§6. Applications to differential operators. The Sobolev theorem

6.1. Differential and pseudodifferential operators on $\mathbb{R}^n$.

As we saw in (5.39)–(5.42), a differential operator $P(D)$ (with constant coefficients) is by Fourier transformation carried over to a multiplication operator $M_p\colon f \mapsto pf$, where $p(\xi)$ is a polynomial. One can extend this idea to the more general functions $p(\xi) \in \mathcal{O}_M$, obtaining a class of operators which we call pseudodifferential operators.

Definition 6.1. *Let $p(\xi) \in \mathcal{O}_M$. The associated pseudodifferential operator* $\operatorname{Op}(p(\xi))$*, also called $P(D)$, is defined by*

$$\operatorname{Op}(p)u \equiv P(D)u = \mathcal{F}^{-1}(p(\xi)\hat{u}(\xi)) \,, \tag{6.1}$$

it maps $\mathcal{S}$ into $\mathcal{S}$ and $\mathcal{S}'$ into $\mathcal{S}'$ (continuously). The function $p(\xi)$ is called **the symbol** *of* $\operatorname{Op}(p)$.

As observed, differential operators with constant coefficients are covered by this definition; but it is interesting that also the solution operator in Example 5.19 is of this type, since it equals $\operatorname{Op}(\langle\xi\rangle^{-2})$.

For these pseudodifferential operators one has the extremely simple rule of calculus:

$$\operatorname{Op}(p)\operatorname{Op}(q) = \operatorname{Op}(pq), \tag{6.2}$$

since $\operatorname{Op}(p)\operatorname{Op}(q)u = \mathcal{F}^{-1}(p\mathcal{F}\mathcal{F}^{-1}(q\mathcal{F}u)) = \mathcal{F}^{-1}(pq\mathcal{F}u)$. In other words, *composition of operators* corresponds to *multiplication of symbols*. Moreover, if p is a function in $\mathcal{O}_M$ for which $1/p$ belongs to $\mathcal{O}_M$, then the operator $\operatorname{Op}(p)$ has the inverse $\operatorname{Op}(1/p)$:

$$\operatorname{Op}(p)\operatorname{Op}(1/p) = \operatorname{Op}(1/p)\operatorname{Op}(p) = I. \tag{6.3}$$

For example, $1 - \Delta = \operatorname{Op}(\langle\xi\rangle^2)$ has the inverse $\operatorname{Op}(\langle\xi\rangle^{-2})$, cf. Example 5.19.

Remark 6.2. We here use the notation pseudodifferential operator for all operators that are obtained by Fourier transformation from multiplication operators in $\mathcal{S}$ (and $\mathcal{S}'$). In practical applications, one usually considers restricted classes of symbols with special properties. On the other hand, one allows symbols depending on x also, associating the operator $\operatorname{Op}(p(x,\xi))$ defined by

$$[\operatorname{Op}(p(x,\xi))u](x) = (2\pi)^{-n} \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi \,; \tag{6.4}$$

to the symbol $p(x,\xi)$. This is consistent with the fact that when P is a differential operator of the form

$$P(x,D)u = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u \;, \tag{6.5}$$

then $P(x,D) = \operatorname{Op}(p(x,\xi))$, where *the symbol* is

$$p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \;. \tag{6.6}$$

Allowing "variable coefficients" makes the theory much more complicated, in particular because the identities (6.2) and (6.3) then no longer hold in an exact way, but in a certain approximative sense, depending on which symbol class one considers. The systematic theory of pseudodifferential operators plays an important role in the modern matematical litterature, as a general framework around differential operators and their solution operators. It is technically more complicated than what we are doing at present, and will be taken up later, in Chapter 7.

Let us consider the L_2-realizations of a pseudodifferential operator $P(D)$. In this "constant-coefficient" case we can appeal to Theorem 12.13 on multiplication operators in L_2.

Theorem 6.3. *Let $p(\xi)\in\mathcal{O}_M$ and let $P(D)$ be the associated pseudodifferential operator* $\operatorname{Op}(p)$. *The realization $P(D)_{\max}$ of $P(D)$ in $L_2(\mathbb{R}^n)$ with domain*

$$D(P(D)_{\max}) = \{\, u\in L_2(\mathbb{R}^n) \mid P(D)u\in L_2(\mathbb{R}^n)\,\} \;, \tag{6.8}$$

is densely defined (with $\mathcal{S}\subset D(P(D)_{\max})$) and closed. Let $P(D)_{\min}$ denote the closure of $P(D)\big|_{C_0^\infty(\mathbb{R}^n)}$; then

$$P(D)_{\max} = P(D)_{\min} \;. \tag{6.9}$$

Furthermore, $(P(D)_{\max})^* = P'(D)_{\max}$, where $P'(D) = \operatorname{Op}(\overline{p})$.

Proof. We write P for $P(D)$ and P' for $P'(D)$. It follows immediately from the Plancherel-Parseval theorem (Theorem 5.5) that

$$\begin{aligned} P_{\max} &= \mathcal{F}^{-1} M_p \mathcal{F}\;; \quad \text{with}\\ D(P_{\max}) &= \mathcal{F}^{-1} D(M_p) = \mathcal{F}^{-1}\{\, f\in L_2(\mathbb{R}^n) \mid pf\in L_2(\mathbb{R}^n)\,\} \;, \end{aligned}$$

where M_p is the multiplication operator in $L_2(\mathbb{R}^n)$ defined as in Theorem 12.13. In particular, $P_{\max}$ is a closed, densely defined operator, and $\mathcal{S}\subset$

$D(M_p)$ implies $\mathcal{S} \subset D(P(D)_{\max})$. We shall now first show that $P_{\max}$ and $P'_{\min}$ are adjoints of one another. This goes in practically the same way as in Section 4.1: For $u \in \mathcal{S}'$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ one has:

$$\begin{aligned}\langle Pu, \overline{\varphi}\rangle &= \langle \mathcal{F}^{-1} p \mathcal{F} u, \overline{\varphi}\rangle = \langle p\mathcal{F}u, \mathcal{F}^{-1}\overline{\varphi}\rangle \\ &= \langle u, \mathcal{F} p \mathcal{F}^{-1}\overline{\varphi}\rangle = \langle u, \overline{\mathcal{F}^{-1}\overline{p}\mathcal{F}\varphi}\rangle = \langle u, \overline{P'\varphi}\rangle \ ,\end{aligned} \tag{6.10}$$

using that $\overline{\mathcal{F}} = (2\pi)^n \mathcal{F}^{-1}$. We see from this on one hand that when $u \in D(P_{\max})$, i.e., u and $Pu \in L_2$, then

$$(Pu, \varphi) = (u, P'\varphi) \quad \text{ for all } \varphi \in C_0^\infty \ ,$$

so that

$$P_{\max} \subset (P'\big|_{C_0^\infty})^* \text{ and } P'\big|_{C_0^\infty} \subset (P_{\max})^* \ ,$$

and thereby

$$P'_{\min} = \text{ closure of } P'\big|_{C_0^\infty} \subset (P_{\max})^* \ .$$

On the other hand we see from (6.10) that when $u \in D((P'\big|_{C_0^\infty})^*)$, i.e., there exists $v \in L_2$ so that $(u, P'\varphi) = (v, \varphi)$ for all $\varphi \in C_0^\infty$, then v equals Pu, i.e.,

$$(P'\big|_{C_0^\infty})^* \subset P_{\max} \ .$$

Thus $P_{\max} = (P'\big|_{C_0^\infty})^* = (P'_{\min})^*$ (cf. Corollary 12.6). So Lemma 4.3 extends to the present situation.

But now we can furthermore use that $(M_p)^* = M_{\overline{p}}$ by Theorem 12.13, which by Fourier transformation is carried over to

$$(P_{\max})^* = P'_{\max} \ .$$

In detail:

$$(P_{\max})^* = (\mathcal{F}^{-1} M_p \mathcal{F})^* = \mathcal{F}^* M_p^* (\mathcal{F}^{-1})^* = \overline{\mathcal{F}} M_{\overline{p}} \overline{\mathcal{F}}^{-1} = \mathcal{F}^{-1} M_{\overline{p}} \mathcal{F} = P'_{\max} \ ,$$

using that $\mathcal{F}^* = \overline{\mathcal{F}} = (2\pi)^n \mathcal{F}^{-1}$.

Since $(P_{\max})^* = P'_{\min}$, it follows that $P'_{\max} = P'_{\min}$, showing that the maximal and the minimal operators coincide, for all these multiplication operators and Fourier transformed multiplication operators. $\square$

Theorem 6.4. *One has for the operators introduced in Theorem* 6.3:

1° $P(D)_{\max}$ *is a bounded operator in* $L_2(\mathbb{R}^n)$ *if and only if* $p(\xi)$ *is bounded, and the norm satisfies*

$$\|P(D)_{\max}\| = \sup\{\, |p(\xi)| \mid \xi \in \mathbb{R}^n \,\}. \tag{6.11}$$

2° $P(D)_{\max}$ *is selfadjoint in* $L_2(\mathbb{R}^n)$ *if and only if* p *is real.*

3° $P(D)_{\max}$ *has the lower bound*

$$m(P(D)_{\max}) = \inf\{\, \operatorname{Re} p(\xi) \mid \xi \in \mathbb{R}^n \,\} \geq -\infty. \tag{6.12}$$

Proof. 1°. We have from Theorem 12.13 and the subsequent remarks that M_p is a bounded operator in $L_2(\mathbb{R}^n)$ when p is a bounded function on $\mathbb{R}^n$, and that the norm in that case is precisely $\sup\{|p(\xi)| \mid \xi \in \mathbb{R}^n\}$. If p is unbounded on $\mathbb{R}^n$, one has on the other hand that since p is continuous (hence bounded on compact sets), $C_N = \sup\{|p(\xi)| \mid |\xi| \leq N\} \to \infty$ for $N \to \infty$. Now C_N equals the norm of the operator of multiplication by p on $L_2(B(0,N))$. For every $R > 0$, we can by choosing N so large that $C_N \geq R$ find functions $f \in L_2(B(0,N))$ (thereby in $L_2(\mathbb{R}^n)$ by extension by 0) with norm 1 and $\|M_p f\| \geq R$. Thus M_p is an unbounded operator in $L_2(\mathbb{R}^n)$. This shows that M_p is bounded if and only if p is bounded.

Statement 1° now follows immediately by use of the Plancherel-Parseval theorem, observing that $\|P(D)u\|/\|u\| = \|\mathcal{F}P(D)u\|/\|\mathcal{F}u\| = \|p\hat{u}\|/\|\hat{u}\|$ for $u \neq 0$.

2°. Since $M_p = M_{\overline{p}}$ if and only if $p = \overline{p}$ by Theorem 12.13 ff., the statement follows in view of the Plancherel-Parseval theorem.

3°. Since the lower bound of M_p is $m(M_p) = \inf\{\operatorname{Re} p(\xi) \mid \xi \in \mathbb{R}^n\}$ (cf. Exercise 12.36), it follows from the Plancherel-Parseval theorem that $P(D)_{\max}$ has the lower bound (6.12). Here we use that $(P(D)u,u)/\|u\|^2 = (\mathcal{F}P(D)u, \mathcal{F}u)/\|\mathcal{F}u\|^2 = (p\hat{u},\hat{u})/\|\hat{u}\|^2$ for $u \in D(P(D)_{\max}) \setminus \{0\}$. ☐

Note that $P(D)_{\max}$ is the zero operator if and only if p is the zero function.

It follows in particular from this theorem that *for all differential operators with constant coefficients on* $\mathbb{R}^n$*, the maximal realization equals the minimal realization;* we have earlier obtained this for first-order operators (cf. Exercise 4.2, where one could use convolution by h_j and truncation), and for $I - \Delta$ (hence for Δ) at the end of Section 5.3.

Since $|\xi|^2$ is real and has lower bound 0, we get as a special case of Theorem 6.4 the result (which could also be inferred from the considerations in Example 5.19):

Corollary 6.5. *The maximal and minimal realizations of* $-\Delta$ *in* $L_2(\mathbb{R}^n)$ *coincide. It is a selfadjoint operator with lower bound* 0.

6.2. Sobolev spaces of arbitrary real order. The Sobolev Theorem.

One of the applications of Fourier transformation is that it can be used in the analysis of regularity of solutions of differential equations $P(D)u = f$, even when existence or uniqueness results are not known on beforehand. In Example 5.19 we found that any solution $u \in \mathcal{S}'$ of $(1-\Delta)u = f$ with $f \in L_2$ must belong to $H^2(\mathbb{R}^n)$. We shall now consider the Sobolev spaces in relation to the Fourier transformation.

We first introduce some auxiliary weighted L_p spaces.

Definition 6.6. *For each $s \in \mathbb{R}$ and each $p \in [1, \infty]$, we denote by $L_{p,s}(\mathbb{R}^n)$ (or just $L_{p,s}$) the Banach space*

$$L_{p,s}(\mathbb{R}^n) = \{ u \in L_{1,\mathrm{loc}}(\mathbb{R}^n) \mid \langle x \rangle^s u(x) \in L_p(\mathbb{R}^n) \}$$
$$\text{with norm } \|u\|_{L_{p,s}} = \|\langle x \rangle^s u(x)\|_{L_p(\mathbb{R}^n)} .$$

For $p = 2$, this is a Hilbert space (namely $L_2(\mathbb{R}^n, \langle x \rangle^{2s} dx)$) with the scalar product

$$(f, g)_{L_{2,s}} = \int_{\mathbb{R}^n} f(x)\overline{g}(x)\langle x \rangle^{2s}\, dx.$$

Note that multiplication by $\langle x \rangle^t$ defines an isometry of $L_{p,s}$ onto $L_{p,s-t}$ for every $p \in [1, \infty]$ and $s, t \in \mathbb{R}$.

One frequently needs the following inequality.

Lemma 6.6a. (The Peetre inequality) *For any $s \in \mathbb{R}$,*

$$\langle x - y \rangle^s \le c_s \langle x \rangle^s \langle y \rangle^{|s|} \qquad \text{for } s \in \mathbb{R} , \tag{6.13}$$

with a positive constant c_s.

Proof. First observe that

$$1 + |x - y|^2 \le 1 + (|x| + |y|)^2 \le c(1 + |x|^2)(1 + |y|^2);$$

this is easily seen to hold with $c = 2$, and with a little more care one can show it with $c = 4/3$. This implies

$$\langle x - y \rangle^s \le c^{s/2} \langle x \rangle^s \langle y \rangle^s \qquad \text{when } s \ge 0 ,$$
$$\langle x - y \rangle^s = \frac{\langle x - y \rangle^{-|s|} \langle x - y + y \rangle^{|s|}}{\langle x \rangle^{|s|}} \le c^{|s|/2} \langle x \rangle^s \langle y \rangle^{|s|} , \qquad \text{when } s \le 0 .$$

Hence (6.13) holds with

$$c_s = c^{|s|/2}, \quad c = 4/3. \tag{6.13a}$$

□

In the following we shall use M_f again to denote multiplication by f, with domain adapted to varying needs. Because of the inequalities (5.2) we have:

Lemma 6.6b. *For $m \in \mathbb{N}_0$, u belongs to $H^m(\mathbb{R}^n)$ if and only if $\hat{u}$ belongs to $L_{2,m}(\mathbb{R}^n)$. The scalar product*

$$(u,v)_{m,\wedge} = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}\langle\xi\rangle^{2m}\, d\xi = (2\pi)^{-n}(\hat{u},\hat{v})_{L_{2,m}}$$

defines a norm $\|u\|_{m,\wedge} = (u,u)^{\frac{1}{2}}_{m,\wedge}$ equivalent with the norm introduced in Definition 4.5 (cf. (5.2) for C_m):

$$\begin{aligned} &\|u\|_m \le \|u\|_{m,\wedge} \le C_m^{\frac{1}{2}}\|u\|_m, \text{ for } m \ge 0,\\ &\|u\|_0 = \|u\|_{0,\wedge}. \end{aligned} \tag{6.14}$$

Proof. In view of the inequalities (5.2) and the Parseval-Plancherel theorem,

$$\begin{aligned} u \in H^m(\mathbb{R}^n) &\iff \sum_{|\alpha|\le m} |\xi^\alpha \hat{u}(\xi)|^2 \in L_1(\mathbb{R}^n)\\ &\iff (1+|\xi|^2)^m|\hat{u}(\xi)|^2 \in L_1(\mathbb{R}^n)\\ &\iff \hat{u} \in L_{2,m}(\mathbb{R}^n)\ . \end{aligned}$$

The inequalities between the norms follow straightforwardly. □

The norm $\|\cdot\|_{m,\wedge}$ is interesting since it is easy to generalize to noninteger or even negative values of m. Consistently with Definition 4.5 we introduce:

Definition 6.7. *For each $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ is defined by*

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle\xi\rangle^s\hat{u}(\xi) \in L_2(\mathbb{R}^n)\} = F^{-1}L_{2,s}(\mathbb{R}^n); \tag{6.15}$$

it is a Hilbert space with the scalar product and norm

$$(u,v)_{s,\wedge} = (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}\langle\xi\rangle^{2s}d\xi\ , \quad \|u\|_{s,\wedge} = (2\pi)^{-n/2}\|\langle\xi\rangle^s\hat{u}(\xi)\|_{L_2}\ . \tag{6.16}$$

The Hilbert space property of $H^s(\mathbb{R}^n)$ follows from the fact that $F = (2\pi)^{-n/2}\mathcal{F}$ by definition gives an *isometry*

$$H^s(\mathbb{R}^n) \xrightarrow{\sim} L_{2,s}(\mathbb{R}^n),$$

cf. (5.16) and Definitions 5.15, 6.6. Since $M_{\langle\xi\rangle^s}$ is an isometry of $L_{2,s}(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n)$, we have the following *commutative diagram of isometries*:

$$\begin{array}{ccc} H^s(\mathbb{R}^n) & \xrightarrow{F} & L_{2,s}(\mathbb{R}^n)\\ \Big\downarrow{\scriptstyle \mathrm{Op}(\langle\xi\rangle^s)} & & \Big\downarrow{\scriptstyle \langle\xi\rangle^s}\ , \\ L_2(\mathbb{R}^n) & \xrightarrow[F]{} & L_2(\mathbb{R}^n) \end{array} \tag{6.17}$$

where $\langle\xi\rangle^s F = F\,\mathrm{Op}(\langle\xi\rangle^s)$.

The operator $\mathrm{Op}(\langle\xi\rangle^s)$ will be denoted Ξ^s, and we clearly have:

$$\Xi^s = \mathrm{Op}(\langle\xi\rangle^s)\,, \quad \Xi^{s+t} = \Xi^s\Xi^t \quad \text{for } \ s,t\in\mathbb{R}\,. \tag{6.18}$$

Observe that $\Xi^{2M} = (1-\Delta)^M$ when M is integer ≥ 0, whereas Ξ^s is a pseudodifferential operator for other values of s. Note that Ξ^s is an isometry of $H^t(\mathbb{R}^n)$ onto $H^{t-s}(\mathbb{R}^n)$ for all $t\in\mathbb{R}$, when the norms $\|\cdot\|_{t,\wedge}$ and $\|\cdot\|_{t-s,\wedge}$ are used. We now easily find:

Lemma 6.8. *Let $s\in\mathbb{R}$.*

1° *Ξ^s defines a homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$, and of $\mathcal{S}'$ onto $\mathcal{S}'$, with inverse Ξ^{-s}.*

2° *$\mathcal{S}$ is dense in $L_{2,s}$ and in $H^s(\mathbb{R}^n)$. $C_0^\infty(\mathbb{R}^n)$ is likewise dense in these spaces.*

Proof. As noted earlier, $\mathcal{S}$ is dense in $L_2(\mathbb{R}^n)$, since C_0^∞ is so. Since $\langle\xi\rangle^s\in\mathcal{O}_M$, $M_{\langle\xi\rangle^s}$ maps $\mathcal{S}$ continuously into $\mathcal{S}$, and $\mathcal{S}'$ continuously into $\mathcal{S}'$, for all s; and since $M_{\langle\xi\rangle^{-s}}$ clearly acts as an inverse both for $\mathcal{S}$ and $\mathcal{S}'$, $M_{\langle\xi\rangle^s}$ defines a homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$, and of $\mathcal{S}'$ onto $\mathcal{S}'$. By inverse Fourier transformation it follows that Ξ^s defines a homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$, and of $\mathcal{S}'$ onto $\mathcal{S}'$, with inverse Ξ^{-s}. The denseness of $\mathcal{S}$ in L_2 now implies the denseness of $\mathcal{S}$ in $L_{2,s}$ by use of $M_{\langle\xi\rangle^{-s}}$, and the denseness of $\mathcal{S}$ in H^s by use of Ξ^{-s} (cf. the isometry-diagram (6.17)). For the last statement, note that the topology of $\mathcal{S}$ is stronger than that of $L_{2,s}$ resp. H^s, any s. A $u\in H^s$, say, can be approximated by $\varphi\in\mathcal{S}$ in the metric of H^s, and φ can be approximated by $\psi\in C_0^\infty(\mathbb{R}^n)$ in the metric of $\mathcal{S}$ (cf. Lemma 5.9). $\square$

The statement 2° is for s integer ≥ 0 also covered by Theorem 4.10.

Note that we now have established continuous injections

$$\mathcal{S}\subset H^{s'}\subset H^s\subset L_2\subset H^{-s}\subset H^{-s'}\subset\mathcal{S}'\,, \quad \text{for } \ s'>s>0\,, \tag{6.19}$$

so that the H^s spaces to some extent "fill in" between $\mathcal{S}$ and L_2, resp. between L_2 and $\mathcal{S}'$. However,

$$\mathcal{S}\underset{\neq}{\subset}\bigcap_{s\in\mathbb{R}} H^s \ \text{ and } \ \mathcal{S}'\underset{\neq}{\supset}\bigcup_{s\in\mathbb{R}} H^s\,, \tag{6.20}$$

which follows since we correspondingly have that

$$\mathcal{S}\underset{\neq}{\subset}\bigcap_{s\in\mathbb{R}} L_{2,s}\,, \quad \mathcal{S}'\underset{\neq}{\supset}\bigcup_{s\in\mathbb{R}} L_{2,s}\,, \tag{6.21}$$

where the functions in $\bigcap_{s\in\mathbb{R}} L_{2,s}$ of course need not be differentiable, and the elements in $\mathcal{S}'$ are not all functions. More information on $\bigcap_{s\in\mathbb{R}} H^s$ is given below in (6.26). (There exists another scale of spaces where one combines polynomial growth conditions with differentiability, whose intersection resp. union equals $\mathcal{S}$ resp. $\mathcal{S}'$. Exercise 6.37 treats $\mathcal{S}(\mathbb{R})$.)

We shall now study how the Sobolev spaces are related to spaces of continuously differentiable functions; the main result is the Sobolev Theorem.

Theorem 6.9. (THE SOBOLEV THEOREM) *Let m be an integer ≥ 0, and let $s > m + n/2$. Then (cf.* (C.9)*)*

$$H^s(\mathbb{R}^n) \subset C^m_{L_\infty}(\mathbb{R}^n)\,, \tag{6.22}$$

with continuous injection, i.e., there is a constant $C > 0$ such that for $u \in H^s(\mathbb{R}^n)$,

$$\sup\{\,|D^\alpha u(x)| \mid x \in \mathbb{R}^n\,,\ |\alpha| \leq m\,\} \leq C\|u\|_{s,\wedge}\,. \tag{6.23}$$

Proof. For $\varphi \in \mathcal{S}$ one has for $s = m + t$, $t > n/2$ and $|\alpha| \leq m$, cf. (5.2),

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} |D^\alpha \varphi(x)| &= \sup|(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\xi^\alpha\hat\varphi(\xi)\,d\xi| \\ &\leq (2\pi)^{-n}\int_{\mathbb{R}^n} |\hat\varphi(\xi)|\langle\xi\rangle^{m+t}\langle\xi\rangle^{-t}\,d\xi \\ &\leq (2\pi)^{-n}\|\hat\varphi\|_{L_{2,s}}\Big(\int_{\mathbb{R}^n}\langle\xi\rangle^{-2t}\,d\xi\Big)^{\frac12} = C\|\varphi\|_{s,\wedge}\,,\end{aligned} \tag{6.24}$$

since the integral of $\langle\xi\rangle^{-2t}$ is finite when $t > n/2$. This shows (6.23) for $\varphi \in \mathcal{S}$. When $u \in H^s$, there exists according to Lemma 6.8 a sequence $\varphi_k \in \mathcal{S}$ so that $\|u - \varphi_k\|_{s,\wedge} \to 0$ for $k \to \infty$. By (6.24), φ_k is a Cauchy sequence in $C^m_{L_\infty}(\mathbb{R}^n)$, and since this space is a Banach space, there is a limit $v \in C^m_{L_\infty}(\mathbb{R}^n)$. Both the convergence in H^s and the convergence in $C^m_{L_\infty}$ imply convergence in $\mathcal{S}'$, thus $u = v$ as elements of $\mathcal{S}'$, and thereby as locally integrable functions. This shows the injection (6.22), with (6.23). $\square$

The theorem will be illustrated by an application:

Theorem 6.10. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ with $\hat u \in L_{2,\mathrm{loc}}(\mathbb{R}^n)$. Then one has for $s \in \mathbb{R}$,*

$$\begin{aligned}\Delta u \in H^s(\mathbb{R}^n) &\iff u \in H^{s+2}(\mathbb{R}^n)\,, \quad \textit{and} \\ \Delta u \in C^\infty_{L_2}(\mathbb{R}^n) &\iff u \in C^\infty_{L_2}(\mathbb{R}^n)\,.\end{aligned} \tag{6.25}$$

Here

$$\bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^n) = C^\infty_{L_2}(\mathbb{R}^n). \tag{6.26}$$

Proof. We start by showing the first line in (6.25). When $u \in H^{s+2}$, then $\Delta u \in H^s$, since $1 - \Delta = \Xi^2$. Conversely, when $\Delta u \in H^s$ and $\hat{u} \in L_{2,\mathrm{loc}}$, then

$$\langle \xi \rangle^s |\xi|^2 \hat{u}(\xi) \in L_2 \quad \text{and} \quad 1_{|\xi| \le 1} \hat{u} \in L_2,$$

which implies that

$$\langle \xi \rangle^{s+2} \hat{u}(\xi) \in L_2,$$

i.e., $u \in H^{s+2}$.

We now observe that

$$C^\infty_{L_2}(\mathbb{R}^n) \subset \bigcap_{s \in \mathbb{N}_0} H^s(\mathbb{R}^n) = \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^n)$$

by definition, whereas

$$\bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^n) \subset C^\infty_{L_\infty}(\mathbb{R}^n) \cap C^\infty_{L_2}(\mathbb{R}^n) \subset C^\infty_{L_2}(\mathbb{R}^n)$$

follows by the Sobolev theorem. These inclusions imply (6.26), and then the validity of the first line in (6.25) for all $s \in \mathbb{R}$ implies the second line. $\square$

Remark 6.11. Theorem 6.10 clearly shows that the Sobolev spaces are very well suited to describe the regularity of solutions of $-\Delta u = f$. The same *cannot* be said of the spaces of continuously differentiable functions, for here we have $u \in C^2(\mathbb{R}^n) \implies \Delta u \in C^0(\mathbb{R}^n)$ without the converse implication being true. An example in dimension $n = 3$ (found in N. M. Günther [G57] page 82 ff.) is the function

$$f(x) = \begin{cases} \dfrac{1}{\log|x|}\Big(\dfrac{3x_1^2}{|x|^2} - 1\Big)\chi(x) & \text{for } x \neq 0\ , \\ 0 & \text{for } x = 0\ , \end{cases}$$

which is continuous with compact support, and is such that $u = \dfrac{1}{4\pi}\dfrac{1}{|x|} * f$ is in $C^1(\mathbb{R}^3) \setminus C^2(\mathbb{R}^3)$ and solves $-\Delta u = f$ in the distribution sense. (Here $u \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$, cf. Theorem 6.25 later.)

There is another type of (Banach) spaces which is closer to the C^k spaces than the Sobolev spaces and work well in the study of Δ, namely the Hölder spaces $C^{k,\sigma}$ with $\sigma \in]0,1[$, where

$$C^{k,\sigma}(\Omega) = \{\, u \in C^k(\Omega) \mid |D^\alpha u(x) - D^\alpha u(y)| \le C|x-y|^\sigma \ \text{ for } \ |\alpha| \le k \,\}\ ,$$

cf. also Exercise 4.17. Here one finds that $\Delta u \in C^{k,\sigma} \iff u \in C^{k+2,\sigma}$, at least locally. These spaces are useful also in studies of nonlinear problems (but are on the other hand not very easy to handle in connection with Fourier transformation). Elliptic differential equations in $C^{k,\sigma}$ spaces are treated for example in the books of R. Courant and D. Hilbert [CH63], D. Gilbarg and N. Trudinger [GT77]; the key word is "Schauder estimates".

The Sobolev theorem holds also for nice subsets of $\mathbb{R}^n$.

Corollary 6.12. *When $\Omega = \mathbb{R}^n_+$, or Ω is bounded, smooth and open, then one has for integer m and $l \geq 0$, with $l > m + n/2$:*

$$H^l(\Omega) \subset C^m_{L_\infty}(\overline{\Omega}), \text{ with } \sup\{\, |D^\alpha u(x)| \mid x \in \overline{\Omega}, |\alpha| \leq m \,\} \leq C_l \|u\|_l. \quad (6.27)$$

Proof. Here we use Theorem 4.12, which shows the existence of a continuous map $E\colon H^l(\Omega) \to H^l(\mathbb{R}^n)$ such that $u = (Eu)\big|_\Omega$. When $u \in H^l(\Omega)$, Eu is in $H^l(\mathbb{R}^n)$ and hence in $C^m_{L_\infty}(\mathbb{R}^n)$ by Theorem 6.9; then $u = (Eu)\big|_\Omega \in C^m_{L_\infty}(\overline{\Omega})$, and

$$\begin{aligned}\sup\big\{\, |D^\alpha u(x)| \mid x \in \overline{\Omega}, |\alpha| \leq m \big\} &\leq \sup\{|D^\alpha Eu(x)| \mid x \in \mathbb{R}^n, |\alpha| \leq m\,\} \\ &\leq C_l \|Eu\|_{l,\wedge} \leq C'_l \|u\|_{H^l(\Omega)}\,. \qquad \square\end{aligned}$$

6.3. Dualities between Sobolev spaces. The Structure Theorem.

We shall now investigate the Sobolev spaces with negative exponents. The main point is that they will be viewed as *dual spaces* of the Sobolev spaces with positive exponent! For the $L_{2,s}$ spaces, this is very natural, and the corresponding interpretation is obtained for the H^s spaces by application of F^{-1}. We here use the sesquilinear duality; i.e., the dual space is the space of continuous, *conjugate-linear* — also called *antilinear* — functionals.

Theorem 6.13. *Let $s \in \mathbb{R}$.*

1° *$L_{2,-s}$ can be identified with the dual space of $L_{2,s}$ by an isometric isomorphism, such that the function $u \in L_{2,-s}$ is identified with the functional $\Lambda \in (L_{2,s})^*$ precisely when*

$$\int u(\xi)\overline{\varphi}(\xi)\, d\xi = \Lambda(\varphi) \quad \text{for } \varphi \in \mathcal{S}\,. \quad (6.28)$$

2° *$H^{-s}(\mathbb{R}^n)$ can be identified with the dual space of $H^s(\mathbb{R}^n)$, by an isometric isomorphism, such that the distribution $u \in H^{-s}(\mathbb{R}^n)$ is identified with the functional $\Lambda \in (H^s(\mathbb{R}^n))^*$ precisely when*

$$\langle u, \overline{\varphi}\rangle = \Lambda(\varphi) \quad \text{for } \varphi \in \mathcal{S}\,. \quad (6.29)$$

Proof. 1°. When $u \in L_{2,-s}$, it defines a continuous antilinear functional Λ_u on $L_{2,s}$ by

$$\Lambda_u(v) = \int u(\xi)\overline{v}(\xi)\, d\xi \quad \text{for } v \in L_{2,s}\,,$$

since

$$|\Lambda_u(v)| = \Big|\int \langle\xi\rangle^{-s} u(\xi) \langle\xi\rangle^{s} \overline{v}(\xi)\, d\xi\Big| \leq \|u\|_{L_{2,-s}} \|v\|_{L_{2,s}}, \quad (6.30)$$

by the Cauchy-Schwarz inequality. Note here that

$$\begin{aligned}\|\Lambda_u\|_{L_{2,s}^*} &= \sup_{v\in L_{2,s}\setminus\{0\}} \frac{|\Lambda_u(v)|}{\|v\|_{L_{2,s}}} = \sup_{\langle\xi\rangle^s v\in L_2\setminus\{0\}} \frac{|\int\langle\xi\rangle^{-s}u(\xi)\langle\xi\rangle^s\overline{v}(\xi)\,d\xi|}{\|\langle\xi\rangle^s v\|_{L_2}}\\ &= \|\langle\xi\rangle^{-s}u\|_{L_2} = \|u\|_{L_{2,-s}},\end{aligned}$$

by the sharpness of the Cauchy-Scwartz inequality, so the mapping $u\mapsto\Lambda_u$ is an isometry, in particular injective. To see that it is an isometric isomorphism as stated in the theorem, we then just have to show its surjectiveness. So let Λ be given as a continuous functional on $L_{2,s}$; then we get by composition with the isometry $M_{\langle\xi\rangle^{-s}} : L_2 \to L_{2,s}$ a continuous functional

$$\Lambda' = \Lambda M_{\langle\xi\rangle^{-s}}$$

on L_2. Because of the identification of L_2 with its own dual space, there exists a function $f\in L_2$ such that $\Lambda'(v) = (f,v)$ for all $v\in L_2$. Then we have for $v\in L_{2,s}$,

$$\Lambda(v) = \Lambda(\langle\xi\rangle^{-s}\langle\xi\rangle^s v) = \Lambda'(\langle\xi\rangle^s v) = (f,\langle\xi\rangle^s v) = \int f(\xi)\langle\xi\rangle^s\overline{v}(\xi)d\xi\ ,$$

which shows that $\Lambda=\Lambda_u$ with $u=\langle\xi\rangle^s f\in L_{2,-s}$. Since $\mathcal{S}$ is dense in $L_{2,s}$, this identification of u with Λ is determined already by (6.28).

2°. The proof of this part now just consists of a “translation” of all the consideration under 1°, by use of F^{-1} and its isometry- and homeomorphism properties. □

For the duality between H^{-s} and H^s we shall use the notation

$$\underset{H^{-s}\ H^s}{\langle u,\overline{v}\rangle}\ ,\ \langle u,\overline{v}\rangle_{H^{-s},H^s} \text{ or just } \langle u,\overline{v}\rangle\ ,\ \text{for } u\in H^{-s},\ v\in H^s, \tag{6.31}$$

since it coincides with the scalar product in $L_2(\mathbb{R}^n)$ and with the distribution duality, when these are defined. Note that we have shown (cf. (6.30)):

$$|\langle u,\overline{v}\rangle| \le \|u\|_{-s,\wedge}\|v\|_{s,\wedge} \text{ when } u\in H^{-s}\ ,\ v\in H^s; \tag{6.32}$$

this is sometimes called the Schwartz inequality (with t) after Laurent Schwartz. Observe also (with the notation of 2°):

$$\begin{aligned}\|u\|_{-s,\wedge} &= \|\Lambda_u\|_{(H^s)^*} = \sup\Big\{\frac{|\Lambda_u(v)|}{\|v\|_{s,\wedge}} \mid v\in H^s\setminus\{0\}\Big\} \\ &= \sup\Big\{\frac{|\langle u,\overline{v}\rangle|}{\|v\|_{s,\wedge}} \mid v\in H^s\setminus\{0\}\Big\} = \sup\Big\{\frac{|\langle u,\varphi\rangle|}{\|\varphi\|_{s,\wedge}} \mid \varphi\in\mathcal{S}\setminus\{0\}\Big\}\ .\end{aligned} \tag{6.33}$$

As an example of the importance of "negative Sobolev spaces", consider the variational construction from Theorem 12.18 and its corollary, applied to the situation where $H = L_2(\mathbb{R}^n)$, $V = H^1(\mathbb{R}^n)$ and $a(u,v) = \sum_{j=1}^n (\partial_j u, \partial_j v)_{L_2} = (u,v)_1 - (u,v)_0$. The imbedding of H into V^* considered there corresponds exactly to the imbedding of $L_2(\mathbb{R}^n)$ into $H^{-1}(\mathbb{R}^n)$! The operator $\widetilde{\mathcal{A}}$ then goes from $H^1(\mathbb{R}^n)$ to $H^{-1}(\mathbb{R}^n)$ and restricts to A going from $D(A)$ to $L_2(\mathbb{R}^n)$. We know from the end of Section 4 that A acts like $-\Delta$ in the distribution sense, with domain $D(A) = H^1 \cap D(A_{\max})$ dense in H^1 (and clearly, $H^2 \subset D(A)$). Then $\widetilde{\mathcal{A}}$, extending A to a mapping from H^1 to H^{-1}, likewise acts like $-\Delta$ in the distribution sense. Finally we have from Theorem 6.10 that $D(A) \subset H^2$, so in fact, $D(A) = H^2$. To sum up, we have inclusions

$$D(A) = H^2 \subset V = H^1 \subset H = L_2 \subset V^* = H^{-1},$$

for the variational realization of $-\Delta$ on the full space $\mathbb{R}^n$.

Having the full scale of Sobolev spaces available, we can apply differential operators (with smooth coefficients) without limitations:

Lemma 6.14. *Let $s \in \mathbb{R}$.*

1° *For each $\alpha \in \mathbb{N}_0^n$, D^α is a continuous operator from $H^s(\mathbb{R}^n)$ into $H^{s-|\alpha|}(\mathbb{R}^n)$.*

2° *For each $f \in \mathcal{S}(\mathbb{R}^n)$, the multiplication by f is a continuous operator from $H^s(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$.*

Proof. 1°. That D^α maps $H^s(\mathbb{R}^n)$ continuously into $H^{s-|\alpha|}(\mathbb{R}^n)$ is seen from the fact that since $|\xi^\alpha| \le \langle\xi\rangle^{|\alpha|}$ (cf. (5.2)),

$$\begin{aligned}\|D^\alpha u\|_{s-|\alpha|,\wedge} &= (2\pi)^{-n/2}\|\langle\xi\rangle^{s-|\alpha|}\xi^\alpha \hat{u}(\xi)\|_0 \\ &\le (2\pi)^{-n/2}\|\langle\xi\rangle^s \hat{u}(\xi)\|_0 = \|u\|_{s,\wedge} \text{ for } u \in H^s(\mathbb{R}^n)\ .\end{aligned}$$

2°. Let us first consider integer values of s. Let $s \in \mathbb{N}_0$, then it follows immediately from the Leibniz formula that one has for a suitable constant c'_s:

$$\|fu\|_s \le c'_s \sup\{\,|D^\alpha f(x)| \mid x \in \mathbb{R}^n\,,\ |\alpha| \le s\,\}\,\|u\|_s\ , \tag{6.34}$$

which shows the continuity in this case. For $u \in H^{-s}(\mathbb{R}^n)$, we now use Theorem 6.13, (6.14) and (6.34):

$$\begin{aligned}|\langle fu, \varphi\rangle| &= |\langle u, f\varphi\rangle| \le \|u\|_{-s,\wedge}\|f\varphi\|_{s,\wedge} \le \|u\|_{-s,\wedge} C_s^{\frac12}\|f\varphi\|_s \\ &\le \|u\|_{-s,\wedge} C_s^{\frac12} c'_s \sup\{\,|D^\alpha f(x)| \mid x \in \mathbb{R}^n\,,\ |\alpha| \le s\,\}\,\|\varphi\|_s \\ &\le \|u\|_{-s,\wedge} C_s^{\frac12} c'_s \sup\{\,|D^\alpha f(x)| \mid x \in \mathbb{R}^n\,,\ |\alpha| \le s\,\}\,\|\varphi\|_{s,\wedge} \\ &= C\|u\|_{-s,\wedge}\|\varphi\|_{s,\wedge},\end{aligned}$$

whereby $fu \in H^{-s}(\mathbb{R}^n)$ with

$$\|fu\|_{-s,\wedge} \le C\|u\|_{-s,\wedge}$$

(cf. (6.33)). This shows the continuity in H^{-s} for s integer ≥ 0.

When s is noninteger, the proof is more technical. We can appeal to convolution (5.34) and use the Peetre inequality (6.13) in the following way: Let $u \in H^s$. Since $f \in \mathcal{S}$, there are inequalities

$$|\hat f(\xi)| \le C'_N \langle\xi\rangle^{-N}$$

for all $N \in \mathbb{R}$. Then we get that

$$\begin{aligned}
\|fu\|^2_{s,\wedge} &= (2\pi)^{-n} \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} |\widehat{fu}(\xi)|^2 \, d\xi \\
&\le (2\pi)^{-3n} \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \Big(\int_{\mathbb{R}^n} |\hat f(\xi-\eta)\hat u(\eta)| \, d\eta \Big)^2 \, d\xi \\
&\le (2\pi)^{-3n} (C'_N)^2 \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \langle\xi\rangle^{s} \langle\xi-\eta\rangle^{-N} |\hat u(\eta)| \, d\eta \Big)^2 \, d\xi \\
&\le (2\pi)^{-3n} (C'_N)^2 c_s \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \langle\xi-\eta\rangle^{|s|-N} \langle\eta\rangle^{s} |\hat u(\eta)| \, d\eta \Big)^2 \, d\xi,
\end{aligned}$$

where we choose N so large that $\langle\zeta\rangle^{|s|-N}$ is integrable, and apply the Cauchy-Schwarz inequality:

$$\begin{aligned}
&\le c' \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \langle\xi-\eta\rangle^{|s|-N} \, d\eta \Big) \Big(\int_{\mathbb{R}^n} \langle\xi-\eta\rangle^{|s|-N} \langle\eta\rangle^{2s} |\hat u(\eta)|^2 \, d\eta \Big) \, d\xi \\
&= c'' \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle\xi-\eta\rangle^{|s|-N} \langle\eta\rangle^{2s} |\hat u(\eta)|^2 \, d\eta \, d\xi \\
&= c'' \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle\zeta\rangle^{|s|-N} \langle\eta\rangle^{2s} |\hat u(\eta)|^2 \, d\eta \, d\zeta = c''' \|u\|^2_{s,\wedge}. \qquad \square
\end{aligned}$$

It can sometimes be useful to observe that for m integer > 0, the proof shows that

$$\|fu\|_m \le \|f\|_{L_\infty} \|u\|_m + C \sup_{|\beta| \le m-1} \|D^\beta f\|_{L_\infty} \|u\|_{m-1}. \tag{6.35}$$

The spaces H^{-s}, $s > 0$, contain more proper distributions, the larger s is taken.

Example 6.14a. The δ-distribution satisfies:

$$\delta \in H^{-s}(\mathbb{R}^n) \iff s > n/2, \tag{6.36}$$

and its α'th derivative $D^\alpha \delta$ is in H^{-s} precisely when $s > |\alpha| + n/2$. This follows from the fact that $\mathcal{F}(D^\alpha \delta) = \xi^\alpha$ (cf. (5.37)) is in $L_{2,-s}$ if and only if $|\alpha| - s < -n/2$.

For more general distributions we have:

Theorem 6.15. *Let $u \in \mathcal{E}'(\Omega)$, identified with a subspace of $\mathcal{E}'(\mathbb{R}^n)$ by extension by 0, and let N be such that for some C_N,*

$$|\langle u, \varphi \rangle| \le C_N \sup \{ |D^\alpha \varphi(x)| \mid x \in \mathbb{R}^n \, , \, |\alpha| \le N \} \, , \tag{6.37}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$; so u is of order N. Then $u \in H^{-s}(\mathbb{R}^n)$ for $s > N + n/2$.

Proof. We have by Theorem 3.12 and its proof that when $u \in \mathcal{E}'(\mathbb{R}^n)$, then u is of some finite order N, for which there exists a constant C_N such that (6.37) holds (regardless of the location of the support of φ), cf. (3.34). By (6.23) we now get that

$$|\langle u, \overline{\varphi} \rangle| \le C'_s \|\varphi\|_{s,\wedge} \;\text{ for }\; s > N + n/2 \, , \text{ when } u \in C_0^\infty(\mathbb{R}^n), \tag{6.38}$$

whereby $u \in H^{-s}$ according to Theorem 6.13 (since $C_0^\infty(\mathbb{R}^n)$ is dense in H^s, cf. Lemma 6.8). $\square$

Note that both for $\mathcal{E}'$ and for the H^s spaces, the Fourier transformed space consists of (locally square integrable) *functions*. For $\mathcal{E}'$ this follows from Remark 5.18 or Theorem 6.15; for the H^s spaces it is seen from the definition. Then Theorem 6.10 can be applied directly to the elements of $\mathcal{E}'(\mathbb{R}^n)$, and more generally to the elements of $\bigcup_{t \in \mathbb{R}} H^t(\mathbb{R}^n)$.

We can now finally give an easy proof of the structure theorem that was announced in Chapter 3 (around formula (3.17)).

Theorem 6.16. (The Structure Theorem.) *Let Ω be open $\subset \mathbb{R}^n$ and let $u \in \mathcal{E}'(\Omega)$. Let V be an open neighborhood of* $\operatorname{supp} u$ *with $\overline{V}$ compact $\subset \Omega$, and let M be an integer $> (N+n)/2$, where N is the order of u (as in Theorem 6.15). There exists a system of continuous functions f_α with support in V for $|\alpha| \le 2M$ such that*

$$u = \sum_{|\alpha| \le 2M} D^\alpha f_\alpha \, . \tag{6.39}$$

Moreover, there exists a continuous function g on $\mathbb{R}^n$ such that $u = (1-\Delta)^M g$ (and one can obtain that $g \in H^{n/2+1-\varepsilon}(\mathbb{R}^n)$ for any $\varepsilon > 0$).

Proof. We have according to Theorem 6.15 that $u \in H^{-s}$ for $s = N+n/2+\varepsilon$ (for any $\varepsilon \in]0,1[$). Now $H^{-s} = \Xi^t H^{t-s}$ for all t. Taking $t = 2M > N + n$, we have that $t - s \geq N + n + 1 - N - n/2 - \varepsilon = n/2 + 1 - \varepsilon$, so that $H^{t-s} \subset C^0_{L_\infty}(\mathbb{R}^n)$, by the Sobolev theorem. Hence

$$H^{-s} = \Xi^{2M} H^{2M-s} = (1-\Delta)^M H^{t-s} \subset (1-\Delta)^M C^0_{L_\infty}(\mathbb{R}^n) \ ,$$

and then (by the bijectiveness of $I - \Delta = \Xi^2$) there exists a $g \in H^{t-s} \subset H^{n/2+1-\varepsilon} \subset C^0_{L_\infty}$ such that

$$u = (1-\Delta)^M g = \sum_{|\alpha|\leq M} C_{M,\alpha} D^{2\alpha} g;$$

in the last step we used (5.2). Now let $\eta \in C_0^\infty(V)$ with $\eta = 1$ on a neighborhood of $\operatorname{supp} u$. Then $u = \eta u$, so we have for any $\varphi \in C_0^\infty(\Omega)$:

$$\begin{aligned}\langle u, \varphi\rangle &= \langle u, \eta\varphi\rangle = \langle \sum_{|\alpha|\leq M} C_{M,\alpha} D^{2\alpha} g, \eta\varphi\rangle = \sum_{|\alpha|\leq M} C_{M,\alpha} \langle g, (-D)^{2\alpha}(\eta\varphi)\rangle \\ &= \sum_{|\alpha|\leq M} \sum_{\beta\leq 2\alpha} C_{M,\alpha} C_{2\alpha,\beta} \langle g, (-D)^{2\alpha-\beta}\eta\, (-D)^\beta \varphi\rangle, \\ &= \sum_{|\alpha|\leq M, \beta\leq 2\alpha} C_{M,\alpha} C_{2\alpha,\beta} \langle D^\beta[(-D)^{2\alpha-\beta}\eta\, g], \varphi\rangle,\end{aligned}$$

by Leibniz' formula. This can be rearranged in the form $\langle \sum_{|\beta|\leq 2M} D^\beta f_\beta, \varphi\rangle$ with f_β continuous and supported in V since η and its derivatives are supported in V, and this shows (6.39). $\square$

As an immediate consequence we get the following result for arbitrary distributions:

Corollary 6.17. *Let Ω be open $\subset \mathbb{R}^n$, let $u \in \mathcal{D}'(\Omega)$, and let Ω' be an open subset of Ω with $\overline{\Omega'}$ compact $\subset \Omega$. Let $\zeta \in C_0^\infty(\Omega)$ with $\zeta = 1$ on Ω', and let N be the order of $\zeta u \in \mathcal{E}'(\Omega)$ (as in Theorem 6.15). When V is a neighborhood of* $\operatorname{supp}\zeta$ *in Ω and M is an integer $> (N+n)/2$, then there exists a system of continuous functions with compact support in V such that $\zeta u = \sum_{|\alpha|\leq 2M} D^\alpha f_\alpha$; in particular,*

$$u = \sum_{|\alpha|\leq 2M} D^\alpha f_\alpha \quad \text{on} \quad \Omega' \ . \tag{6.40}$$

Based on this corollary and a partition of unity as in Theorem 2.16 one can for any $u \in \mathcal{D}'(\Omega)$ construct a system $(g_\alpha)_{\alpha\in\mathbb{N}_0^n}$ of continuous functions g_α on Ω, which is *locally finite* (only finitely many functions are different from 0 on each compact subset of Ω), such that $u = \sum_{\alpha\in\mathbb{N}_0^n} D^\alpha g_\alpha$.

6.4. Regularity theory for elliptic differential equations.

When $P(x,D)$ is an m'th order differential operator (6.5) with symbol (6.6), the part of order m is called the *principal part*:

$$P_m(x,D) = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha \ , \tag{6.41}$$

and the associated symbol is called the *principal symbol*

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha \ ; \tag{6.42}$$

the latter is also sometimes called the *characteristic polynomial*. The operator $P(x,D)$ is said to be *elliptic* on M ($M \subset \mathbb{R}^n$), when

$$p_m(x,\xi) \neq 0 \ \text{ for } \ \xi \in \mathbb{R}^n \setminus \{0\}\,, \ \text{ all } \ x \in M\ . \tag{6.43}$$

(This extends the definition given in (5.39)ff. for constant-coefficient operators.) We recall that the Laplace operator, whose symbol and principal symbol equal $-|\xi|^2$, is elliptic on $\mathbb{R}^n$.

The argumentation in Theorem 6.10 can easily be extended to general elliptic operators with constant coefficients a_α:

Theorem 6.18. 1° *Let $P(D) = \operatorname{Op}(p(\xi))$, where $p(\xi) \in \mathcal{O}_M$ and there exist $m \in \mathbb{R}$, $c > 0$ and $r \geq 0$ such that*

$$|p(\xi)| \geq c\langle\xi\rangle^m \text{ for } |\xi| \geq r. \tag{6.44}$$

For $s \in \mathbb{R}$ one then has: When $u \in \mathcal{S}'$ with $\hat{u} \in L_{2,\mathrm{loc}}$, then

$$P(D)u \in H^s(\mathbb{R}^n) \implies u \in H^{s+m}(\mathbb{R}^n). \tag{6.45}$$

2° *In particular, this holds when $P(D)$ is an elliptic differential operator of order $m \in \mathbb{N}$ with constant coefficients.*

Proof. 1°. That $P(D)u \in H^s(\mathbb{R}^n)$ means that $\langle\xi\rangle^s p(\xi)\hat{u}(\xi) \in L_2(\mathbb{R}^n)$. Therefore we have when $\hat{u}(\xi) \in L_{2,\mathrm{loc}}(\mathbb{R}^n)$, using (6.44):

$$1_{\{|\xi|\geq r\}}\langle\xi\rangle^{s+m}\hat{u}(\xi) \in L_2(\mathbb{R}^n), \quad 1_{\{|\xi|\leq r\}}\hat{u}(\xi) \in L_2(\mathbb{R}^n),$$

and hence that $\langle\xi\rangle^{s+m}\hat{u}(\xi) \in L_2(\mathbb{R}^n)$, i.e., $u \in H^{s+m}(\mathbb{R}^n)$.

2°. Now let $p(\xi)$ be the symbol of an elliptic differential operator of order $m \in \mathbb{N}$, i.e., $p(\xi)$ is a polynomial of degree m, where the principal part

$p_m(\xi) \neq 0$ for all $\xi \neq 0$. Then $|p_m(\xi)|$ has a positive minimum on the unit sphere $\{ \xi \in \mathbb{R}^n \mid |\xi| = 1 \}$,

$$c_0 = \min\{ |p_m(\xi)| \mid |\xi| = 1 \} > 0,$$

and because of the homogeneity,

$$|p_m(\xi)| \geq c_0 |\xi|^m \text{ for all } \xi \in \mathbb{R}^n.$$

Since $p(\xi) - p_m(\xi)$ is of degree $\leq m - 1$,

$$\frac{|p(\xi) - p_m(\xi)|}{|\xi|^m} \to 0 \text{ for } |\xi| \to \infty.$$

Choose $r \geq 1$ so that this fraction is $\leq c_0/2$ for $|\xi| \geq r$. Since $\langle \xi \rangle^m \leq 2^{m/2} |\xi|^m$ for $|\xi| \geq 1$, we obtain that

$$\begin{aligned} |p(\xi)| &\geq |p_m(\xi)| - |p(\xi) - p_m(\xi)| \geq \frac{c_0}{2} |\xi|^m \\ &\geq \frac{c_0}{2^{1+m/2}} \langle \xi \rangle^m, \text{ for } |\xi| \geq r. \end{aligned}$$

This shows (6.44). □

Corollary 6.19. *When $P(D)$ is an elliptic differential operator of order m with constant coefficients, one has for each $s \in \mathbb{R}$, when $u \in \mathcal{S}'$ with $\hat{u} \in L_{2,\mathrm{loc}}$:*

$$P(D)u \in H^s(\mathbb{R}^n) \iff u \in H^{s+m}(\mathbb{R}^n).$$

Proof. The implication $\Longleftarrow$ is an immediate consequence of Lemma 6.14, while $\Longrightarrow$ follows from Theorem 6.18. □

We have furthermore for the minimal realization, in the case of constant coefficients:

Theorem 6.20. *Let $P(D)$ be elliptic of order m on $\mathbb{R}^n$, with constant coefficients. Let Ω be an open subset of $\mathbb{R}^n$. The minimal realization $P_{\min}$ of $P(D)$ in $L_2(\Omega)$ satisfies*

$$D(P_{\min}) = H_0^m(\Omega) \,. \tag{6.46}$$

When $\Omega = \mathbb{R}^n$, $D(P_{\min}) = D(P_{\max}) = H^m(\mathbb{R}^n)$, with equivalent norms.

Proof. For $\Omega = \mathbb{R}^n$ we have already shown in Theorem 6.3 that $D(P_{\min}) = D(P_{\max})$, and the identification of this set with $H^m(\mathbb{R}^n)$ follows from Corollary 6.19. That the graph-norm and the H^m-norm are equivalent, follows e.g. when we note that by the Parseval-Plancherel theorem,

$$\|u\|_0^2 + \|Pu\|_0^2 = (2\pi)^{-n}(\|\hat{u}\|_0^2 + \|\widehat{Pu}\|_0^2) = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |p(\xi)|^2)|\hat{u}(\xi)|^2 \, d\xi \ ,$$

and combine this with the estimates in Theorem 6.18, implying that there are positive constants c' and C' so that

$$c'\langle\xi\rangle^{2m} \leq 1 + |p(\xi)|^2 \leq C'\langle\xi\rangle^{2m} \ , \text{ for } \xi \in \mathbb{R}^n.$$

(One could also deduce the equivalence of norms from the easy fact that the graph norm is dominated by the H^m-norm, and both norms define a Hilbert space (since $P(D)_{\max}$ is closed). For then the identity mapping $i \colon H^m(\mathbb{R}^n) \to D(P(D)_{\max})$ is both continuous and surjective, hence must be a homeomorphism by the open mapping principle (Theorem B.14).)

For the assertion concerning the realization in $L_2(\Omega)$ we now observe that the closure of $C_0^\infty(\Omega)$ in graph norm and in H^m-norm must be identical; this shows (6.46). □

For differential operators with variable coefficients it takes some further efforts to show regularity of solutions of elliptic differential equations. We shall just give a relatively easy proof in the case where the principal part has constant coefficients.

Here we need *locally defined* Sobolev spaces.

Definition 6.21. *Let $s \in \mathbb{R}$, and let Ω be open $\subset \mathbb{R}^n$. The space $H^s_{\mathrm{loc}}(\Omega)$ is defined as the set of distributions $u \in \mathcal{D}'(\Omega)$ for which $\varphi u \in H^s(\mathbb{R}^n)$ for all $\varphi \in C_0^\infty(\Omega)$ (where φu as usual is understood to be extended by zero outside Ω).*

Concerning multiplication by φ, see Lemma 6.14. The lemma implies that in order to show that a distribution $u \in \mathcal{D}'(\Omega)$ belongs to $H^s_{\mathrm{loc}}(\Omega)$, it suffices to show e.g. that $\eta_l u \in H^s(\mathbb{R}^n)$ for each of the functions η_l introduced in Corollary 2.14 (for a given $\varphi \in C_0^\infty(\Omega)$ one takes l so large that $\operatorname{supp}\varphi \subset K_l$; then $\varphi u = \varphi\eta_l u$). It is also sufficient in order for $u \in \mathcal{D}'(\Omega)$ to lie in $H^s_{\mathrm{loc}}(\Omega)$ that there for any $x \in \Omega$ exists a neighborhood ω and a nonnegative test function $\psi \in C_0^\infty(\Omega)$ with $\psi = 1$ on ω such that $\psi u \in H^s(\mathbb{R}^n)$. To see this, note that for each l, K_{l+1} can be covered by a finite system of such neighborhoods $\omega_1, \dots, \omega_N$, and

$$1 \leq \psi_1(x) + \cdots + \psi_N(x) \leq N \ \text{ for } \ x \in K_{l+1} \ ,$$

so that
$$\eta_l u = \sum_{j=1}^{N} \frac{\eta_l}{\psi_1 + \cdots + \psi_N} \psi_j u \in H^s(\mathbb{R}^n) .$$

The space $H^s_{\mathrm{loc}}(\Omega)$ is a Fréchet space with the topology defined by the seminorms
$$p_l(u) = \|\eta_l u\|_{H^s(\mathbb{R}^n)} \ \text{ for } \ l = 1, 2, \dots . \tag{6.47}$$

Remark 6.22. For completeness we mention that $H^s_{\mathrm{loc}}(\Omega)$ has the dual space $H^{-s}_{\mathrm{comp}}(\Omega)$ (which it is itself the dual space of), in a similar way as in Theorem 6.13 (and Exercises 2.3 and 2.7). Here
$$H^t_{\mathrm{comp}}(\Omega) = \bigcup_{l=1}^{\infty} H^t_{K_l} , \tag{6.48}$$
where $H^t_{K_l}$ is the closed subspace of $H^t(\mathbb{R}^n)$ consisting of the elements with support in K_l; the space $H^t_{\mathrm{comp}}(\Omega)$ is provided with the inductive limit topology (Appendix B).

Using Lemma 6.14, we find:

Lemma 6.23. *Let $s \in \mathbb{R}$. When $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}_0^n$, then the operator $u \mapsto f D^\alpha u$ is a continuous mapping of $H^s_{\mathrm{loc}}(\Omega)$ into $H^{s-|\alpha|}_{\mathrm{loc}}(\Omega)$.*

Proof. When $u \in H^s_{\mathrm{loc}}(\Omega)$, one has for each $j = 1, \dots, n$, each $\varphi \in C_0^\infty(\Omega)$, that
$$\varphi(D_j u) = D_j(\varphi u) - (D_j \varphi) u \in H^{s-1}(\mathbb{R}^n) ,$$
since $\varphi u \in H^s(\mathbb{R}^n)$ implies $D_j(\varphi u) \in H^{s-1}(\mathbb{R}^n)$, and $D_j \varphi \in C_0^\infty(\Omega)$. Thus D_j maps the space $H^s_{\mathrm{loc}}(\Omega)$ into $H^{s-1}_{\mathrm{loc}}(\Omega)$, and it is found by iteration that D^α maps $H^s_{\mathrm{loc}}(\Omega)$ into $H^{s-|\alpha|}_{\mathrm{loc}}(\Omega)$. Since $f\varphi \in C_0^\infty(\Omega)$ when $\varphi \in C_0^\infty(\Omega)$, we see that $f D^\alpha u \in H^{s-|\alpha|}_{\mathrm{loc}}(\Omega)$. The continuity is verified in the usual way. □

Observe moreover the following obvious consequence of the Sobolev theorem:

Corollary 6.24. *For Ω open $\subset \mathbb{R}^n$ one has:*
$$\bigcap_{s \in \mathbb{R}} H^s_{\mathrm{loc}}(\Omega) = C^\infty(\Omega) . \tag{6.49}$$

Now we shall show the regularity theorem:

Theorem 6.25. *Let Ω be open $\subset \mathbb{R}^n$, and let $P = P(x,D)$ be an* **elliptic** *differential operator of order $m > 0$ on Ω, with constant coefficients in the principal part and C^∞-coefficients in the other terms. Then one has for any $s \in \mathbb{R}$, when $u \in \mathcal{D}'(\Omega)$:*

$$Pu \in H^s_{\mathrm{loc}}(\Omega) \iff u \in H^{s+m}_{\mathrm{loc}}(\Omega); \tag{6.50}$$

in particular,

$$Pu \in C^\infty(\Omega) \iff u \in C^\infty(\Omega) \,. \tag{6.51}$$

Proof. The implication $\Longleftarrow$ in (6.51) is obvious, and it follows in (6.50) from Lemma 6.23. Now let us show $\Longrightarrow$ in (6.50). It is given that P is of the form

$$P(x,D) = P_m(D) + Q(x,D) \,, \tag{6.52}$$

where $P_m(D) = \mathrm{Op}(p_m(\xi))$ is an elliptic m'th order differential operator with constant coefficients and Q is a differential operator of order $m-1$ with C^∞ coefficients.

Let u satisfy the left-hand side of (6.50), and let $x \in \Omega$. According to the descriptions of $H^t_{\mathrm{loc}}(\Omega)$ we just have to show that there is a neighborhood ω of x and a function $\psi \in C_0^\infty(\Omega)$ which is 1 on ω such that $\psi u \in H^{s+m}(\mathbb{R}^n)$.

We first choose $r > 0$ such that $\underline{B}(x,r) \subset \Omega$. Let $V_j = B(x, r/j)$ for $j = 1, 2, \dots$. As in Corollary 2.14, we can for each j find a function $\psi_j \in C_0^\infty(V_j)$ with $\psi_j = 1$ on V_{j+1}. Then in particular, $\psi_j \psi_{j+1} = \psi_{j+1}$.

Since $\psi_1 u$ can be considered as a distribution on $\mathbb{R}^n$ with compact support, $\psi_1 u$ is of finite order, and there exists by Theorem 6.15 a number $M \in \mathbb{Z}$ such that $\psi_1 u \in H^{-M}(\mathbb{R}^n)$. We will show inductively that

$$\psi_{j+1} u \in H^{-M+j}(\mathbb{R}^n) \cup H^{s+m}(\mathbb{R}^n) \quad \text{for } j = 1, 2, \dots \,. \tag{6.53}$$

When j gets so large that $-M + j \geq s + m$, then $\psi_{j+1} u \in H^{s+m}(\mathbb{R}^n)$, and the desired information has been obtained, with $\omega = V_{j+2}$ and $\psi = \psi_{j+1}$.

The induction step goes as follows: Let it be given that

$$\psi_j u \in H^{-M+j-1} \cup H^{s+m} \,, \text{ and } Pu \in H^s_{\mathrm{loc}}(\Omega) \,. \tag{6.54}$$

Now we write

$$Pu = P_m(D)u + Q(x,D)u \,,$$

and observe that in view of the Leibniz formula, we have for each l:

$$\psi_l P u = P_m(D)\psi_l u + S_l(x,D)u \,, \tag{6.55}$$

where $S_l(x, D) = (\psi_l P_m - P_m \psi_l) + \psi_l Q$ is a differential operator *of order* $m-1$, which has *coefficients supported in* $\operatorname{supp}\psi_l \subset V_l$. We then get

$$P_m \psi_{j+1} u = \psi_{j+1} P u - S_{j+1} u = \psi_{j+1} P u - S_{j+1} \psi_j u \;, \tag{6.56}$$

since ψ_j is 1 on V_{j+1}, which contains the supports of ψ_{j+1} and the coefficients of S_{j+1}. According to the given information (6.54) and Lemma 6.23,

$$S_{j+1}\psi_j u \in H^{-M+j-1-m+1} \cup H^{s+m-m+1} = H^{-M+j-m} \cup H^{s+1} \;,$$

and $\psi_{j+1} P u \in H^s$, so that, all taken together,

$$P_m \psi_{j+1} u \in H^{-M+j-m} \cup H^s \;. \tag{6.57}$$

Now we can apply Corollary 6.19 to P_m, which allows us to conclude that

$$\psi_{j+1} u \in H^{-M+j} \cup H^{s+m} \;.$$

This shows that (6.54) implies (6.53), and the induction works as claimed.
The last implication in (6.51) now follows from Corollary 6.24. □

An argumentation as in the above proof is often called a "bootstrap"-argument, which relates the method to one of the adventures of Münchhausen, where he (on horseback) was stuck in a swamp and dragged himself and the horse up step by step by pulling at his bootstraps.

We get in particular from the case $s = 0$:

Corollary 6.26. *When P is an elliptic differential operator on Ω of order m, with constant coefficients in the principal symbol, then*

$$D(P_{\max}) \subset H^m_{\mathrm{loc}}(\Omega) \;. \tag{6.58}$$

The corollary implies that the realizations T and T_1 of $-\Delta$ introduced in Theorems 4.27 and 4.28 have domains contained in $H^2_{\mathrm{loc}}(\Omega)$; the so-called "interior regularity". There remains the question of "regularity up to the boundary", which can be shown for nice domains by a larger effort.

The theorem and its corollary can also be shown for elliptic operators with all coefficients variable. Classical proofs in positive integer-order Sobolev spaces use approximation of u by difference quotients (and allow some relaxation of the smoothness assumptions on the coefficients, depending on how high a regularity one wants to show). There is also an elegant modern proof that involves construction of an approximate inverse operator (called a parametrix) by the help of pseudodifferential operator theory. This is taken up in Chapter 7, see Corollary 7.12.

One finds in general that $D(P_{\max})$ is *not* contained in $H^m(\Omega)$ when $\Omega \neq \mathbb{R}^n$ (unless the dimension n is equal to 1); see Exercises 4.5 and 6.2 for examples.

Besides the general *regularity* question for solutions of elliptic differential equations treated above, the question of *existence* of solutions can be conveniently discussed in the framework of Sobolev spaces and Fourier integrals. There is a nice introduction to partial differential equations building on distribution theory in F. Treves [T75]. The books of L. Hörmander [H83], [H85] (vol. I–IV) can be recommended for those who want a much deeper knowledge of the modern theory of linear differential operators. Let us also mention the books of J.-L. Lions and E. Magenes [LM68] on elliptic and parabolic boundary value problems, the book of D. Gilbarg and N. Trudinger [GT77] on linear and nonlinear elliptic problems in general spaces, and the book of L. C. Evans [E98] on PDE in general; the latter starts from scratch and uses only distribution theory in disguise (speaking instead of weak solvability), and has a large section on nonlinear problems.

Remark 6.27. The theory of elliptic problems has further developments in several directions. Let us point to the following two:

1° *The Schrödinger operator.* Hereby is usually meant a realization of the differential operator $P_V = -\Delta + V$ on $\mathbb{R}^n$, where V is a multiplication operator (by a function $V(x)$ called the potential function). As we have seen (for $V = 0$), $P_0\big|_{C_0^\infty(\mathbb{R}^n)}$ is essentially selfadjoint in $L_2(\mathbb{R}^n)$ (Corollary 6.5). It is important to define classes of potentials V for which P_V with domain $C_0^\infty(\mathbb{R}^n)$ is essentially selfadjoint too, and to describe numerical ranges, spectra and other properties of these operators. The operators enter in quantum mechanics and in particular in *scattering theory*, where one investigates the connection between $\exp(itP_0)$ and $\exp(itP_V)$ (defined by functional analysis).

2° *Boundary value problems* in dimension $n \geq 2$. One here considers the Laplace operator and other elliptic operators on smooth open subsets Ω of $\mathbb{R}^n$. The statements in Chapter 4 give a beginning of this theory.

One can show that the boundary mapping (also called a trace operator)

$$\gamma_j : u \mapsto \Big(\frac{\partial}{\partial n}\Big)^j u\big|_{\partial\Omega} \,,$$

defined on $C^m(\overline{\Omega})$, can be extended to a continuous map from the Sobolev space $H^m(\Omega)$ to the space $H^{m-j-\frac12}(\partial\Omega)$ when $m > j$; here $H^s(\partial\Omega)$ is defined as in Section 6.2 when $\partial\Omega = \mathbb{R}^{n-1}$, and is more generally defined by the help of local coordinates. Theorems 4.17 and 4.25 have in the case $n \geq 2$ the generalization that $H_0^m(\Omega)$ consists of those H^m-functions u for which

$\gamma_j u = 0$ for $j = 0, 1, \dots, m-1$. Conditions on these boundary values can be given a sense in $H^m(\Omega)$. As briefly indicated at the end of Section 4.4 for second-order operators, one can develop a theory of selfadjoint or variational realizations of elliptic operators on Ω determined by boundary conditions. More on this in Chapters 9 for a constant-coefficient case, and in Chapters 7 and 11 for variable-coefficient cases.

For a second-order elliptic operator A we have from Corollary 6.26 that the domains of its realizations are contained in $H^2_{\mathrm{loc}}(\Omega)$. Under special hypotheses concerning the boundary condition and the smoothness of Ω, one can show with a greater effort that the domains are in fact contained in $H^2(\Omega)$; this belongs to the regularity theory for boundary value problems. A particular case is treated in Chapter 9; a technique for general cases is developed in Chapters 10 and 11.

Having such realizations available, one can furthermore discuss evolution equations with a time-parameter:

$$\begin{gathered} \partial_t u(x,t) + Au(x,t) = f(x,t) \text{ for } t > 0, \\ u(x,0) = g(x) \end{gathered}$$

(with boundary conditions); here the semiboundedness properties of variational operators allow a construction of solutions by use of the semigroup theory established in functional analysis (more about this e.g. in books of K. Yoshida [Y68] and A. Friedman [F69]). Semigroups are in the present book taken up in Chapter 14.

Exercises for Chapter 6, miscellaneous exercises.

6.1. Show that when $u \in \mathcal{S}'$ with $\hat{u} \in L_{2,\mathrm{loc}}(\mathbb{R}^n)$, then:

$$u \in H^s(\mathbb{R}^n) \iff \Delta^2 u \in H^{s-4}(\mathbb{R}^n).$$

(Δ^2 is called the biharmonic operator.)

6.2. For $\varphi(x') \in \mathcal{S}(\mathbb{R}^{n-1})$, we can define the function (with notation as in (1.13)–(1.14))

$$u_\varphi(x', x_n) = \mathcal{F}^{-1}_{\xi' \to x'}(\hat{\varphi}(\xi')e^{-\langle \xi' \rangle x_n}) \text{ for } x = (x', x_n) \in \mathbb{R}^n_+,$$

by use of Fourier transformation in the x'-variable only.

(a) Show that $(\langle \xi' \rangle^2 - \partial^2_{x_n})(\hat{\varphi}(\xi')e^{-\langle \xi' \rangle x_n}) = 0$, and hence that $(I - \Delta)u_\varphi = 0$ on $\mathbb{R}^n_+$.

(b) Show that if a sequence φ_k in $\mathcal{S}(\mathbb{R}^{n-1})$ converges in $L_2(\mathbb{R}^{n-1})$ to a function ψ, then u_{φ_k} is a Cauchy sequence in $L_2(\mathbb{R}^n_+)$. (Calculate the norm of $u_{\varphi_k} - u_{\varphi_l}$ by use of the Parseval-Plancherel theorem in the x'-variable.)

(c) Denoting the limit of u_{φ_k} in $L_2(\mathbb{R}^n_+)$ by v, show that v is in the maximal domain for $I - \Delta$ (and for Δ) on $\Omega = \mathbb{R}^n_+$.

(*Comment.* One can show that ψ is the boundary value of v in a general sense, consistent with that of Theorem 4.24. Then if $v \in H^2(\mathbb{R}^n_+)$, ψ must be in $H^1(\mathbb{R}^{n-1})$, cf. Exercise 4.21. So if ψ is taken $\notin H^1(\mathbb{R}^{n-1})$ then $v \notin H^2(\mathbb{R}^n_+)$, and we have an example of a function in the maximal domain which is not in $H^2(\mathbb{R}^n_+)$. The tools for a complete clarification of these phenomena are given in Chapter 9.)

6.3. (a) Show that when $u \in \mathcal{E}'(\mathbb{R}^n)$ (or $\bigcup_t H^t(\mathbb{R}^n)$) and $\psi \in C_0^\infty(\mathbb{R}^n)$, then $u * \psi \in C^\infty(\mathbb{R}^n)$. (One can use (5.34).)

(b) Show that when $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R}^n)$, then $u * \psi \in C^\infty(\mathbb{R}^n)$. (One can write

$$u = \eta_1 u + \sum_{j=1}^{\infty} (\eta_{j+1} - \eta_j)u \ ,$$

where η_j is as in Corollary 2.14; the sum is locally finite, i.e., finite on compact subsets. Then $u * \psi = \eta_1 u * \psi + \sum (\eta_{j+1} - \eta_j)u * \psi$ is likewise locally finite.)

6.4. Show that the heat equation for $x \in \mathbb{R}^n$,

$$\frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t) = 0 \ , \qquad t > 0,$$
$$u(x,0) = \varphi(x) \ ,$$

for each $\varphi \in \mathcal{S}(\mathbb{R}^n)$ has a solution of the form

$$u(x,t) = c_1 t^{-n/2} \int_{\mathbb{R}^n} \exp(-c_2|x-y|^2/t)\varphi(y)\,dy;$$

determine the constants c_1 and c_2.

6.5. (a) Show that $\delta * f = f$ for $f \in \mathcal{S}(\mathbb{R}^n)$.
(b) Show that the function $H(s)H(t)$ on $\mathbb{R}^2$ (with points (s,t)) satisfies

$$\frac{\partial^2}{\partial s \partial t} H(s)H(t) = \delta \quad \text{in } \mathcal{S}'(\mathbb{R}^2)\ . \tag{1}$$

(c) Show that the function $U(x,y) = H(x+y)H(x-y)$ on $\mathbb{R}^2$ (with points (x,y)) is a solution of the differential equation

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} = 2\delta \text{ in } \mathcal{S}'(\mathbb{R}^2)\ . \tag{2}$$

(A coordinate change $s = x+y$, $t = x-y$, may be useful.)
(d) Show that when $f(x,y) \in \mathcal{S}(\mathbb{R}^2)$, then $u = \frac{1}{2}U * f$ is a C^∞ solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f \text{ on } \mathbb{R}^2\ . \tag{3}$$

6.6. Let $P(D)$ be a differential operator with constant coefficients. A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a fundamental solution (or an elementary solution) of $P(D)$ if E satisfies

$$P(D)E = \delta\ .$$

(a) Show that when E is a fundamental solution of $P(D)$ in $\mathcal{S}'$, then

$$P(D)(E * f) = f \quad \text{for } f \in \mathcal{S}$$

(cf. Exercise 6.5 (a)), i.e., the equation $P(D)u = f$ has the solution $u = E * f$ for $f \in \mathcal{S}$.
(b) Find fundamental solutions of $-\Delta$ and of $-\Delta + 1$ in $\mathcal{S}'(\mathbb{R}^3)$ (cf. Section 5.4 and Exercise 5.4).
(c) Show that on $\mathbb{R}^2$, $\frac{1}{2}H(x+y)H(x-y)$ is a fundamental solution of $P(D) = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. (Cf. Exercise 6.5.)

(*Comment.* Point (c) illustrates the fact that fundamental solutions exist for much more general operators than those whose symbol is invertible, or is so outside a bounded set (e.g., elliptic operators). In fact, Ehrenpreis and Malgrange showed in the 1950's that any nontrivial constant coefficient partial differential operator has a fundamental solution, see proofs in [R74, Sect. 8.1] or [H83, Sect. 7.3]. The latter book gives many important examples.)

Miscellaneous exercises (exam problems).

The following problems have been used for examinations at Copenhagen University in courses in "Modern Analysis" since the 1980's, drawing on material from Chapters 1–6 and 12, and in some cases Appendix B.

6.7. (Concerning the definition of fundamental solution, see Exercise 6.6.)
(a) Show that when f and g are locally integrable functions on $\mathbb{R}$ with supports satisfying

$$\operatorname{supp} f \subset [a, \infty[\,, \quad \operatorname{supp} g \subset [b, \infty[\,,$$

where a and $b \in \mathbb{R}$, then $f * g$ is a locally integrable function on $\mathbb{R}$, with $\operatorname{supp}(f * g) \subset [a + b, \infty[$. (Write the convolution integral.)
(b) Let λ_1 and $\lambda_2 \in \mathbb{C}$. Find

$$E(x) = (H(x)e^{\lambda_1 x}) * (H(x)e^{\lambda_2 x}) \,,$$

where $H(x)$ is the Heaviside function ($H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \leq 0$).
(c) Let $P(t)$ be a second order polynomial with the factorization $P(t) = (t - \lambda_1)(t - \lambda_2)$. Show that

$$[(\delta' - \lambda_1 \delta) * (\delta' - \lambda_2 \delta)] * E(x) = \delta \,,$$

and thereby that E is a fundamental solution of the operator

$$P\left(\frac{d}{dx}\right) = \frac{d^2}{dx^2} - (\lambda_1 + \lambda_2)\frac{d}{dx} + \lambda_1 \lambda_2 \,.$$

(d) Find out whether there exists a fundamental solution of $P\left(\frac{d}{dx}\right)$ with support in $\,]-\infty, 0\,]$.

(e) Find the solution of the problem

$$(*) \qquad \begin{cases} (P(\frac{d}{dx})u)(x) = f(x) \quad \text{for } \ x > 0 \ , \\ u(0) = 0 \ , \\ u'(0) = 0 \ , \end{cases}$$

where f is a given continuous function on $[\,0, \infty\,[$.

6.8. Let $\mathfrak{t}$ denote the vector space of real sequences $\underline{a} = (a_k)_{k\in\mathbb{N}}$. For each $N \in \mathbb{Z}$ one defines $\ell_{2,N}$ as the subspace of $\mathfrak{t}$ consisting of sequences $\underline{a}$ for which

$$\|a\|_N = \Big(\sum_{k\in\mathbb{N}} k^{2N}|a_k|^2\Big)^{\frac{1}{2}} < \infty \ . \tag{1}$$

Let $\mathfrak{s}$ denote the set $\bigcap_{N\geq 0} \ell_{2,N}$. Moreover, write

$$\langle \underline{a}, \underline{b} \rangle = \sum_{k\in\mathbb{N}} a_k b_k \ , \tag{2}$$

when this series is convergent.

(a) Let $\ell_{2,N}$ be provided with the topology determined by the norm $\|\cdot\|_N$, and investigate which of the following properties hold for the topological vector space $\ell_{2,N}$: locally convex, locally bounded, complete, Banach space, Fréchet space.

(b) Let $\mathfrak{s}$ be provided with the topology determined by the sequence of norms $\|\cdot\|_N$, $N = 0, 1, 2, \ldots$, and investigate which of the following properties hold for the topological vector space $\mathfrak{s}$: locally convex, locally bounded, complete, Banach space, Fréchet space.

(c) Let N be an integer ≥ 0. Show that $(\ell_{2,N})^*$ can be identified with the space $\ell_{2,-N}$ in such a way that when $\Lambda \in (\ell_{2,N})^*$ is identified with the sequence $\underline{a} = (a_k)_{k\in\mathbb{N}}$, then

$$\Lambda(\underline{b}) = \langle \underline{a}, \underline{b} \rangle \tag{3}$$

for all $\underline{b} = (b_k)_{k\in\mathbb{N}}$ in $\ell_{2,N}$.

(d) Show that the dual space $\mathfrak{s}^*$ of $\mathfrak{s}$ can be identified with the space $\bigcup_{N\geq 0} \ell_{2,-N}$.

(e) Show that the operator T from $\mathfrak{t}$ into $\mathfrak{t}$ defined by:

$$T[(a_k)_{k\in\mathbb{N}}] = \Big(\frac{1}{k}a_k + k^3 a_{k+1}\Big)_{k\in\mathbb{N}} \ ,$$

defines a continuous operator from $\mathfrak{s}$ into $\mathfrak{s}$.

6.9. Let u denote the distribution on $\mathbb{R}$:

$$u = \delta_0 - \delta_1.$$

(a) Show that there exists a continuous function f on $\mathbb{R}$, for which

$$u = f'',$$

and indicate such one.

(b) Show that there exists a triple of continuous functions g_0, g_1 and g_2 on $\mathbb{R}$ with compact support such that

$$u = g_0 + g_1' + g_2'',$$

and find such a triple.

6.10. Let $a(x)$ be a real C^∞-function on $\mathbb{R}$, satisfying

$$c_1 \geq a(x) \geq c_2, \qquad |a'(x)| \leq c_3,$$

for all $x \in \mathbb{R}$, with positive constants c_1, c_2 and c_3. Let S_0 be the operator $-\frac{d}{dx}a\frac{d}{dx} : u \mapsto -(au')'$ with domain $D(S_0) = C_0^\infty(\mathbb{R})$.

(a) Show that S_0 is a symmetric operator in $L_2(\mathbb{R})$ with lower bound 0.

(b) Show that the Friedrichs extension S of S_0 is the operator $-\frac{d}{dx}a\frac{d}{dx}$ with domain $D(S) = H^2(\mathbb{R})$.

(c) Let furthermore $b(x)$ be a real C^∞-function, with $|b(x)| \leq a(x)$ for all x and $b'(x)$ bounded. Let $s_1(u, v)$ be the sesquilinear form

$$s_1(u, v) = \int_{\mathbb{R}} (a(x) + ib(x))u'(x)\,\overline{v'(x)}\,dx,$$

defined on $H^1(\mathbb{R}) \subset L_2(\mathbb{R})$. Show that $s_1(u, v)$ satisfies the conditions for application of the Lax–Milgram theorem (with $H = L_2(\mathbb{R})$ and $V = H^1(\mathbb{R})$), and determine the associated operator S_1. Show that its numerical range satisfies:

$$\nu(S_1) \subset \{\, z \in \mathbb{C} \mid |\operatorname{Im} z| \leq \operatorname{Re} z \,\}.$$

6.11. Let a and b be real numbers, and let $u(x, y)$ be a function in $L_2(\mathbb{R}^2)$ satisfying the differential equation

$$(a\frac{\partial}{\partial x} + b\frac{\partial^2}{\partial x^2})u + \frac{\partial^2}{\partial y^2}u = f, \qquad (*)$$

where f is a function in $L_2(\mathbb{R}^2)$.

(a) Show that if $b > 0$, then $u \in H^2(\mathbb{R}^2)$.

(b) Show that if $b = 0$ and $a \neq 0$, then $u \in H^1(\mathbb{R}^2)$.

From here on, consider $(*)$ for u and f in $L_{2,\mathrm{loc}}(\mathbb{R}^2)$.

(c) Let $a = 0$ and $b = -1$. Show that the function $u(x, y) = H(x - y)$ is a solution of $(*)$ with $f = 0$, for which $\partial_x u$ and $\partial_y u$ do not belong to $L_{2,\mathrm{loc}}(\mathbb{R}^2)$, and hence $u \notin H^1_{\mathrm{loc}}(\mathbb{R}^2)$. ($H$ denotes the Heaviside function.)

6.12. Let λ denote the topology on $C_0^\infty(\mathbb{R}^n)$ defined by the seminorms $\varphi \mapsto \sup_{x\in\mathbb{R}^n, |\alpha|\le m} |\partial^\alpha \varphi(x)|$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, with $m = 0, 1, 2, \dots$.

(a) Show that for $\beta \in \mathbb{N}_0^n$, ∂^β is a continuous mapping of $(C_0^\infty(\mathbb{R}^n), \lambda)$ into $(C_0^\infty(\mathbb{R}^n), \lambda)$.

Let $\mathcal{D}'_\lambda(\mathbb{R}^n)$ denote the dual space of $(C_0^\infty(\mathbb{R}^n), \lambda)$.

(b) Show that $\mathcal{D}'_\lambda(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.

(c) Show that for any function in $L_1(\mathbb{R}^n)$, the corresponding distribution belongs to $\mathcal{D}'_\lambda(\mathbb{R}^n)$.

(d) Show that any distribution with compact support on $\mathbb{R}^n$ belongs to $\mathcal{D}'_\lambda(\mathbb{R}^n)$.

(e) Show that every distribution in $\mathcal{D}'_\lambda(\mathbb{R}^n)$ is temperate and even belongs to one of the Sobolev spaces $H^t(\mathbb{R}^n)$, $t \in \mathbb{R}$.

(f) Show that the distribution given by the function 1 is temperate, but does not belong to any of the Sobolev spaces $H^t(\mathbb{R}^n)$, $t \in \mathbb{R}$.

6.13. 1. Let there be given two Hilbert spaces V and H and a continuous linear map J of V into H. Let V_0 be a dense subspace of V. Assume that $J(V)$ is dense in H, and that there for any u in V_0 exists a constant $c_u \in [0, \infty[$ such that $|(u, v)_V| \le c_u \|Jv\|_H$ for all v in V_0.

(a) Show that J and J^* both are injective.

(b) Show that $J^{*-1}J^{-1}$ is a selfadjoint operator on H.

(c) Show that for u in V_0, $y \mapsto (J^{-1}y, u)_V$ is a continuous linear functional on $J(V)$.

(d) Show that $J(V_0)$ is contained in the domain of $J^{*-1}J^{-1}$.

2. Consider the special case where $H = L_2(\mathbb{R}^2)$, $V = L_{2,1}(\mathbb{R}^2)$, $V_0 = L_{2,2}(\mathbb{R}^2)$ and J is the identity map of $L_{2,1}(\mathbb{R}^2)$ into $L_2(\mathbb{R}^2)$. (Recall Definition 6.6.)

Show that $J^*(L_2(\mathbb{R}^2)) = L_{2,2}(\mathbb{R}^2)$, and find $J^{*-1}J^{-1}$.

3. Consider finally the case where $H = L_2(\mathbb{R}^2)$, $V = H^1(\mathbb{R}^2)$, $V_0 = H^2(\mathbb{R}^2)$ and J is the identity map of $H^1(\mathbb{R}^2)$ into $L_2(\mathbb{R}^2)$. Find $J^{*-1}J^{-1}$.

6.14. Let b denote the function $b(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$.

Let A denote the differential operator $A = \frac{1}{i}\frac{d}{dx} + b$ on the interval $I =]-a, a[\subset \mathbb{R}$, $a \in]0, \infty]$.

(a) Find the domain of the maximal realization A_{max}.

(b) Find the domain of the minimal realization A_{min}.

(c) Show that A has a selfadjoint realization.

(d) Show that if $\lambda \in \mathbb{R}$, $f \in D(A_{\text{max}})$ and $A_{\text{max}} f = \lambda f$, then $\bar{f} f$ is a constant function.

(e) Assume that $a = \infty$, i.e. $I = \mathbb{R}$. Show that A_{max} has no eigenvalues.

6.15. Let $n \in \mathbb{N}$ and an open nonempty subset Ω of $\mathbb{R}^n$ be given.

Let $\mathcal{D}'_F(\Omega)$ denote the set of distributions of finite order on Ω.

(a) Show that for β in $\mathbb{N}_0^n$ and Λ in $\mathcal{D}'_F(\Omega)$, $\partial^\beta \Lambda$ is in $\mathcal{D}'_F(\Omega)$.

(b) Show that for f in $C^\infty(\Omega)$ and Λ in $\mathcal{D}'_F(\Omega)$, $f\Lambda$ is in $\mathcal{D}'_F(\Omega)$.

(c) Show that any temperate distribution on $\mathbb{R}^n$ is of finite order.

(d) Give an example of a distribution in $\mathcal{D}'_F(\mathbb{R})$, which is not temperate.

(e) Show that when φ belongs to $C_0^\infty(\mathbb{R}^n)$ and Λ is a distribution on $\mathbb{R}^n$, then $\varphi * \Lambda$ is a distribution of order 0 on $\mathbb{R}^n$.

6.16. Let b denote the function $b(x) = e^{-\frac{x^2}{2}}$, $x \in [0, 2]$.

Let A_0 denote the operator in $H = L_2([0, 2])$ with domain

$$D(A_0) = \{f \in C^2([0,2]) \mid f(0) - 2f'(0) - f'(2) = 0, f(2) + e^2 f'(0) + 5f'(2) = 0\}$$

and action $A_0 f = -bf'' + xbf' + f$ for f in $D(A_0)$.

Let V denote the subspace of $\mathbb{C}^4$ spanned by the vectors

$$\begin{pmatrix} -e^2 & 2 \\ 1 & 2 \\ 0 & 2 \\ & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

(a) Show that

$$D(A_0) = \left\{ f \in C^2([0,2]) \mid \begin{pmatrix} f(0) \\ f(2) \\ f'(0) \\ f'(2) \end{pmatrix} \in V \right\}.$$

(b) Show that A_0 can be extended to a selfadjoint operator A in H.

6.17. (a) Show that the equations

$$\mathrm{PF}\left(\frac{1}{|x|}\right)(\varphi) = \lim_{\varepsilon\to 0^+}\left[\int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{|x|}\,dx + \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{|x|}\,dx + 2\varphi(0)\log\varepsilon\right],$$

for $\varphi \in C_0^\infty(\mathbb{R})$, define a distribution $\mathrm{PF}(\frac{1}{|x|})$ on $\mathbb{R}$.

(b) Show that $\mathrm{PF}(\frac{1}{|x|})$ is a temperate distribution of order ≤ 1.

(c) Show that the restriction of $\mathrm{PF}(\frac{1}{|x|})$ to $\mathbb{R}\setminus\{0\}$ is a distribution given by a locally integrable function on $\mathbb{R}\setminus\{0\}$.

(d) Find the distribution $x\,\mathrm{PF}(\frac{1}{|x|})$ on $\mathbb{R}$.

(e) Show that there is a constant c such that the Fourier transform of $\mathrm{PF}(\frac{1}{|x|})$ is the distribution given by the locally integrable function $c - 2\log|\xi|$ on $\mathbb{R}$. (Do not try to find c, it is not easy!).

6.18. In this exercise we consider the Laplace operator $\Delta = \partial_1^2 + \partial_2^2$ on $\mathbb{R}^2$.
(a) Show that $H^2_{\mathrm{loc}}(\mathbb{R}^2)$ is contained in $C^0(\mathbb{R}^2)$.
(b) Let u be a distribution on $\mathbb{R}^2$.

Assume that there exists a continuous function h on $\mathbb{R}^2$ such that $\langle u, \Delta\varphi\rangle = \int_{\mathbb{R}^2} h\varphi\,dx$ for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Show that there exists a continuous function k on $\mathbb{R}^2$ such that $\langle u, \varphi\rangle = \int_{\mathbb{R}^2} k\varphi\,dx$ for all $\varphi \in C_0^\infty(\mathbb{R}^2)$.

6.19. Let I denote the interval $]-\pi,\pi[$, and – as usual – let $\mathcal{D}'(I)$ be the space of distributions on I with the weak* topology.
(a) Show that for any given $r \in]0,1]$, the sequence

$$\left(\frac{1}{2\pi}\sum_{n=-N}^{N} r^{|n|}e^{-int}\right)_{N\in\mathbb{N}}$$

converges to a limit P_r in $\mathcal{D}'(I)$, and that $P_1(\varphi) = \langle P_1, \varphi\rangle = \varphi(0)$, $\varphi \in C_0^\infty(I)$.
(*Hint* for (a) and (b): Put $c_n(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\varphi(\theta)d\theta$, $\varphi \in C_0^\infty(I)$; you can utilize that $\sum_{n=-\infty}^{\infty}|c_n(\varphi)| < \infty$ when $\varphi \in C_0^\infty(I)$.)

(b) Show that $r \mapsto P_r$ is a continuous map of $]0,1]$ into $\mathcal{D}'(I)$.

(c) Show that when r converges to 1 from the left, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos\theta+r^2}\varphi(\theta)d\theta$$

converges to $\varphi(0)$ for each φ in $C_0^\infty(I)$.

6.20. Let Λ denote a distribution on $\mathbb{R}$. For any given f in $C(\mathbb{R})$ and x in $\mathbb{R}$ we define $\tau(x)f$ in $C(\mathbb{R})$ by

$$(\tau(x)f)(y) = f(y+x), \quad y \in \mathbb{R}.$$

Define

$$(T\varphi)(x) = \Lambda(\tau(x)\varphi) = \langle \Lambda, \tau(x)\varphi \rangle, \quad x \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}).$$

(a) Show that $T\varphi$ is a continuous function on $\mathbb{R}$ for each φ in $C_0^\infty(\mathbb{R})$.

(b) The space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ is topologized by the increasing sequence $(p_n)_{n\in\mathbb{N}}$ of seminorms defined by

$$p_n(f) = \sup_{|x|\le n} |f(x)| \ , \ n \in \mathbb{N} \ , \quad f \in C(\mathbb{R}).$$

Show that T is a continuous linear map of $C_0^\infty(\mathbb{R})$ into $C(\mathbb{R})$.

(c) Show that $T(\tau(y)\varphi) = \tau(y)(T\varphi)$ for y in $\mathbb{R}$ and φ in $C_0^\infty(\mathbb{R})$.

(d) Show that every continuous linear map S of $C_0^\infty(\mathbb{R})$ into $C(\mathbb{R})$ with the property that $S(\tau(y)\varphi) = \tau(y)(S\varphi)$ for all y in $\mathbb{R}$ and φ in $C_0^\infty(\mathbb{R})$, is given by $(S\varphi)(x) = \langle M, \tau(x)\varphi \rangle$, $\varphi \in C_0^\infty(\mathbb{R})$, $x \in \mathbb{R}$, for some distribution M on $\mathbb{R}$.

6.21. Let n be a natural number.

The space $C_{L_2}^\infty(\mathbb{R}^n)$ of functions f in $C^\infty(\mathbb{R}^n)$ with $\partial^\alpha f$ in $L_2(\mathbb{R}^n)$ for each multiindex $\alpha \in \mathbb{N}_0^n$ is topologized by the increasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}_0}$ of norms defined by

$$\|f\|_0^2 = \int_{\mathbb{R}^n} |f(x)|^2 \, dx \ \text{ and } \ \|f\|_k^2 = \sum_{|\alpha|\le k} \|\partial^\alpha f\|_0^2 \ , \ k \in N \ , \ f \in C_{L_2}^\infty(\mathbb{R}^n) \ .$$

(a) Show that any distribution Λ in one of the Sobolev spaces $H^t(\mathbb{R}^n)$, $t \in \mathbb{R}$, by restriction defines a continuous linear functional on $C_{L_2}^\infty(\mathbb{R}^n)$.

(b) Let M be a continuous linear functional on $C^\infty_{L_2}(\mathbb{R}^n)$. Show that there exists one and only one distribution Λ in $\bigcup_{t\in\mathbb{R}} H^t(\mathbb{R}^n)$ such that $\Lambda(\varphi) = M(\varphi)$ when $\varphi \in C^\infty_{L_2}(\mathbb{R}^n)$.

(c) Let M be a linear functional on $C^\infty_{L_2}(\mathbb{R}^n)$. Show that M is continuous if and only if there exists a finite family $(f_i, \alpha_i)_{i\in I}$ of pairs, with $f_i \in L_2(\mathbb{R}^n)$ and $\alpha_i \in \mathbb{N}_0^n$, $i \in I$, such that $M(\varphi) = \sum_{i\in I} \int_{\mathbb{R}^n} f_i \partial^{\alpha_i}\varphi \, dx$ for φ in $C^\infty_{L_2}(\mathbb{R}^n)$.

6.22. Let a be a real number. Let A denote the differential operator on $\mathbb{R}^2$ given by

$$A = 2D_1^4 + a(D_1^3 D_2 + D_1 D_2^3) + 2D_2^4.$$

(a) Show that for an appropriate choice of a, the operator is not elliptic.

In the rest of the problem $a = 1$.

(b) Show that A is elliptic.

(c) Show that the equation $u + Au = f$ has a unique solution in $S'(\mathbb{R}^n)$ for each f in $L_2(\mathbb{R}^2)$, and that the solution is a function in $C^2(\mathbb{R}^2)$.

(d) What is the domain of definiton of the maximal realization $A_{\max}$ of A in $L_2(\mathbb{R}^2)$?

6.23. Let — as usual — χ denote a function in $C_0^\infty(\mathbb{R})$ with the properties:

$$\begin{aligned} &0 \le \chi \le 1 \\ &\chi(x) = 0 \text{ for } x \notin]-2, 2[\\ &\chi(x) = 1 \text{ for } x \in [-1, 1]\,. \end{aligned}$$

Define

$$\kappa_n(x) = \begin{cases} \chi(nx - 3)\,, & x < \frac{4}{n} \\ 1 \quad , & \frac{3}{n} < x < 6 \\ \chi(x - 6) \quad , & 5 < x \end{cases}$$

for $n = 1, 2, 3, \cdots$.

(a) Explain why κ_n is a well-defined function in $C_0^\infty(\mathbb{R})$ for each n in $\mathbb{N}$. Show that the sequence of functions $(e^{-n}\kappa_n)_{n\in\mathbb{N}}$ converges to 0 in $C_0^\infty(\mathbb{R})$.

(b) Show that there exists no distribution u on $\mathbb{R}$ with the property that the restriction of u to $]0, \infty[$ equals the distribution given by the locally integrable function $x \mapsto e^{\frac{6}{x}}$ on $]0, \infty[$.

6.24. Let $n \in \mathbb{N}$ be given. For an arbitrary function φ on $\mathbb{R}^n$, define $\check\varphi = S\varphi$ by $\check\varphi(x) = \varphi(-x)$, $x \in \mathbb{R}^n$. For u in $\mathcal{D}'(\mathbb{R}^n)$, define $\check u$ by $\langle \check u, \varphi\rangle = \langle u, \check\varphi\rangle$, $\varphi \in C_0^\infty(\mathbb{R}^n)$. For $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$ set $u * \varphi = \varphi * u$. Similarly, set $u*\varphi=\varphi*u$ for $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and u in $\mathcal{S}'(\mathbb{R}^n)$. For f in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$, denote the corresponding distribution on $\mathbb{R}^n$ by Λ_f or f.

(a) Show that $(\partial^\alpha\Lambda_f)^\vee=(-\partial)^\alpha\Lambda_{\check f}$ for f continuous on $\mathbb{R}^n$ and $\alpha\in\mathbb{N}_0^n$. Show that for u in $\mathcal{E}'(\mathbb{R}^n)$, $\check u$ is in $\mathcal{E}'(\mathbb{R}^n)$ with support $\operatorname{supp}(\check u)=-\operatorname{supp}(u)$.

(b) Explain the fact that when f is continuous with compact support in $\mathbb{R}^n$ and α in $\mathbb{N}_0^n$, and φ in $C_0^\infty(\mathbb{R}^n)$, then the distribution $\varphi*\partial^\alpha\Lambda_f$ is given by the function $(\partial^\alpha\varphi)*f$ in $C_0^\infty(\mathbb{R}^n)$. Show that for u in $\mathcal{E}'(\mathbb{R}^n)$ and φ i $C_0^\infty(\mathbb{R}^n)$, $\varphi*u$ is given by a function (that we shall also denote $\varphi*u=u*\varphi$); show that $\varphi\mapsto\varphi*u$ defines a continuous mapping of $C_0^\infty(\mathbb{R}^n)$ into $C_0^\infty(\mathbb{R}^n)$.

(c) Show that for $\varphi\in C_0^\infty(\mathbb{R}^n)$ and $v\in\mathcal{D}'(\mathbb{R}^n)$,

$$\langle\varphi*v),\psi\rangle=\langle v,\psi*(\Lambda_\varphi)^\vee\rangle$$

when $\psi\in C_0^\infty(\mathbb{R}^n)$. Show that for $u\in\mathcal{E}'(\mathbb{R}^n)$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$,

$$\langle\varphi*u,\psi\rangle=\langle\Lambda_\varphi,\psi*\check u\rangle$$

for $\psi\in C_0^\infty(\mathbb{R}^n)$.

(d) Show that for $u\in\mathcal{E}'(\mathbb{R}^n)$ and $v\in\mathcal{D}'(\mathbb{R}^n)$, the expression $\langle u*v,\psi\rangle=\langle v,\psi*\check u\rangle$, $\psi\in C_0^\infty(\mathbb{R}^n)$, defines a distribution $u*v$ in $\mathcal{D}'(\mathbb{R}^n)$, the convolution of u and v; moreover, $v\mapsto u*v$ defines a continuous linear map of $\mathcal{D}'(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n)$.

(e) Show that for $u\in\mathcal{E}'(\mathbb{R}^n)$, $v\in\mathcal{D}'(\mathbb{R}^n)$ and $\alpha\in\mathbb{N}_0^n$,

$$\partial^\alpha(u*v)=(\partial^\alpha u)*v=u*(\partial^\alpha v).$$

(f) Assume in this question that $n=1$. Find, for $j\in\mathbb{N}_0$, the convolution of the j'th derivative of the distribution $\delta:\varphi\mapsto\varphi(0)$, $\varphi\in C_0^\infty(\mathbb{R})$, and the distribution corresponding to the Heaviside function $H=1_{]0,\infty[}$.

(g) Show that for u and v in $\mathcal{E}'(\mathbb{R}^n)$, $u*v$ is in $\mathcal{E}'(\mathbb{R}^n)$ with $\operatorname{supp}(u*v)\subset\operatorname{supp}(u)+\operatorname{supp}(v)$. Moreover, the Fourier transformation carries convolution into a product:

$$\mathcal{F}(u*v)=\mathcal{F}(u)\mathcal{F}(v).$$

(One can use here that for u in $\mathcal{E}'(\mathbb{R}^n)$, $\mathcal{F}u$ is given by a function (also denoted $\mathcal{F}u$) in $C^\infty(\mathbb{R}^n)$).

(h) Show that for u and v in $\mathcal{E}'(\mathbb{R}^n)$ and w in $\mathcal{D}'(\mathbb{R}^n)$,

$$(u*v)*w=u*(v*w),$$

and $\delta*w=w$.

(j) Let $P(D)$ denote a partial differential operator with constant coefficients on $\mathbb{R}^n$. Assume that the distribution v on $\mathbb{R}^n$ is a fundamental solution, i.e. $P(D)v = \delta$. Show that if f is a distribution on $\mathbb{R}^n$, and f — or v — has compact support, then the distribution $f * v$ — or $v * f$ — is a solution u of the equation $P(D)u = f$.

6.25. Let Ω denote $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Consider the differential operator A on Ω given by

$$\begin{aligned} A\varphi = &-(1+\cos^2 x)\varphi''_{x,x} - (1+\sin^2 x + i\cos^2 y)\varphi''_{y,y} \\ &+ (2\cos x \sin x)\varphi'_x + (i2\cos y \sin y)\varphi'_y + \varphi, \end{aligned}$$

when $\varphi \in C_0^\infty(\Omega)$.

(a) Show that $H^2(\Omega)$ is contained in the domain of the maximal realization $A_{\max}$ of A on $L_2(\Omega)$, and that $H_0^2(\Omega)$ is contained in the domain of the minimal realisation $A_{\min}$ of A on $L_2(\Omega)$.

(b) Show that the sesquilinear form

$$\{\varphi, \psi\} \mapsto (A\varphi, \psi)\,,\ \varphi, \psi \in C_0^\infty(\Omega)\,,$$

has one and only one extension to a bounded sesquilinear form on $H_0^1(\Omega)$, and that this form is $H_0^1(\Omega)$-coercive.

(c) Show that $H^2(\Omega) \cap H_0^1(\Omega)$ is contained in the domain of the corresponding variational operator $\widetilde{A}$.

(d) Show that for (a,b) in $\mathbb{R}^2$ satisfying $|b| > 3a$, $\widetilde{A} - a - ib$ is a bijective map of the domain of $\widetilde{A}$ onto $L_2(\Omega)$, with a bounded inverse.

6.26. Let $Q =]0,1[\times]0,1[\subset \mathbb{R}^2$, and consider the sesquilinear form

$$a(u,v) = \int_Q (\partial_1 u \partial_1 \bar{v} + \partial_2 u \partial_2 \bar{v} + u \partial_1 \bar{v})\, dx_1 dx_2.$$

Let $H = L_2(Q)$, $V_0 = H_0^1(Q)$ and $V_1 = H^1(Q)$, and let, respectively, a_0 and a_1 denote a defined on V_0 resp. V_1. One considers the triples (H, V_0, a_0) and (H, V_1, a_1).

(a) Show that a_0 is V_0-elliptic and that a_1 is V_1-coercive, and explain why the Lax-Milgram theorem can be applied to the triples (H, V_0, a_0) and (H, V_1, a_1). The hereby defined operators will be denoted A_0 and A_1.

(*Hint.* One can show that $(u, \partial_1 u)_H$ is purely imaginary, when $u \in C_0^\infty(Q)$.)

(b) Show that A_0 acts like $-\Delta - \partial_1$ in the distribution sense, and that functions $u \in D(A_0) \cap C(\overline{Q})$ satisfy $u|_{\partial Q} = 0$.

(*Hint.* Let $u_k \to u$ in $H_0^1(Q)$, $u_k \in C_0^\infty(Q)$. For a boundary point x which is not a corner, one can by a suitable choice of truncation function reduce the u_k's and u to have support in a small neighborhood $B(x,\delta) \cap \overline{Q}$ and show that u is 0 as an L_2-function on the interval $B(x,\delta/2) \cap \partial Q$, by inequalities as in Theorem 4.24.)

(c) Show that A_1 acts like $-\Delta - \partial_1$ in the distribution sense, and that functions $u \in D(A_1) \cap C^2(\overline{Q})$ satisfy certain first-order boundary conditions on the edges of Q; find them. (Note that the Gauss and Green's formulas hold on Q, with a piecewise continuous definition of the normal vector at the boundary.)

(d) Show that A_0 has its numerical range (and hence its spectrum) contained in the set

$$\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq 2, \quad |\operatorname{Im} \lambda| \leq \sqrt{\operatorname{Re} \lambda} \}.$$

(*Hint.* Note that for $u \in V_0$ with $\|u\|_H = 1$, $|\operatorname{Im} a_0(u,u)| \leq \|D_1 u\|_H$, while $\operatorname{Re} a_0(u,u) \geq \|D_1 u\|_H^2$.)

(e) Investigate the numerical range (and spectrum) of A_1. (One should at least find a convex set as in Corollary 12.20. One may possibly improve this to the set

$$\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\tfrac{1}{2}, \quad |\operatorname{Im} \lambda| \leq \sqrt{2 \operatorname{Re} \lambda + 1} \}.)$$

6.27. Let $\mathcal{L}$ denote the differential operator defined by

$$\mathcal{L}u = -\partial_x(x \partial_x u) + (x+1)u = (1 + \partial_x)[x(1 - \partial_x)u],$$

for $u \in \mathcal{S}'(\mathbb{R})$.

(a) Find the operator $\widehat{\mathcal{L}}$ that $\mathcal{L}$ carries over to by Fourier transformation, in other words, $\widehat{\mathcal{L}} = \mathcal{F}\mathcal{L}\mathcal{F}^{-1}$.

(b) Show that the functions

$$g_k(\xi) = \frac{(1 - i\xi)^k}{(1 + i\xi)^{k+1}}, \quad k \in \mathbb{Z},$$

satisfy

$$\widehat{\mathcal{L}}\, g_k = 2(k+1) g_k$$

(hence are eigenfunctions for $\widehat{\mathcal{L}}$ with eigenvalues $2(k+1)$), and that the system $\{\frac{1}{\sqrt{\pi}} g_k\}_{k \in \mathbb{Z}}$ is orthonormal in $L_2(\mathbb{R})$.

(c) Show that $H(x)e^{-x}$ by convolution with itself m times gives

$$H(x)e^{-x} * \cdots * H(x)e^{-x} = \frac{x^m}{m!} H(x)e^{-x} \quad (m+1 \text{ factors}).$$

(d) Show that $\mathcal{L}$ has a system of eigenfunctions

$$f_k(x) = \mathcal{F}^{-1} g_k = p_k(x) H(x) e^{-x}, \quad k \in \mathbb{N}_0,$$

belonging to eigenvalues $2(k+1)$, where each p_k is a polynomial of degree k. Calculate p_k for $k = 0, 1, 2$.
(*Hint.* One can for example calculate $\mathcal{F}^{-1}\frac{1}{1+i\xi}$ and use point (c).)
(e) Show that further eigenfunctions for $\mathcal{L}$ (with eigenalues $2(-m+1)$) are obtained by taking

$$f_{-m}(x) = f_{m-1}(-x), \quad m \in \mathbb{N},$$

and show that the whole system $\{\sqrt{2}\, f_k\}_{k\in\mathbb{Z}}$ is an orthonormal system in $L_2(\mathbb{R})$.
(f) Show that when $\varphi \in \mathcal{S}(\mathbb{R})$, then

$$\sum_{k\in\mathbb{N}_0} |(f_k, \varphi)_{L_2}| \le C \|\mathcal{L}\varphi\|_{L_2},$$

for a suitable constant C.
(*Hint.* A useful inequality can be obtained by applying the Bessel inequality to $\mathcal{L}\varphi$ and observe that $(f_k, \mathcal{L}\varphi) = (\mathcal{L}f_k, \varphi)$.)
(g) Show that $\Lambda = \sum_{k\in\mathbb{N}_0} f_k$ defines a distribution in $\mathcal{S}'(\mathbb{R})$ by the formula

$$\langle \Lambda, \varphi \rangle = \lim_{N\to\infty} \sum_{k=0}^{N} \langle f_k, \varphi \rangle,$$

and that this distribution is supported in $[0, \infty[$.

(*Comment:* The system $\{\sqrt{2} f_k \mid k \in \mathbb{N}_0\}$ on $\mathbb{R}_+$ is a variant of what is usually called the Laguerre ortonormal system; it is complete in $L_2(\mathbb{R}_+)$.)

6.28. For $a \in \mathbb{R}_+$, let

$$f_a(x) = \frac{a}{\pi} \frac{1}{x^2 + a^2} \quad \text{for} \quad x \in \mathbb{R}.$$

Show that $f_a \to \delta$ in $H^{-1}(\mathbb{R})$ for $a \to 0+$.

6.29. Consider the partial differential operator in two variables

$$A = D_1^4 + D_2^4 + b D_1^2 D_2^2,$$

where b ia a complex constant.

(a) Show that A is elliptic if and only is $b \in \mathbb{C}\setminus]-\infty,-2]$. (One can investigate the cases $b \in \mathbb{R}$ and $b \in \mathbb{C}\setminus\mathbb{R}$ separately.)

(b) Show (by reference to the relevant theorems) that the maximal realization and the minimal realization of A on $\mathbb{R}^2$ coincide, and that they in the elliptic cases have domain $H^4(\mathbb{R}^2)$.

(c) Show that $A_{\max}$ can be defined by the Lax-Milgram theorem from a sesquilinear form $a(u,v)$ on a suitable subspace V of $H = L_2(\mathbb{R}^2)$ (indicate a and V).

Describe polygonal sets containing the numerical range and the spectrum of $A_{\max}$, on one hand when $b \in]-2,\infty[$, on the other hand when $b = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$.

6.30. In the following, $\varphi(x)$ denotes a given function in $L_2(\mathbb{R})$ with compact support.

(a) Show that $\hat\varphi \in H^m(\mathbb{R})$ for all $m \in \mathbb{N}$, and that $\hat\varphi \in C^\infty(\mathbb{R})$.

(b) Show that when $\psi \in L_1(\mathbb{R})$, then $\sum_{l\in\mathbb{Z}} \psi(\xi + 2\pi l)$ defines a function in $L_1(\mathbb{T})$, and

$$\int_{\mathbb{R}} \psi(\xi)\,d\xi = \int_0^{2\pi} \sum_{l\in\mathbb{Z}} \psi(\xi + 2\pi l)\,d\xi.$$

(We recall that $L_p(\mathbb{T})$ $(1 \le p < \infty)$ denotes the space of (equivalence classes of) functions in $L_{p,\mathrm{loc}}(\mathbb{R})$ with period 2π; it is a Banach space when provided with the norm $\big(\frac{1}{2\pi}\int_0^{2\pi} |\psi(\xi)|^p\,d\xi\big)^{1/p}$.)

Show that the series $\sum_{l\in\mathbb{Z}} |\hat\varphi(\xi + 2\pi l)|^2$ converges uniformly towards a continuous function $g(\xi)$ with period 2π.

(*Hint.* Using an inequality from Chapter 4 one can show that $\sup_{\xi\in[2\pi l,2\pi(l+1)]} |\hat\varphi(\xi)|^2 \le c\|1_{[2\pi l,2\pi(l+1)]}\hat\varphi\|^2_{H^1([2\pi l,2\pi(l+1)])}$.)

(c) Show the identities, for $n \in \mathbb{Z}$,

$$\begin{aligned}\int_{\mathbb{R}} \varphi(x-n)\overline{\varphi(x)}\,dx &= \tfrac{1}{2\pi}\int_{\mathbb{R}} |\hat\varphi(\xi)|^2 e^{-in\xi}\,d\xi \\ &= \tfrac{1}{2\pi}\int_0^{2\pi} \sum_{l\in\mathbb{R}} |\hat\varphi(\xi+2\pi l)|^2 e^{-in\xi}\,d\xi.\end{aligned}$$

(d) Show that the following statements (i) and (ii) are equivalent:

(i) The system of functions $\{\,\varphi(x-n) \mid n \in \mathbb{Z}\,\}$ is an orthonormal system in $L_2(\mathbb{R})$.

(ii) The function $g(\xi)$ defined in (b) is constant $= 1$.

(*Hint.* Consider the Fourier series of g.)

(*Comment.* The results of this exercise are used in the theory of wavelets.)

6.31. (a) For each $j \in \mathbb{N}$, define the distribution u_j by

$$u_j = \sum_{k=1}^{2^j-1} \tfrac{1}{2^j}\delta_{\frac{k}{2^j}}.$$

Show that $u_j \in \mathcal{D}'(\,]0,1[\,)$, and that $u_j \to 1$ in $\mathcal{D}'(\,]0,1[\,)$ for $j \to \infty$.

(b) Considering $\mathbb{R}^2$ with coordinates (x,y), denote $(x^2+y^2)^{1/2} = r$. Let $v_1(x,y) = \log r$ and $v_2(x,y) = \partial_x \log r$ for $(x,y) \neq (0,0)$, setting them equal to 0 for $(x,y) = (0,0)$; show that both functions belong to $L_{1,\mathrm{loc}}(\mathbb{R}^2)$.

Identifying $\log r$ with v_1 as an element of $\mathcal{D}'(\mathbb{R}^2)$, show that the derivative in the distribution sense $\partial_x \log r$ can be identified with the function v_2, and that both distributions have order 0.

6.32. Let $f \in L_1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f(x)\,dx = 1$. For each $j \in \mathbb{N}$, define f_j by

$$f_j(x) = j^n f(jx).$$

Show that $f_j \to \delta$ in $\mathcal{S}'(\mathbb{R}^n)$.

6.33. Consider the differential operator A in $H = L_2(\mathbb{R}^3_+)$ defined by

$$A = -\partial_1^2 - \partial_2^2 - \partial_3^2 + \partial_2\partial_3 + 1,$$

and the sesquilinear form $a(u,v)$ on $V = H^1(\mathbb{R}^3_+)$ defined by

$$a(u,v) = (\partial_1 u, \partial_1 v) + (\partial_2 u, \partial_2 v) + (\partial_3 u, \partial_3 v) - (\partial_2 u, \partial_3 v) + (u,v),$$

where $\mathbb{R}^3_+ = \{(x_1,x_2,x_3) \mid x_3 > 0\}$.

(a) Show that A is elliptic of order 2, and that a is V-elliptic.

(b) Let A_1 be the variational operator defined from the triple (H,V,a). Show that A_1 is a realization of A.

(c) What is the boundary condition satisfied by the functions $u \in D(A_1) \cap C^\infty_{(0)}(\overline{\mathbb{R}}^3_+)$?

6.34. With B denoting the unit ball in $\mathbb{R}^n$, consider the two functions

$$u = 1_B, \quad v = 1_{\mathbb{R}^n \setminus B}.$$

For each of the distributions u, $\partial_1 u$, v and $\partial_1 v$, find out whether it belongs to a Sobolev space $H^s(\mathbb{R}^n)$, and indicate such a space in the affirmative cases.

6.35. Let f be the function on $\mathbb{R}$ defined by:

$$f(x) = \begin{cases} 1 & \text{for } |x| > \pi/2, \\ 1 + \cos x & \text{for } |x| \le \pi/2. \end{cases}$$

(a) Find f', f'', f''' and $\widehat{f}$. (Recall that $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$.)
(b) For each of these distributions, determine whether it is an $L_{1,\mathrm{loc}}(\mathbb{R})$-function, and in case not, find what the order of the distribution is.

6.36. Let Ω be the unit disk $B(0,1)$ in $\mathbb{R}^2$ with points denoted (x,y), let $H = L_2(\Omega)$, and let $V = H_0^1(\Omega)$. Consider the sesquilinear form $a(u,v)$ with domain V, defined by

$$a(u,v) = \int_\Omega ((2+x)\partial_x u\, \partial_x \overline{v} + (2+y)\partial_y u\, \partial_y \overline{v})\, dxdy.$$

(a) Show that a is bounded on V and V-coercive. Is it V-elliptic?
(b) Show that the variational operator A_0 defined from the triple (H, V, a) is selfadjoint in H.
(c) Show that A_0 is a realization of a partial differential operator; which one?
(d) The functions $u \in D(A_0)$ have boundary value zero; indicate why.

6.37. For each nonnegative integer m, define the space $K^m(\mathbb{R})$ of distributions on $\mathbb{R}$ by:

$$K^m(\mathbb{R}) = \{u \in L_2(\mathbb{R}) \mid x^j D^k u(x) \in L_2(\mathbb{R}) \text{ for } j + k \le m\};$$

here j and k denote nonnegative integers.
(a) Provided with the norm

$$\|u\|_{K^m} = \Big(\sum_{j+k\le m} \|x^j D^k u(x)\|^2_{L_2(\mathbb{R})}\Big)^{\frac{1}{2}},$$

$K^m(\mathbb{R})$ is a Hilbert space; indicate why.
(b) Show that $\mathcal{F}(K^m(\mathbb{R})) = K^m(\mathbb{R})$.
(c) Show that $\bigcap_{m\ge 0} K^m(\mathbb{R}) = \mathcal{S}(\mathbb{R})$.
(*Hint.* One can for example make use of Theorem 4.18.)