S_1S_2u,u) + (\mathcal{R}_1u,u)\\ &= \|Pu\|_0 + (S_2u, S_1^*u) + (\mathcal{R}_1u,u).\end{aligned} \tag{7.56}$$

To handle the first term, we note that by the Sobolev space mapping properties of Q and $\mathcal{R}_2$ we have (similarly to (7.47))

$$\|u\|_{d'}^2 \le (\|QPu\|_{d'} + \|\mathcal{R}_2u\|_{d'})^2 \le C(\|Pu\|_0 + \|u\|_0)^2 \le 2C(\|Pu\|_0^2 + \|u\|_0^2)$$

with $C > 0$, hence

$$\|Pu\|_0^2 \ge (2C)^{-1}\|u\|_{d'}^2 - \|u\|_0^2. \tag{7.57}$$

The last term in (7.56) satisfies

$$|(\mathcal{R}_1u,u)| \le \|\mathcal{R}_1u\|_0\|u\|_0 \le c\|u\|_0^2. \tag{7.58}$$

For the middle term, we estimate

$$|(S_2u, S_1^*u)| \le \|S_2u\|_0\|S_1^*u\|_0 \le c'\|u\|_{d'}\|u\|_{d'-1} \le \tfrac12 c'(\varepsilon^2\|u\|_{d'}^2 + \varepsilon^{-2}\|u\|_{d'-1}^2),$$

any $\varepsilon > 0$. If $d' - 1 > 0$, we refine this further by using that by (7.53),

$$\|u\|_{d'-1}^2 \le \varepsilon' \|u\|_{d'}^2 + C'(\varepsilon')\|u\|_0^2.$$

for any $\varepsilon' > 0$. Taking first ε small and then ε' small enough, we can obtain that

$$|(S_2 u, S_1^* u)| \le (4C)^{-1}\|u\|_{d'}^2 + C''\|u\|_0^2. \tag{7.59}$$

Application of (7.57)–(7.59) in (7.56) gives that

$$\operatorname{Re}(Au, u) \ge (4C)^{-1}\|u\|_{d'}^2 - C'''\|u\|_0^2,$$

an inequality of the desired type. □

The result can be applied to differential operators in the following way (generalizing the applications of the Lax-Milgram lemma given in Section 4.4):

Theorem 7.17. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and let $A = \sum_{|\alpha|\le d} a_\alpha D^\alpha$ be strongly elliptic on a neighborhood Ω_1 of $\overline{\Omega}$ (with $(N \times N)$-matrix formed C^∞ functions $a_\alpha(x)$ on Ω_1). Then d is even, $d = 2m$, and the realization A_γ of A in $L_2(\Omega)^N$ with domain $D(A_\gamma) = H_0^m(\Omega)^N \cap D(A_{\max})$ is a variational operator (hence has its spectrum and numerical range in an angular set* (12.48)*).*

Proof. The order d is even, because $\operatorname{Re} a^0(x,\xi)$ and $\operatorname{Re} a^0(x,-\xi)$ are both positive definite. Theorem 7.16 assures that

$$\operatorname{Re}(Au, u) \ge c_0\|u\|_m^2 - k\|u\|_0^2, \text{ when } u \in C_0^\infty(\Omega)^N, \tag{7.60}$$

with $c_0 > 0$. We can rewrite A in the form

$$Au = \sum_{|\beta|,|\theta|\le m} D^\beta(b_{\beta\theta} D^\theta u);$$

with suitable matrices $b_{\beta\theta}(x)$ that are C^∞ on Ω_1, using the backwards Leibniz formula (7.37). Then define the sesquilinear form $a(u,v)$ by

$$a(u,v) = \sum_{|\beta|,|\theta|\le m} (b_{\beta\theta} D^\theta u, D^\beta v)_{L_2(\Omega)^N}, \text{ for } u, v \in H_0^m(\Omega)^N;$$

it is is bounded on $V = H_0^m(\Omega)^N$ since the $b_{\beta\theta}$ are bounded on Ω. Moreover, by distribution theory,

$$(Au, v) = a(u,v) \text{ for } u \in D(A_{\max}), v \in C_0^\infty(\Omega)^N; \tag{7.61}$$

this identity extends by continuity to $v \in V$ (recall that $H_0^m(\Omega)^N$ is the closure of $C_0^\infty(\Omega)^N$ in m-norm). Then by (7.60), $a(u,v)$ is V-coercive (12.38), with $H = L_2(\Omega)^N$.

Applying the Lax-Milgram construction from Section 12.4 to the triple $\{H, V, a\}$, we obtain a variational operator A_γ. In the following, use the notation of Chapter 4. In view of (7.61), A_γ is a closed extension of $A_{C_0^\infty}$, hence of $A_{\min}$. Similarly, the Hilbert space adjoint $A_\gamma{}^*$ extends $A'_{\min}$, by the same construction applied to the triple $\{H, V, a^*\}$, so A_γ is a realization of A. It follows that $D(A_\gamma) \subset H_0^m(\Omega)^N \cap D(A_{\max})$. By (7.61)ff., this inclusion is an identity. $\square$

We find moreover that $D(A_\gamma) \subset H^{2m}_{\mathrm{loc}}(\Omega)^N$, in view of the ellipticity of A and Corollary 7.12.

Note that there were no smoothness assumptions on Ω whatsoever in this theorem. With some smoothness, one can moreover show that the domain is in $H^{2m}(\Omega)^N$. This belongs to the deeper theory of elliptic boundary value problems (for which a pseudodifferential strategy is presented in Chapters 10 and 11), and will be shown for smooth sets at the end of Chapter 11.

A_γ is regarded as the Dirichlet realization of A, since its domain consists of those functions u in the maximal domain that belong to $H_0^m(\Omega)^N$; when Ω is sufficiently smooth, this means that the Dirichlet boundary values $\{\gamma_0 u, \gamma_1 u, \dots, \gamma_{m-1} u\}$ are 0.

One can also show a version of Theorem 7.17 for suitable ψdo's, cf. [G96, Sect. 1.7].

Exercises for Chapter 7.

7.1. For an arbitrary $s \in \mathbb{R}$, find the asymptotic expansion of the symbol $\langle \xi \rangle^s$ in homogeneous terms.

7.2. Show, by insertion in the formula (7.14) and suitable reductions, that the ψdo defined from the symbol (7.25) is zero.

7.3. Let $p(x,\xi)$ and $p'(x,\xi)$ be polyhomogeneous pseudodifferential symbols, of order d resp. d'. Consider $p'' \sim p \circ p'$, defined according to (7.36) and Definition 7.7b; it is polyhomogeneous of degree $d'' = d + d'$.

(a) Show that $p''_{d''} = p_d \, p'_{d'}$.

(b) Show that

$$p''_{d''-1} = \sum_{j=1}^{n} D_{\xi_j} p_d \, \partial_{x_j} p'_{d'} + p_d \, p'_{d'-1} + p_{d-1} \, p'_{d'}.$$

(c) Find $p''_{d''-2}$.

7.4. Consider the fourth-order operator $a(x)\Delta^2 + b(x)$ on $\mathbb{R}^n$, where a and b are C^∞-functions with $a(x) > 0$ for all x.

(a) Show that A is elliptic. Find a parametrix symbol, where the first three homogeneous terms (of order $-4, -5, -6$) are worked out in detail.

(b) Investigate the special case $a = |x|^2 + 1$, $b = 0$.

7.5. The operator $L_\sigma = -\operatorname{div}\operatorname{grad} + \sigma \operatorname{grad}\operatorname{div} = -\Delta + \sigma \operatorname{grad}\operatorname{div}$, applied to n-vectors, is a case of the Lamé operator. In details

$$L_\sigma u = \begin{pmatrix} -\Delta + \sigma\partial_1^2 & \sigma\partial_1\partial_2 & \dots & \sigma\partial_1\partial_n \\ \sigma\partial_1\partial_2 & -\Delta + \sigma\partial_2^2 & \dots & \sigma\partial_2\partial_n \\ \vdots & \vdots & \ddots & \vdots \\ \sigma\partial_1\partial_n & \sigma\partial_2\partial_n & \dots & -\Delta + \sigma\partial_n^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix};$$

here σ is a real constant. Let $n = 2$ or 3.

(a) For which σ is L_σ elliptic?

(b) For which σ is L_σ strongly elliptic?