§5. Fourier transformation of distributions

5.1. Rapidly decreasing functions.

In the following we study an important tool in the treatment of differential operators, *the Fourier transform*. To begin with, it is useful in the study of differential operators on $\mathbb{R}^n$ with constant coefficients, but it can also be used in the more advanced theory to treat operators on subsets Ω and with variable coefficients.

The space $\mathcal{D}'(\mathbb{R}^n)$ is too large to permit a sensible definition of the Fourier transform. We therefore restrict the attention to a somewhat smaller space of distributions, $\mathcal{S}'(\mathbb{R}^n)$, the dual of a space of test functions $\mathcal{S}(\mathbb{R}^n)$ which is slightly larger than $C_0^\infty(\mathbb{R}^n)$. ($\mathcal{S}$ resp. $\mathcal{S}'$ is often called the Schwartz space of test functions resp. distributions, after Laurent Schwartz.)

It will be convenient to introduce the function

$$\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}\ , \tag{5.1}$$

and its powers $\langle x\rangle^s$, $s\in\mathbb{R}$. Since $|x|/\langle x\rangle \to 1$ for $|x|\to\infty$, $\langle x\rangle$ is of the same order of magnitude as $|x|$, but has the advantage of being a positive C^∞ function on all of $\mathbb{R}^n$. We note that one has for $m\in\mathbb{N}_0$:

$$\langle x\rangle^{2m} = (1+x_1^2+\cdots+x_n^2)^m = \sum_{|\alpha|\le m} C_{m,\alpha}x^{2\alpha}\begin{cases}\le C_m\sum_{|\alpha|\le m}x^{2\alpha},\\ \ge \sum_{|\alpha|\le m}x^{2\alpha},\end{cases} \tag{5.2}$$

with positive integers $C_{m,\alpha}$ and C_m. This is shown by multiplying out the expression $(1+x_1^2+\cdots+x_n^2)^m$; we recall from (A.9) that $C_{m,\alpha}=\frac{m!}{\alpha!(m-|\alpha|)!}$.

Definition 5.1. *The vector space $\mathcal{S}(\mathbb{R}^n)$ (often just denoted $\mathcal{S}$) is defined as the space of C^∞ functions $\varphi(x)$ on $\mathbb{R}^n$ such that $x^\alpha D^\beta\varphi(x)$ is bounded for all multi-indices α and $\beta\in\mathbb{N}_0^n$. $\mathcal{S}$ is provided with the family of seminorms*

$$p_M(\varphi) = \sup\bigl\{\langle x\rangle^M|D^\alpha\varphi(x)| \bigm| x\in\mathbb{R}^n,\ |\alpha|\le M\bigr\},\quad M\in\mathbb{N}_0. \tag{5.3}$$

The functions $\varphi\in\mathcal{S}$ are called **rapidly decreasing functions**.

With this system of seminorms, $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space. The seminorms are in fact norms and form an increasing family, so in particular, the system has the max-property (cf. Remark B.6). We could also have taken the family of seminorms

$$p_{\alpha,\beta}(\varphi) = \sup\{\,|x^\alpha D^\beta\varphi(x)| \mid x\in\mathbb{R}^n\,\}\ , \tag{5.4}$$

where α and β run through $\mathbb{N}_0^n$; it is seen from the inequalities (5.2) that the family of seminorms (5.4) defines the same topology as the family (5.3).

As a local neighborhood basis at 0 we can take the sets

$$V_{M,\frac{1}{N}} = \{\, \varphi \in \mathcal{S} \mid \langle x\rangle^M |D^\alpha \varphi(x)| < \tfrac{1}{N} \text{ for } |\alpha| \le M \,\}, \tag{5.5}$$

for $M \in \mathbb{N}_0$, $N \in \mathbb{N}$. The topology on $C_0^\infty(\mathbb{R}^n)$ is stronger than the topology induced on this space by $\mathcal{S}(\mathbb{R}^n)$, since the sets $V_{M,\frac{1}{N}} \cap C_0^\infty(\mathbb{R}^n)$ are open, convex balanced neighborhoods of 0 in $C_0^\infty(\mathbb{R}^n) = \bigcup C_{K_j}^\infty(\mathbb{R}^n)$; this follows since their intersections with $C_{K_j}^\infty(\mathbb{R}^n)$ are open neighborhoods of 0 in $C_{K_j}^\infty(\mathbb{R}^n)$. (One may observe that the topology induced from $\mathcal{S}$ is metrizable by Theorem B.8, whereas the usual topology on $C_0^\infty(\mathbb{R}^n)$ is not metrizable.)

Lemma 5.2.
1° *For $1 \le p \le \infty$, $\mathcal{S}(\mathbb{R}^n)$ is continuously injected in $L_p(\mathbb{R}^n)$.*
2° *For $1 \le p \le \infty$, $\mathcal{S}(\mathbb{R}^n) \subset C_{L_p}^\infty(\mathbb{R}^n)$.*
3° *For $1 \le p < \infty$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $L_p(\mathbb{R}^n)$.*

Proof. 1°. We have for φ in $\mathcal{S}(\mathbb{R}^n)$, M in $\mathbb{N}_0$ and α in $\mathbb{N}_0^n$ with $|\alpha| \le M$ that

$$|D^\alpha \varphi(x)| \le p_M(\varphi)\langle x\rangle^{-M}$$

for all x in $\mathbb{R}^n$. In particular, $\|\varphi\|_{L_\infty} = \sup |\varphi(x)| = p_0(\varphi)$. For $1 \le p < \infty$,

$$\|\varphi\|_{L_p}^p \le p_M(\varphi)^p \int_{\mathbb{R}^n} \langle x\rangle^{-Mp} dx;$$

here we note that

$$\int_{\mathbb{R}^n} \langle x\rangle^{-Mp}\, dx = \int_{\mathbb{R}^n} (1+|x|^2)^{-Mp/2} dx < \infty \text{ when } M > \tfrac{n}{p}. \tag{5.6}$$

Then $\|\varphi\|_{L_p} \le C_M p_M(\varphi)$ for $M > \frac{n}{p}$, completing the proof of 1°.

Now 2° follows, since $D^\alpha f \in \mathcal{S}$ for all α when $f \in \mathcal{S}$.

3° follows from the fact that the subset $C_0^\infty(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$ is dense in $L_p(\mathbb{R}^n)$ for $p < \infty$, cf. Theorem 2.10. $\square$

It is obvious that multiplication by a polynomial maps $\mathcal{S}$ into $\mathcal{S}$. There are also other C^∞ functions that define multiplication operators in $\mathcal{S}$. L. Schwartz introduced the following space of functions ("opérateurs de multiplication") containing the polynomials:

Definition 5.3. *The vector space $\mathcal{O}_M(\mathbb{R}^n)$ (or just $\mathcal{O}_M$) of* **slowly increasing functions** *on $\mathbb{R}^n$ consists of the functions $p(x) \in C^\infty(\mathbb{R}^n)$ which satisfy: For any $\alpha \in \mathbb{N}_0^n$ there exists $c > 0$ and $a \in \mathbb{R}$ (depending on p and α) such that*

$$|D^\alpha p(x)| \le c\langle x\rangle^a \quad \textit{for all} \;\; x \in \mathbb{R}^n .$$

For example, $\langle x\rangle^t \in \mathcal{O}_M$ for any $t \in \mathbb{R}$; this is seen by repeated application of the rule

$$\partial_j \langle x\rangle^s = s\langle x\rangle^{s-2} x_j. \tag{5.7}$$

The elements of $\mathcal{O}_M$ define *multiplication operators* $M_p \colon f \mapsto pf$ which (by the Leibniz formula) map $\mathcal{S}$ continuously into $\mathcal{S}$. In particular, since $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{O}_M(\mathbb{R}^n)$, we see that $\varphi\psi$ belongs to $\mathcal{S}(\mathbb{R}^n)$ when φ and ψ belong to $\mathcal{S}(\mathbb{R}^n)$.

Clearly, ∂^α and D^α are *continuous operators* in $\mathcal{S}(\mathbb{R}^n)$.

When $f \in L_1(\mathbb{R}^n)$, the Fourier transformed function $(\mathcal{F}f)(\xi)$ is defined by the formula

$$(\mathcal{F}f)(\xi) \equiv \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx\ .$$

We now recall some of the more elementary rules for the Fourier transformation of functions (proofs are included for the convenience of the reader):

Theorem 5.4.

1° *The Fourier transform $\mathcal{F}$ is a continuous linear map of $L_1(\mathbb{R}^n)$ into $C_{L_\infty}(\mathbb{R}^n)$, such that when $f \in L_1(\mathbb{R}^n)$, then*

$$\|\hat{f}\|_{L_\infty} \le \|f\|_{L_1}, \quad \hat{f}(\xi) \to 0 \text{ for } |\xi| \to \infty. \tag{5.8}$$

2° *The Fourier transform is a continuous linear map of $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$, and one has for $f \in \mathcal{S}(\mathbb{R}^n)$, $\xi \in \mathbb{R}^n$:*

$$\mathcal{F}[x^\alpha D_x^\beta f(x)](\xi) = (-D_\xi)^\alpha(\xi^\beta \hat{f}(\xi)), \tag{5.9}$$

for all multi-indices α and $\beta \in \mathbb{N}_0^n$.

3° *Defining the co-Fourier transform $\overline{\mathcal{F}}$ by*

$$\overline{\mathcal{F}} f(\xi) = \int_{\mathbb{R}^n} e^{+ix\cdot\xi} f(x)\, dx, \text{ whereby } \overline{\mathcal{F}f} = \overline{\mathcal{F}}\,\overline{f}$$

(it is likewise continuous from $\mathcal{S}$ to $\mathcal{S}$), one has for $f \in \mathcal{S}(\mathbb{R}^n)$ with $\hat{f} = \mathcal{F}f$:

$$f(x) = (2\pi)^{-n} \int e^{i\xi\cdot x} \hat{f}(\xi)\, d\xi \quad [\equiv (2\pi)^{-n} \overline{\mathcal{F}} \hat{f}], \tag{5.10}$$

so the operator $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n)$ bijectively onto $\mathcal{S}(\mathbb{R}^n)$ with $\mathcal{F}^{-1} = (2\pi)^{-n}\overline{\mathcal{F}}$.

Proof. 1°. The inequality

$$|\hat{f}(\xi)| = |\int e^{ix\cdot\xi} f(x)|\, dx \le \int |f(x)|\, dx$$

shows the first statement in (5.8), so $\mathcal{F}$ maps L_1 into L_∞. When $f \in L_1$, the functions $e^{-ix\cdot\xi} f(x)$ have the integrable majorant $|f(x)|$ for all ξ, so the continuity of $\hat{f}(\xi)$ follows from Lemma 2.8 1°. That $\hat{f}(\xi) \to 0$ for $|\xi| \to \infty$ will be shown further below.

2°. When $f \in L_1$ with $x_j f(x) \in L_1$, the functions of x

$$\partial_{\xi_j}(e^{-ix\cdot\xi} f(x)) = -ix_j e^{-ix\cdot\xi} f(x)$$

have the integrable majorant $|x_j f(x)|$, so it follows from Lemma 2.8 2° that $\partial_{\xi_j} \hat{f}(\xi)$ exists and equals $\mathcal{F}(-ix_j f(x))$. Then also $-D_{\xi_j} \hat{f} = \mathcal{F}(x_j f(x))$. When $f \in \mathcal{S}$, we can apply this rule to all derivatives, obtaining the formula $(-D_\xi)^\alpha \hat{f} = \mathcal{F}(x^\alpha f)$.

When $f \in \mathcal{S}$, we find by integration by parts (cf. (A.20)) that

$$\begin{aligned} &\int e^{-ix\cdot\xi} \partial_{x_j} f(x)\, dx = \lim_{R\to\infty} \int_{|x|\le R} e^{-ix\cdot\xi} \partial_{x_j} f(x)\, dx = \\ &\lim_{R\to\infty} \Big(\int_{|x|\le R} i\xi_j e^{-ix\cdot\xi} f(x)\, dx + \int_{|x|=R} e^{-ix\cdot\xi} f(x) \tfrac{x_j}{|x|}\, dx \Big) = i\xi_j \hat{f}(\xi), \end{aligned}$$

(since $R^{n-1} \sup\{|f(x)| \mid |x| = R\} \to 0$ for $R \to \infty$), showing the formula $i\xi_j \hat{f} = \mathcal{F}(\partial_{x_j} f(x))$. Then also $\mathcal{F}(D_{x_j} f) = \xi_j \hat{f}$. Repeated application gives that $\mathcal{F}(D^\beta f) = \xi^\beta \hat{f}$ for all $\beta \in \mathbb{N}_0^n$.

Formula (5.9) follows for $f \in \mathcal{S}$ by combination of the two facts we have shown. Note here that by 1°, the right hand side is bounded and continuous for all α, β; this implies (in view of the Leibniz formula) that $\hat{f} \in \mathcal{S}$.

This shows that $\mathcal{F}$ maps $\mathcal{S}$ into $\mathcal{S}$; the continuity will be shown below. Let us first complete the proof of 1°: Let $f \in L_1$ and let $\varepsilon > 0$. Since $\mathcal{S}$ is dense in L_1 (Lemma 5.2), there is a $g \in \mathcal{S}$ with $\|f - g\|_{L_1} < \varepsilon/2$. Then by 1°, $|\hat{f}(\xi) - \hat{g}(\xi)| < \varepsilon/2$ for all ξ. Since $\hat{g} \in \mathcal{S}$ by 2°, we can find an $R > 0$ such that $|\hat{g}(\xi)| \le \varepsilon/2$ for $|\xi| \ge R$. Then

$$|\hat{f}(\xi)| \le |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)| < \varepsilon \text{ for } |\xi| \ge R.$$

Now for the continuity: Note that (5.9) implies

$$\begin{aligned} \mathcal{F}((I - \Delta) f) &= \mathcal{F}((I - \partial_{x_1}^2 - \cdots - \partial_{x_n}^2) f(x)) \\ &= (1 + \xi_1^2 + \cdots + \xi_n^2) \hat{f}(\xi) = \langle \xi \rangle^2 \hat{f}. \end{aligned} \tag{5.11}$$

Then we have for each $k \in \mathbb{N}_0$, taking $l = \frac{k}{2}$ if k is even, $l = \frac{k+1}{2}$ if k is odd, and recalling (5.6)):

$$
\begin{aligned}
p_0(\hat{f}) &= \sup |\hat{f}(\xi)| \leq \|\langle x\rangle^{-n-1}\langle x\rangle^{n+1} f\|_{L_1} \leq \|\langle x\rangle^{-n-1}\|_{L_1} p_{n+1}(f), \\
p_k(\hat{f}) &= \sup_{\xi\in\mathbb{R}^n, |\alpha|\leq k} |\langle\xi\rangle^k D_\xi^\alpha \hat{f}(\xi)| \leq \sup_{\xi\in\mathbb{R}^n, |\alpha|\leq k} |\langle\xi\rangle^{2l} D_\xi^\alpha \hat{f}(\xi)| \\
&= \sup_{\xi\in\mathbb{R}^n, |\alpha|\leq k} |\mathcal{F}[(1-\Delta)^l (x^\alpha f(x))]| \leq \sup_{|\alpha|\leq k} \|(1-\Delta)^l (x^\alpha f(x))\|_{L_1} \\
&\leq \|\langle x\rangle^{-n-1}\|_{L_1} \sup_{x\in\mathbb{R}^n, |\alpha|\leq k} |\langle x\rangle^{n+1}(1-\Delta)^l (x^\alpha f(x))| \\
&\leq C p_{k+n+1}(f). \qquad (5.12)
\end{aligned}
$$

This shows that $\mathcal{F}$ is continuous from $\mathcal{S}$ to $\mathcal{S}$.

3°. Observe first that since $\overline{\mathcal{F}} f = \overline{\mathcal{F}\overline{f}}$, the operator $\overline{\mathcal{F}}$ has properties analogous to those of $\mathcal{F}$. To show (5.10), we need to calculate $(2\pi)^{-n}\int e^{i\xi\cdot x}(\int e^{-i\xi\cdot y} f(y)dy)d\xi$ for $f \in \mathcal{S}$. The function $e^{i\xi\cdot(x-y)}f(y)$ is not integrable on $\mathbb{R}^{2n}$, so we cannot simply change the order of integration. Therefore we introduce an integration factor $\psi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, which will be removed later by a passage to the limit. More precisely, we insert a function $\psi(\varepsilon\xi)$ with $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\varepsilon > 0$. Then we find for each fixed x, by use of the Fubini theorem and the change of variables $(\eta, z) = (\varepsilon\xi, (y-x)/\varepsilon)$:

$$
\begin{aligned}
(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\psi(\varepsilon\xi)\hat{f}(\xi)d\xi &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\psi(\varepsilon\xi)\Big(\int_{\mathbb{R}^n} e^{-i\xi\cdot y} f(y)dy\Big)\,d\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-i\xi\cdot(y-x)}\psi(\varepsilon\xi) f(y)\,d\xi\,dy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-i\eta\cdot z}\psi(\eta) f(x+\varepsilon z)\,d\eta\,dz \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{\psi}(z) f(x+\varepsilon z)\,dz\,,
\end{aligned}
$$

since the functional determinant is 1.

For $\varepsilon \to 0$,

$$
e^{i\xi\cdot x}\psi(\varepsilon\xi)\hat{f}(\xi) \to \psi(0)e^{i\xi\cdot x}\hat{f}(\xi) \text{ with } |e^{i\xi\cdot x}\psi(\varepsilon\xi)\hat{f}(\xi)| \leq C|\hat{f}(\xi)|,
$$

where $C = \sup_\xi |\psi(\xi)|$. Moreover,

$$
\hat{\psi}(z) f(x+\varepsilon z) \to \hat{\psi}(z) f(x) \text{ with } |\hat{\psi}(z) f(x+\varepsilon z)| \leq C'|\hat{\psi}(z)|,
$$

where $C' = \sup_y |f(y)|$. By the theorem of Lebesgue we then find:

$$
(2\pi)^{-n}\psi(0)\int e^{i\xi\cdot x}\hat{f}(\xi)\,d\xi = (2\pi)^{-n} f(x)\int \hat{\psi}(z)\,dz.
$$

One can in particular use $\psi(\xi)=e^{-\frac12|\xi|^2}$. In this case, $\psi(0)=1$, and $\hat\psi(z)$ can be shown to satisfy

$$\mathcal{F}(e^{-\frac12|\xi|^2})=(2\pi)^{\frac n2}e^{-\frac12|z|^2}; \tag{5.13}$$

then $\int\hat\psi(z)dz=(2\pi)^{\frac n2}\int e^{-\frac12|z|^2}dz=(2\pi)^n$. This implies formula (5.10). $\square$

We observe moreover that the map $\mathcal{F}\colon L_1(\mathbb{R}^n)\to C_{L_\infty}(\mathbb{R}^n)$ is *injective.* For if $f\in L_1(\mathbb{R}^n)$ is such that $\hat f(\xi)=0$ for all $\xi\in\mathbb{R}^n$, then we have for any $\varphi\in C_0^\infty(\mathbb{R}^n)$, denoting $\mathcal{F}^{-1}\varphi$ by ψ, that

$$\begin{aligned}0&=\int_{\mathbb{R}^n}\hat f(\xi)\psi(\xi)\,d\xi=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)\psi(\xi)\,dxd\xi\\&=\int_{\mathbb{R}^n}f(x)\big(\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\psi(\xi)\,d\xi\big)dx=\int_{\mathbb{R}^n}f(x)\varphi(x)\,dx;\end{aligned}$$

then it follows by the Du Bois-Reymond Lemma (Lemma 3.2) that $f=0$ as an element of $L_1(\mathbb{R}^n)$.

There is the following extension to $L_2(\mathbb{R}^n)$:

Theorem 5.5. (Parseval-Plancherel)

1° *The Fourier transform $\mathcal{F}\colon\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ extends in a unique way to an isometric isomorphism $\mathcal{F}_2$ of $L_2(\mathbb{R}^n,dx)$ onto $L_2(\mathbb{R}^n,(2\pi)^{-n}dx)$. For $f,g\in L_2(\mathbb{R}^n)$,*

$$\begin{aligned}\int f(x)\overline{g(x)}\,dx&=(2\pi)^{-n}\int\mathcal{F}_2f(\xi)\overline{\mathcal{F}_2g(\xi)}\,d\xi\,,\\\int|f(x)|^2dx&=(2\pi)^{-n}\int|\mathcal{F}_2f(\xi)|^2d\xi\,.\end{aligned}\tag{5.14}$$

2° *One has that*

$$\mathcal{F}_2f=\mathcal{F}f\ \text{ for } f\in L_2(\mathbb{R}^n)\cap L_1(\mathbb{R}^n). \tag{5.15}$$

Moreover, when $f\in L_2(\mathbb{R}^n)$ and hence $1_{B(0,N)}f\in L_1(\mathbb{R}^n)$, then the sequence of continuous functions $\mathcal{F}(1_{B(0,N)}f)$ converges in $L_2(\mathbb{R}^n)$ to $\mathcal{F}_2f$ for $N\to\infty$.

Proof. 1°. We first show (5.14) for $f,g\in\mathcal{S}$. By Theorem 5.4 3°,

$$g(x)=(2\pi)^{-n}\int e^{i\xi\cdot x}\hat g(\xi)d\xi\,,$$

so by the Fubini theorem,

$$\begin{aligned}\int f(x)\overline{g(x)}\,dx&=(2\pi)^{-n}\int f(x)\int e^{-i\xi\cdot x}\overline{\hat g(\xi)}\,d\xi dx\\&=(2\pi)^{-n}\int\Big(\int f(x)e^{-i\xi\cdot x}dx\Big)\overline{\hat g(\xi)}\,d\xi=(2\pi)^{-n}\int\hat f(\xi)\overline{\hat g(\xi)}\,d\xi.\end{aligned}$$

This shows the first formula, and the second formula follows by taking $f = g$.

We see from these formulas that $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to L_2(\mathbb{R}^n, (2\pi)^{-n}dx)$ is a linear isometry from $\mathcal{S}(\mathbb{R}^n)$ considered as a dense subspace of $L_2(\mathbb{R}^n, dx)$. Since the target space is complete, $\mathcal{F}$ extends in a unique way to a continuous linear map $\mathcal{F}_2 : L_2(\mathbb{R}^n, dx) \to L_2(\mathbb{R}^n, (2\pi)^{-n}dx)$, which is an isometry. The range $\mathcal{F}_2(L_2(\mathbb{R}^n, dx))$ is then also complete, hence is a closed subspace of $L_2(\mathbb{R}^n, (2\pi)^{-n}dx)$, but since it contains $\mathcal{S}(\mathbb{R}^n)$, which is dense in $L_2(\mathbb{R}^n, (2\pi)^{-n}dx)$, it must equal $L_2(\mathbb{R}^n, (2\pi)^{-n}dx)$. The identities in (5.14) extend by continuity. This shows 1°.

2°. Let $f \in L_2(\mathbb{R}^n)$, then $f_N = 1_{B(0,N)}f$ is clearly in $L_2(\mathbb{R}^n)$, and it is in $L_1(\mathbb{R}^n)$ by (A.25). We first show (5.15) for f_N. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L_2(\mathbb{R}^n)$, there is a sequence $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_j \to f_N$ in $L_2(\mathbb{R}^n)$ for $j \to \infty$. Then since $1_{\complement B(0,N)}$ is a bounded function, $1_{\complement B(0,N)}\varphi_j \to 1_{\complement B(0,N)} f_N = 0$ in $L_2(\mathbb{R}^n)$. With $\chi(x)$ as in (2.3), let $\psi_j = \chi(x/N)\varphi_j$; then also $\psi_j \to f_N$ in $L_2(\mathbb{R}^n)$ for $j \to \infty$. Since f_N and ψ_j are supported in $B(0, 2N)$, we have by the Cauchy-Schwarz inequality:

$$\|f_N - \psi_j\|_{L_1} \le \operatorname{vol}(B(0,2N))^{1/2}\|f_N - \psi_j\|_{L_2},$$

so ψ_j converges to f_N also in $L_1(\mathbb{R}^n)$. Now 1° above and Theorem 5.4 1° give that

$$\begin{aligned} &\mathcal{F}\psi_j \to \mathcal{F}f_N \quad \text{in } C_{L_\infty}(\mathbb{R}^n),\\ &\mathcal{F}\psi_j \to \mathcal{F}_2 f_N \quad \text{in } L_2(\mathbb{R}^n, (2\pi)^{-n}dx)\,. \end{aligned}$$

Then the limit in $C_{L_\infty}(\mathbb{R}^n)$ is a continuous representative of the limit in $L_2(\mathbb{R}^n, (2\pi)^{-n}dx)$. This shows (5.15) for f_N.

Using the second formula in (5.14) and the Lebesgue theorem we find:

$$(2\pi)^{-n}\|\mathcal{F}_2 f - \mathcal{F}_2 f_N\|_{L_2}^2 = \|f - f_N\|_{L_2}^2 \to 0 \quad \text{for } N \to \infty,$$

showing the convergence statement in 2°.

Finally, we obtain (5.15) in general: When $f \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$ and we define f_N by $f_N = 1_{B(0,N)}f$, then $f_N \to f$ in $L_1(\mathbb{R}^n)$ as well as in $L_2(\mathbb{R}^n)$ (by the Lebesgue theorem). Then $\mathcal{F}_2 f_N \to \mathcal{F}_2 f$ in $L_2(\mathbb{R}^n, (2\pi)^{-n}dx)$ and $\mathcal{F}f_N \to \mathcal{F}f$ in $C_{L_\infty}(\mathbb{R}^n)$, so the limits are the same as elements of $L_2(\mathbb{R}^n)$. $\square$

2° shows that the definition of $\mathcal{F}_2$ on L_2 is consistent with the definition of $\mathcal{F}$ on L_1, so we can drop the index 2, writing $\mathcal{F}$ instead of $\mathcal{F}_2$ from now on.

The isometry property can also be expressed in the way that *the operator*

$$F = (2\pi)^{-n/2}\mathcal{F}\colon L_2(\mathbb{R}^n, dx) \to L_2(\mathbb{R}^n, dx) \tag{5.16}$$

is an isometric isomorphism (i.e., F is a unitary operator in the Hilbert space $L_2(\mathbb{R}^n)$). It is because of this isometry property in connection with (5.9) that the L_2-theory for distributions is particularly useful for the treatment of partial differential equations.

Let $f = u$ and $g = \overline{\mathcal{F}v}$, then we have as a special case of (5.14) (using (5.10)):

$$\int \mathcal{F}u\, v\, dx = \int u\, \mathcal{F}v\, dx, \text{ for } u, v \in L_2(\mathbb{R}^n, dx). \tag{5.17}$$

The following rules for *convolution* are known from measure theory: When $f \in L_1(\mathbb{R}^n)$ and $g \in C_{L_\infty}(\mathbb{R}^n)$, then the convolution

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

is defined for all $x \in \mathbb{R}^n$. The function $f * g$ belongs to $C_{L_\infty}(\mathbb{R}^n)$, with

$$\|f * g\|_{L_\infty} \le \|f\|_{L_1} \|g\|_{L_\infty}\,. \tag{5.18}$$

For $f, g \in L_1(\mathbb{R}^n)$, the convolution $(f * g)(x)$ is defined for almost all $x \in \mathbb{R}^n$ and gives an element of $L_1(\mathbb{R}^n)$ (also denoted $f * g$), with

$$\|f * g\|_{L_1} \le \|f\|_{L_1} \|g\|_{L_1}\,. \tag{5.19}$$

The classical result on Fourier transformation of convolutions is:

Theorem 5.6. *When $f, g \in L_1(\mathbb{R}^n)$, then*

$$\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g\,. \tag{5.20}$$

Proof. We find by use of the Fubini theorem and a simple change of variables:

$$\begin{aligned}
\mathcal{F}(f * g)(\xi) &= \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\Big(\int_{\mathbb{R}^n} f(x-y)g(y)dy\Big)\,dx \\
&= \int_{\mathbb{R}^n} g(y)\Big(\int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x-y)dx\Big)\,dy \\
&= \int_{\mathbb{R}^n} g(y)\Big(\int_{\mathbb{R}^n} e^{-i\xi\cdot(x+y)} f(x)dx\Big)\,dy \\
&= \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx \int_{\mathbb{R}^n} e^{-i\xi\cdot y} g(y)\,dy = \mathcal{F}f(\xi)\mathcal{F}g(\xi). \quad \square
\end{aligned}$$

Observe furthermore:

Lemma 5.7. *When φ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, then $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$, and $\psi \mapsto \varphi * \psi$ is a continuous operator on $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Since φ and ψ belong to $L_1(\mathbb{R}^n) \cap C_{L_\infty}(\mathbb{R}^n)$, the rules (5.18) and (5.19) show that $\varphi * \psi \in L_1(\mathbb{R}^n) \cap C_{L_\infty}(\mathbb{R}^n)$. Since $\mathcal{F}(\varphi * \psi) = (\mathcal{F}\varphi) \cdot (\mathcal{F}\psi)$, $\mathcal{F}(\varphi * \psi)$ belongs to $\mathcal{S}(\mathbb{R}^n)$. Since $\varphi * \psi$ is continuous and $\mathcal{F}$ is injective from $L_1(\mathbb{R}^n)$ to $C_{L_\infty}(\mathbb{R}^n)$, mapping $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$, we find that $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$. It is seen from the formula $\varphi * \psi = \mathcal{F}^{-1}((\mathcal{F}\varphi) \cdot (\mathcal{F}\psi))$ that the map $\psi \mapsto \varphi * \psi$ is continuous. $\square$

5.2. Temperate distributions.

Definition 5.8. $\mathcal{S}'(\mathbb{R}^n)$ *(also written as $\mathcal{S}'$) is defined as the vector space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. The elements of $\mathcal{S}'(\mathbb{R}^n)$ are called* **temperate distributions**.[1]

According to Lemma B.7, $\mathcal{S}'$ consists of the linear functionals Λ on $\mathcal{S}$ for which there exists an $M \in \mathbb{N}_0$ and a constant C_M (depending on Λ) such that

$$|\Lambda(\varphi)| \leq C_M p_M(\varphi), \text{ for all } \varphi \in \mathcal{S}. \tag{5.21}$$

$\mathcal{S}'(\mathbb{R}^n)$ is provided with the weak* topology (as around (3.28)); this makes $\mathcal{S}'(\mathbb{R}^n)$ a topological vector space. (Its dual space is $\mathcal{S}(\mathbb{R}^n)$, cf. Section 3.5.)

Note that Definition 5.8 does not require introduction of $\mathcal{LF}$ spaces etc. (Section B.2), but is based solely on the concept of Fréchet spaces. However, it is of interest to set $\mathcal{S}'(\mathbb{R}^n)$ in relation to $\mathcal{D}'(\mathbb{R}^n)$, in particular to justify the use of the word "distribution" in this connection. We first show:

Lemma 5.9. $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ *is a dense subset of $\mathcal{S}(\mathbb{R}^n)$, with a stronger topology.*

Proof. As already noted, $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, and the neighborhood basis (5.5) for $\mathcal{S}$ at zero intersects C_0^∞ with open neighborhoods of 0 there, so that the topology induced on C_0^∞ from $\mathcal{S}$ is weaker than the original toplogy on C_0^∞.

To show the denseness, let $u \in \mathcal{S}(\mathbb{R}^n)$; then we must show that there is a sequence $u_N \to u$ in $\mathcal{S}(\mathbb{R}^n)$ with $u_N \in C_0^\infty(\mathbb{R}^n)$. For this we take

$$u_N(x) = \chi(x/N)u(x) \ , \tag{5.22}$$

[1] The word "temperate" used for the special distributions alludes to the temperate zone (with moderate temperature), the word can also mean "exercising moderation and self-restraint". The word "tempered" is also often used, but it has more to do with temper (mood), or can indicate a modification of a physical condition. The word temperate is used in Hörmander's books [H83], [H85].

cf. (2.3). Here $u_N \in C_0^\infty(\mathbb{R}^n)$; $u_N(x)$ equals $u(x)$ for $|x| \le N$ and equals 0 for $|x| \ge 2N$. Note that (as used also in the proof of Theorem 4.10)

$$|D_x^\beta \chi(x/N)| = |N^{-|\beta|} D_y^\beta \chi(y)|_{y=x/N}| \le C_\beta N^{-|\beta|} \tag{5.23}$$

for each β; here $D_x^\beta \chi(x/N)$ has support in $\{\, x \mid N \le |x| \le 2N \,\}$ when $\beta \neq 0$. For each $M \ge 0$, each α, we have:

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n} |\langle x\rangle^M D^\alpha[(1-\chi(x/N))u(x)]| = \sup_{|x|\ge N} |\langle x\rangle^M D^\alpha[(1-\chi(x/N))u(x)]| \\
&\le \langle N\rangle^{-1} \sup_{|x|\ge N} |\langle x\rangle^{M+1}((\chi(x/N)-1)D^\alpha u + \sum_{0\ne\beta\le\alpha} \tbinom{\alpha}{\beta} D^\beta\chi(x/N) D^{\alpha-\beta}u)| \\
&\le C_{M,\alpha}\langle N\rangle^{-1} \to 0 \text{ for } N \to \infty.
\end{aligned} \tag{5.24}$$

It follows that $\chi(x/N)u \to u$ in $\mathcal{S}$ for $N \to \infty$. $\square$

In particular, a functional $\Lambda \in \mathcal{S}'(\mathbb{R}^n)$ restricts to a continuous functional on $\mathcal{D}(\mathbb{R}^n)$, also documented by the fact that (5.21) implies, when φ is supported in a compact set K:

$$\begin{aligned}
|\Lambda(\varphi)| \le C_M p_M(\varphi) &= C_M \sup\{\, \langle x\rangle^M |D^\alpha\varphi(x)| \bigm| x \in \mathbb{R}^n,\ |\alpha| \le M \,\} \\
&\le C'_M \sup\{\, |D^\alpha\varphi(x)| \bigm| x \in K,\ |\alpha| \le M \,\},
\end{aligned} \tag{5.24a}$$

with $C'_M = C_M \sup_{x\in K}\langle x\rangle^M$.

Theorem 5.9a. *The map $J\colon \Lambda \mapsto \Lambda'$ from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ defined by restriction of Λ to $\mathcal{D}(\mathbb{R}^n)$,*

$$\langle \Lambda', \varphi\rangle = \Lambda(\varphi) \textit{ for } \varphi \in \mathcal{D}(\mathbb{R}^n),$$

is injective, and hence allows an identification of $J\mathcal{S}'(\mathbb{R}^n)$ with a subspace of $\mathcal{D}'(\mathbb{R}^n)$, also called $\mathcal{S}'(\mathbb{R}^n)$:

$$\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)\,. \tag{5.26}$$

Proof. The map $J\colon \Lambda \mapsto \Lambda'$ is injective because of Lemma 5.9, since

$$\langle \Lambda', \varphi\rangle = 0 \quad \text{for all } \ \varphi \in C_0^\infty(\mathbb{R}^n) \tag{5.25}$$

implies that Λ is 0 on a dense subset of $\mathcal{S}$, hence equals the 0-functional (since Λ is continuous on $\mathcal{S}$). $\square$

When $\Lambda \in \mathcal{S}'$, we now write $\Lambda(\varphi)$ as $\langle \Lambda, \varphi\rangle$, also when $\varphi \in \mathcal{S}$. Note that the elements of $\mathcal{D}'(\mathbb{R}^n)$ that lie in $\mathcal{S}'(\mathbb{R}^n)$ are exactly those for which there exists an M and a C_M such that

$$|\Lambda(\varphi)| \le C_M p_M(\varphi) \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^n), \tag{5.26a}$$

independently of the support of φ. For, they are continuous on $\mathcal{D}$ with respect to the topology of $\mathcal{S}$, hence extend (in view of Lemma 5.9) to continuous functionals on $\mathcal{S}$.

One may observe that $J\colon \mathcal{S}' \to \mathcal{D}'$ is precisely *the adjoint* of the continuous injection $\iota\colon \mathcal{D} \to \mathcal{S}$.

Lemma 5.10. *For $1 \le p \le \infty$ and f in $L_p(\mathbb{R}^n)$, the map $\varphi \mapsto \int_{\mathbb{R}^n} f\varphi\,dx$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ defines a temperate distribution. In this way one has for each p a continuous injection of $L_p(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$.*

Proof. Denote the map $f \mapsto \int_{\mathbb{R}^n} f\varphi\,dx$ by Λ_f. Let as usual p' be given by $\frac{1}{p} + \frac{1}{p'} = 1$, with $1' = \infty$ and $\infty' = 1$. According to the Hölder inequality,

$$|\Lambda_f(\varphi)| \le \|f\|_{L_p} \|\varphi\|_{L_{p'}},$$

and here we have by Lemma 5.2:

$$\|\varphi\|_{L_{p'}} \begin{cases} \le C_M p_M(\varphi), \text{ if } p' < \infty,\ M > \frac{n}{p'}, \\ = p_0(\varphi), \text{ if } p' = \infty. \end{cases}$$

Hence Λ_f is a continuous functional on $\mathcal{S}$, and therefore belongs to $\mathcal{S}'$. Then Λ_f also defines an element of $\mathcal{D}'$, denoted $J\Lambda_f$ above. Since $J\Lambda_f(\varphi) = \int_{\mathbb{R}^n} f\varphi\,dx$ for $\varphi \in C_0^\infty(\mathbb{R}^n)$, we know from the Du Bois-Reymond lemma that the map $f \mapsto J\Lambda_f$ is injective. Then $f \mapsto \Lambda_f$ must likewise be injective. $\square$

In particular, $\mathcal{S}(\mathbb{R}^n)$ is continuously injected into $\mathcal{S}'(\mathbb{R}^n)$.

Example 5.11. Here are some examples of elements of $\mathcal{S}'$.

1° Besides the already mentioned functions $u \in L_p(\mathbb{R}^n)$, $p \in [1,\infty]$, *all functions $v \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$ with $|v(x)| \le C\langle x\rangle^N$* for some N are in $\mathcal{S}'$. (Note that these functions include the $\mathcal{O}_M$-functions, in particular the polynomials, but they need not be differentiable.) We see this by observing that for such a function v,

$$\begin{aligned} |\langle v, \varphi\rangle| = |\int v\varphi dx| &\le C \int \langle x\rangle^{-n-1}\,dx\, \sup\{\langle x\rangle^{N+n+1}|\varphi(x)| \mid x \in \mathbb{R}^n\} \\ &\le C' p_{N+n+1}(\varphi), \text{ for } \varphi \in \mathcal{S}. \end{aligned}$$

2° The *δ-distribution and its derivatives $D^\alpha\delta$* are in $\mathcal{S}'$, since

$$|\langle D^\alpha\delta, \varphi\rangle| = |D^\alpha\varphi(0)| \le p_{|\alpha|}(\varphi), \text{ for } \varphi \in \mathcal{S}.$$

3° *Distributions with compact support.* We have shown earlier that a distribution u with compact support satisfies an estimate (3.34); the seminorm in this expression is $\le p_{N_j}$, so the distribution is a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$. Hence

$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)\,. \tag{5.27}$$

There exist distributions in $\mathcal{D}'(\mathbb{R}^n) \setminus \mathcal{S}'(\mathbb{R}^n)$. An example is e^x (for $n = 1$), cf. Exercise 5.1. In fact it grows too fast for $x \to \infty$; this illustrates the use of the word "temperate" in the name for $\mathcal{S}'$, indicating that the elements of $\mathcal{S}'$ grow in a controlled way for $|x| \to \infty$.

5.3. The Fourier transform on $\mathcal{S}'$.

The operations D^α and M_p (multiplication by p) for $p \in C^\infty(\mathbb{R}^n)$ are defined on $\mathcal{D}'(\mathbb{R}^n)$ (Definition 3.5). Concerning their action on $\mathcal{S}'$, we have:

Lemma 5.12.
1° *D^α maps $\mathcal{S}'$ continuously into $\mathcal{S}'$ for all $\alpha \in \mathbb{N}_0^n$.*
2° *When $p \in \mathcal{O}_M$, M_p maps $\mathcal{S}'$ continuously into $\mathcal{S}'$,*

Proof. For $\alpha \in \mathbb{N}_0^n$, $p \in \mathcal{O}_M$, we set

$$\begin{aligned} \langle D^\alpha u, \varphi\rangle &= \langle u, (-D)^\alpha \varphi\rangle\,, \\ \langle pu, \varphi\rangle &= \langle u, p\varphi\rangle, \text{ for } \varphi \in \mathcal{S}(\mathbb{R}^n). \end{aligned} \tag{5.28}$$

Because of the continuity of the maps in $\mathcal{S}$, it follows that these formulas define distributions in $\mathcal{S}'$, which agree with the original definitions on $\mathcal{D}'(\mathbb{R}^n)$. The continuity of the hereby defined maps in $\mathcal{S}'$ is shown in a similar way as in Theorem 3.8. $\square$

We find, similarly to Theorem 3.15:

Lemma 5.13. *For φ in $\mathcal{S}(\mathbb{R}^n)$ and u in $\mathcal{S}'(\mathbb{R}^n)$, the prescription*

$$\langle \varphi * u, \psi\rangle = \langle u, \check{\varphi} * \psi\rangle\,, \quad \psi \in \mathcal{S}(\mathbb{R}^n) \tag{5.29}$$

defines a temperate distribution $\varphi * u$; and $u \mapsto \varphi * u$ is a continuous operator in $\mathcal{S}'(\mathbb{R}^n)$. Moreover,

$$\begin{aligned} &D^\alpha(\varphi * u) = (D^\alpha \varphi) * u = \varphi * D^\alpha u, \text{ for } \varphi \in \mathcal{S},\ u \in \mathcal{S}',\ \alpha \in \mathbb{N}_0^n, \\ &(\varphi * \psi) * u = \varphi * (\psi * u), \text{ for } \varphi, \psi \in \mathcal{S},\ u \in \mathcal{S}'. \end{aligned} \tag{5.30}$$

Both in multiplication and in convolution formulas we shall henceforth allow the smooth factor to be written to the right, setting $u \cdot \varphi = \varphi \cdot u$ and $u * \varphi = \varphi * u$.

Remark 5.14. With the notation of Section 3.5, one finds by a bi-annihilator argument (as for $\mathcal{D}$ and $\mathcal{D}'$) that $\mathcal{S}(\mathbb{R}^n)$ is a dense subset of $\mathcal{S}'(\mathbb{R}^n)$. An operator A on $\mathcal{S}(\mathbb{R}^n)$ therefore has at most one extension to a continuous operator on $\mathcal{S}'(\mathbb{R}^n)$. It is then moreover seen that if a continuous operator A on $\mathcal{S}(\mathbb{R}^n)$ has a corresponding continuous operator B on $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} (A\varphi)\psi\,dz = \int_{\mathbb{R}^n} \varphi(B\psi)\,dx$ for all φ, ψ in $\mathcal{S}(\mathbb{R}^n)$, then A has a unique extension to a continuous operator on $\mathcal{S}'(\mathbb{R}^n)$, namely $B^\times$. Moreover, if the restrictions of A and B to $C_0^\infty(\mathbb{R}^n)$ map $C_0^\infty(\mathbb{R}^n)$ continuously into $C_0^\infty(\mathbb{R}^n)$, then the restriction to $\mathcal{S}'(\mathbb{R}^n)$ of the already defined extension of $A|_{C_0^\infty(\mathbb{R}^n)}$

to a $\sigma(\mathcal{D}'(\mathbb{R}^n)\,,\ C_0^\infty(\mathbb{R}^n))$ continuous operator on $\mathcal{D}'(\mathbb{R}^n)$ is precisely the extension of A to a $\sigma(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$ continuous operator on $\mathcal{S}'(\mathbb{R}^n)$.

The operators of differentiation, multiplication and convolution introduced on $\mathcal{S}'$ above can be considered from this point of view.

We shall finally introduce the very important *generalization of the Fourier transform* and show how it interacts with the other maps:

Definition 5.15. *For $u \in \mathcal{S}'$, the prescription*

$$\langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle \quad \textit{for all} \;\; \varphi \in \mathcal{S} \tag{5.31}$$

defines a temperate distribution $\mathcal{F}u$ (also denoted $\hat{u}$); and $\mathcal{F}\colon u \mapsto \mathcal{F}u$ is a continuous operator on $\mathcal{S}'(\mathbb{R}^n)$. We also define $F = (2\pi)^{-n/2}\mathcal{F}$.

The definition is of course chosen such that it is consistent with the formula (5.17) for the case where $u \in \mathcal{S}$. It is also consistent with the definition of $\mathcal{F}_2$ on $L_2(\mathbb{R}^n)$, since $\varphi_k \to u$ in $L_2(\mathbb{R}^n)$ implies $\mathcal{F}\varphi_k \to \mathcal{F}_2 u$ in $\mathcal{S}'$. Similarly, the definition is consistent with the definition on $L_1(\mathbb{R}^n)$. That $\mathcal{F}$ is a *continuous* operator on $\mathcal{S}'$ is seen as in Theorem 3.8 or by use of Remark 5.14.

The operator $\overline{\mathcal{F}}$ is similarly extended to $\mathcal{S}'$, on the basis of the identity

$$\langle \overline{\mathcal{F}}u, \varphi \rangle = \langle u, \overline{\mathcal{F}}\varphi \rangle; \tag{5.32}$$

and since

$$(2\pi)^{-n}\overline{\mathcal{F}}\mathcal{F} = (2\pi)^{-n}\mathcal{F}\overline{\mathcal{F}} = I \tag{5.33}$$

on $\mathcal{S}$, this identity is likewise carried over to $\mathcal{S}'$, so we obtain:

Theorem 5.16. *$\mathcal{F}$ is a homeomorphism of $\mathcal{S}'$ onto $\mathcal{S}'$, with inverse $\mathcal{F}^{-1} = (2\pi)^{-n}\overline{\mathcal{F}}$.*

This extension of $\mathcal{F}$ to an operator on $\mathcal{S}'$ gives an enormous freedom in the use of the Fourier transform. We obtain directly from the theorems for $\mathcal{F}$ on $\mathcal{S}$, Lemma 5.9 and the definitions of the generalized operators:

Theorem 5.17. *For all $u \in \mathcal{S}'$, one has when $\alpha \in \mathbb{N}_0^n$ and $\varphi \in \mathcal{S}$:*

$$\begin{aligned}
&\text{(i)} \quad \mathcal{F}(D^\alpha u) = \xi^\alpha \mathcal{F}u,\\
&\text{(ii)} \quad \mathcal{F}(x^\alpha u) = (-D_\xi)^\alpha \mathcal{F}u,\\
&\text{(iii)} \quad \mathcal{F}(\varphi * u) = (\mathcal{F}\varphi)\cdot(\mathcal{F}u),\\
&\text{(iv)} \quad \mathcal{F}(\varphi \cdot u) = (2\pi)^{-n}(\mathcal{F}\varphi) * (\mathcal{F}u).
\end{aligned} \tag{5.34}$$

Let us study some special examples.

For $u = \delta$,

$$\langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle = \hat{\varphi}(0) = \int \varphi(x)\, dx = \langle 1, \varphi \rangle \ , \ \text{for } \varphi \in \mathcal{S},$$

hence

$$\mathcal{F}[\delta] = 1 \ . \tag{5.35}$$

Since clearly also $\overline{\mathcal{F}}[\delta] = 1$ (cf. (5.32)), we get from the inversion formula (5.33) that

$$\mathcal{F}[1] = (2\pi)^n \delta \ . \tag{5.36}$$

An application of Theorem 5.17 then gives:

$$\begin{aligned} \mathcal{F}[D^\alpha \delta] &= \xi^\alpha \\ \mathcal{F}[(-x)^\alpha] &= (2\pi)^n D_\xi^\alpha \delta \ . \end{aligned} \tag{5.37}$$

Remark 5.18. We have shown that $\mathcal{F}$ defines a homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$, of L_2 onto L_2 and of $\mathcal{S}'$ onto $\mathcal{S}'$. One can ask for the image by $\mathcal{F}$ of other spaces. For example, $\mathcal{F}(C_0^\infty(\mathbb{R}^n))$ must be a certain subspace of $\mathcal{S}$; but this is *not* contained in $C_0^\infty(\mathbb{R}^n)$. On the contrary, if $\varphi \in C_0^\infty(\mathbb{R}^n)$, then $\hat{\varphi}$ can only have compact support if $\varphi = 0$! For $n = 1$ we can give a quick explanation of this: When $\varphi \in C_0^\infty(\mathbb{R})$, then $\hat{\varphi}(\zeta)$ can be defined for *all* $\zeta \in \mathbb{C}$ by the formula

$$\hat{\varphi}(\zeta) = \int_{\operatorname{supp}\varphi} e^{-ix\zeta} \varphi(x) dx \ ,$$

and this function $\hat{\varphi}(\zeta)$ is *holomorphic* in $\zeta = \xi + i\eta \in \mathbb{C}$, since $(\partial_\xi + i\partial_\eta)\hat{\varphi}(\xi + i\eta) = 0$ (the Cauchy-Riemann equation), as is seen by differentiation under the integral sign. (One could also appeal to Morera's Theorem.) Now if $\hat{\varphi}(\zeta)$ is identically 0 on an open, nonempty interval of the real axis, then $\hat{\varphi} = 0$ everywhere. The argument may be extended to $n > 1$.

Even for *distributions* u with compact support, $\hat{u}(\zeta)$ is a *function* of ζ which can be defined for all $\zeta \in \mathbb{C}^n$. For, one can show that $\hat{u}$ coincides with the function

$$\hat{u}(\zeta) = \langle u, \psi(x) e^{-ix\cdot\zeta} \rangle \ \Big[= \langle \underset{\mathcal{E}'}{u}, \underset{\mathcal{E}}{e^{-ix\cdot\zeta}} \rangle \Big], \tag{5.38}$$

where $\psi(x)$ is a function $\in C_0^\infty(\mathbb{R}^n)$ which is 1 on a neighborhood of $\operatorname{supp} u$. It is seen as in Exercise 3.14 that this function $\hat{u}(\zeta)$ is C^∞ as a function of $(\xi_1, \eta_1, \dots, \xi_n, \eta_n) \in \mathbb{R}^{2n}$ ($\zeta_j = \xi_j + i\eta_j$), with

$$\partial_{\xi_j} \hat{u}(\zeta) = \langle u, \psi(x) \partial_{\xi_j} e^{-ix\cdot\zeta} \rangle,$$

and similarly for ∂_{η_j}. Since $e^{-ix\cdot\zeta}$ satisfies the Cauchy-Riemann equation in each complex variable ζ_j, so does $\hat{u}(\zeta)$, so $\hat{u}(\zeta)$ is a holomorphic function of

$\zeta_j \in \mathbb{C}$ for each j. Then it follows also here that the support of $\hat{u}(\zeta)$ cannot be compact unless $u = 0$.

The spaces of holomorphic functions obtained by applying $\mathcal{F}$ to $C_0^\infty(\mathbb{R}^n)$ resp. $\mathcal{E}'(\mathbb{R}^n)$ may be characterized by their growth properties in ζ (the Paley-Wiener Theorem, see e.g. the book of W. Rudin [R74], Theorems 7.22, 7.23, or the book of L. Hörmander [H63], Theorem 1.7.7).

For partial differential operators with constant coefficients, the Fourier transform gives a remarkable simplification. When

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha \tag{5.39}$$

is a differential operator on $\mathbb{R}^n$ with coefficients $a_\alpha \in \mathbb{C}$, the equation

$$P(D)u = f \tag{5.40}$$

(with u and $f \in \mathcal{S}'$) is by Fourier transformation carried over to the multiplication equation

$$p(\xi)\hat{u}(\xi) = \hat{f}(\xi) \tag{5.41}$$

where $p(\xi)$ is the polynomial

$$p(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha \,; \tag{5.42}$$

it is called *the symbol of* $P(D)$.

The m'th order part of $P(D)$ is called *the principal part* (often denoted $P_m(D)$), and its associated symbol p_m *the principal symbol*, i.e.,

$$P_m(D) = \sum_{|\alpha| = m} a_\alpha D^\alpha \;, \quad p_m(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha \;. \tag{5.43}$$

It is often so that it is the principal part that determines the solvability properties of (5.40). The operator $P(D)$ is in particular called *elliptic* if $p_m(\xi) \ne 0$ for $\xi \ne 0$. Note that $p_m(\xi)$ is a homogeneous polynomial in ξ of degree m.

Example 5.19. ("The world's simplest example") Consider the operator $P = 1 - \Delta$ on $\mathbb{R}^n$. By Fourier transformation, the equation

$$(1 - \Delta)u = f \quad \text{on } \mathbb{R}^n \tag{5.44}$$

is carried into the equation

$$(1+|\xi|^2)\hat u = \hat f \quad \text{on } \ \mathbb{R}^n\ , \tag{5.45}$$

and this leads by division with $1+|\xi|^2 = \langle\xi\rangle^2$ to

$$\hat u = \langle\xi\rangle^{-2}\hat f\ .$$

Thus (5.44) has the solution

$$u = \mathcal{F}^{-1}(\langle\xi\rangle^{-2}\mathcal{F}f).$$

We see that for any f given in $\mathcal{S}'$ there is one and only one solution $u \in \mathcal{S}'$, and if f belongs to $\mathcal{S}$ then the solution u belongs to $\mathcal{S}$. When f is given in $L_2(\mathbb{R}^n)$, we see from (5.45) that $(1+|\xi|^2)\hat u(\xi) \in L_2$. This implies not only that $u \in L_2$ (since $\hat u \in L_2$), but even that $D_j u$ and $D_iD_ju \in L_2$ for $i,j = 1,\dots,n$. Indeed, $\xi_j\hat u$ and $\xi_i\xi_j\hat u$ are in L_2 since $|\xi_j| \le 1+|\xi|^2$ and $|\xi_i\xi_j| \le \frac12(|\xi_i|^2+|\xi_j|^2) \le |\xi|^2$; here we have used the elementary inequality

$$2ab \le a^2+b^2 \quad \text{for} \quad a,b\in\mathbb{R}\ , \tag{5.46}$$

which follows from $(a-b)^2 \ge 0$.

Thus we obtain:

$$u\in\mathcal{S}'(\mathbb{R}^n) \ \text{ with }\ (1-\Delta)u\in L_2(\mathbb{R}^n) \implies u\in H^2(\mathbb{R}^n)\ . \tag{5.47}$$

Conversely, it is clear that $u\in H^2(\mathbb{R}^n) \implies (1-\Delta)u\in L_2(\mathbb{R}^n)$, so that in fact,

$$u\in H^2(\mathbb{R}^n) \iff (1-\Delta)u\in L_2(\mathbb{R}^n)\ . \tag{5.48}$$

In particular, the maximal operator in $L_2(\mathbb{R}^n)$ for $A = 1-\Delta$ has $D(A_{\max}) = H^2(\mathbb{R}^n)$. This resembles to some extent what we found for ordinary differential operators in Section 4.3, and it demonstrates clearly the usefulness of the Fourier transform. Note that our estimates show that the graph norm on $D(A_{\max})$ is equivalent with the $H^2(\mathbb{R}^n)$-norm. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $H^2(\mathbb{R}^n)$ (Corollary 4.11), we see that $A_{\min} = A_{\max}$ here, and the operator is selfadjoint as an unbounded operator in $L_2(\mathbb{R}^n)$. (More on maximal and minimal operators on $\mathbb{R}^n$ in Theorem 6.3ff below.)

It should be noted that $1-\Delta$ is an unusually "nice" operator, since the polynomial $1+|\xi|^2$ is positive everywhere. As soon as there are zeroes, the theory becomes more complicated. For example, the wave operator $\partial_t^2-\Delta_x$ on $\mathbb{R}^{n+1}$ with symbol $-\tau^2+|\xi|^2$ requires rather different techniques. Even for the Laplace operator Δ, whose symbol $-|\xi|^2$ has just one zero $\xi = 0$, it is less simple to discuss exact solutions.

At any rate, the Laplace operator is elliptic, and it is fairly easy to show qualitative properties of the solutions of the equation $-\Delta u = f$ by use of the Fourier transform. We return to this and a further discussion of differential operators in Chapter 6. First we shall study some properties of the Fourier transform which for example lead to exact results for the equation $-\Delta u = f$.

5.4. Homogeneity.

When calculating the Fourier transform of specific functions, one can sometimes profit from symmetry properties. We shall give some useful examples.

The idea is to use the interaction of the Fourier transform with suitable coordinate changes; here we take the *orthogonal transformations* $y = Ox$ and the *dilations* $y = \lambda x (= \mu_\lambda(x))$, described in Example 3.21. As mentioned there, the associated maps are given by

$$\begin{aligned} [T(O)u](y) &= u(O^{-1}y) = u(x)\ , \quad \text{when } \ y = Ox\ , \\ [T(\mu_\lambda)u](y) &= u(y/\lambda) = u(x)\ , \quad \text{when } \ y = \lambda x\ ; \end{aligned} \tag{5.49a}$$

they clearly map $\mathcal{S}$ into $\mathcal{S}$ and $\mathcal{S}'$ into $\mathcal{S}'$ (where they are interpreted as in (3.58) and (3.56)). For test functions $\psi \in \mathcal{S}$ we now find (using that $(O^*)^{-1} = O$):

$$\begin{aligned} \mathcal{F}[T(O)\psi](\xi) &= \int e^{-iy\cdot\xi}\psi(O^{-1}y)dy \\ &= \int e^{-iOx\cdot\xi}\psi(x)dx = \int e^{-ix\cdot O^*\xi}\psi(x)dx \\ &= \mathcal{F}[\psi](O^*\xi) = [T((O^*)^{-1})\hat{\psi}](\xi) = [T(O)\hat{\psi}](\xi)\ ; \end{aligned} \tag{5.49}$$

$$\begin{aligned} \mathcal{F}[T(\mu_\lambda)\psi](\xi) &= \int e^{-iy\cdot\xi}\psi(y/\lambda)dy \\ &= \int e^{-i\lambda x\cdot\xi}\psi(x)|\lambda^n|dx = |\lambda^n|\mathcal{F}[\psi](\lambda\xi) = [|\lambda^n|T(\mu_{1/\lambda})\hat{\psi}](\xi)\ . \end{aligned} \tag{5.50}$$

This leads to the general rules for $u \in \mathcal{S}'$:

$$\begin{aligned}
\langle \mathcal{F}[T(O)u], \psi\rangle &= \langle T(O)u, \mathcal{F}\psi\rangle \\
&= \langle u, T(O^{-1})\mathcal{F}\psi\rangle = \langle u, \mathcal{F}[T(O^*)\psi]\rangle \\
&= \langle \mathcal{F}u, T(O^*)\psi\rangle = \langle T((O^*)^{-1})\mathcal{F}u, \psi\rangle = \langle T(O)\mathcal{F}u, \psi\rangle \ ; \\
\langle \mathcal{F}[T(\mu_\lambda)u], \psi\rangle &= \langle T(\mu_\lambda)u, \mathcal{F}\psi\rangle \\
&= \langle u, |\lambda^n| T(\mu_{1/\lambda})\mathcal{F}\psi\rangle = \langle u, \mathcal{F}[T(\mu_\lambda)\psi]\rangle \\
&= \langle \mathcal{F}u, T(\mu_\lambda)\psi\rangle = \langle |\lambda^n| T(\mu_{1/\lambda})\mathcal{F}u, \psi\rangle \ .
\end{aligned}$$

(The rules could also have been obtained from (5.49), (5.50) by extension by continuity, cf. Remark 5.14.) We have shown:

Theorem 5.20. *Let O be an orthogonal transformation in $\mathbb{R}^n$ and let μ_λ be the multiplication by the scalar $\lambda \in \mathbb{R}\setminus\{0\}$. The associated coordinate change maps $T(O)$ and $T(\mu_\lambda)$ in $\mathcal{S}'$ are connected with the Fourier transform in the following way:*

$$\mathcal{F}[T(O)u] = T((O^*)^{-1})\mathcal{F}u = T(O)\mathcal{F}u \ , \tag{5.51}$$

$$\mathcal{F}[T(\mu_\lambda)u] = |\lambda^n| T(\mu_{1/\lambda})\mathcal{F}u \ , \tag{5.52}$$

for $u \in \mathcal{S}'$.

The theorem is used in the treatment of functions with special invariance properties under such coordinate changes.

Those functions $u(x)$ which only depend on the distance $|x|$ to 0, may be characterized as the functions that are *invariant under all orthogonal transformations*, i.e. for which

$$T(O)u = u \ \text{ for all orthogonal transformations } \ O \tag{5.53}$$

(since the orthogonal transformations are exactly those transformations in $\mathbb{R}^n$ which preserve $|x|$). We shall analogously for $u \in \mathcal{S}'(\mathbb{R}^n)$ say that u depends only on the distance $|x|$ to 0 when (5.53) holds.

A function is *homogeneous of degree* r, when $u(ax) = a^r u(x)$ holds for all $a > 0$ and all $x \in \mathbb{R}^n \setminus \{0\}$, that is,

$$T(\mu_{1/a})u = a^r u \ , \quad \text{for all } \ a > 0 \ . \tag{5.54}$$

We say analogously that a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree r when (5.54) holds.

Theorem 5.20 easily implies:

Corollary 5.21. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$, and let $r \in \mathbb{R}$.*

1° *If u only depends on the distance $|x|$ to 0, then the same holds for $\hat{u}$.*

2° *If u is homogeneous of degree r, then $\hat{u}$ is homogeneous of degree $-r-n$.*

Proof. 1°. The identities (5.53) carry over to similar identities for $\hat{u}$ according to (5.51).

2°. When u is homogeneous of degree r, then we have according to (5.52) and (5.54):

$$T(\mu_{1/a})\mathcal{F}u = a^{-n}\mathcal{F}[T(\mu_a)u] = a^{-n}\mathcal{F}[a^{-r}u] = a^{-n-r}\mathcal{F}u \,,$$

which shows that $\mathcal{F}u$ is homogeneous of degree $-n-r$. □

Let us apply the theorem to the functions $u(x) = |x|^{-r}$, which have both properties: They are homogeneous of degree $-r$ and depend only on $|x|$.

Let $n/2 < r < n$; then u can be integrated into 0 whereas u^2 can be integrated out to ∞. Then we can write $u = \chi u + (1-\chi)u$, where $\chi u \in L_1$ and $(1-\chi)u \in L_2$. It follows that $\hat{u} \in C_{L_\infty}(\mathbb{R}^n) + L_2(\mathbb{R}^n) \subset L_{2,\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'$. We see from Corollary 5.21 that $\hat{u}(\xi)$ is a function which only depends on $|\xi|$ and is homogeneous of degree $r-n$.

To determine $\hat{u}$ more precisely, we shall consider the function $v(\xi)$ defined by

$$v(\xi) = |\xi|^{n-r}\hat{u}(\xi).$$

It is in $L_{2,\mathrm{loc}} \cap \mathcal{S}'$ (since $r < n$), and is *homogeneous of degree* 0 and depends only on $|\xi|$ (i.e., it is invariant under dilations and orthogonal transformations).

If v is known to be continuous on $\mathbb{R}^n \setminus \{0\}$, the invariance implies that $v(\xi) = v(\eta)$ for all points ξ and $\eta \in \mathbb{R}^n \setminus \{0\}$, so v equals a constant $c_{n,r}$ on $\mathbb{R}^n \setminus \{0\}$. Since v is a locally integrable function, it identifies with the constant function $c_{n,r}$ on $\mathbb{R}^n$. This gives the formula for $\hat{u}$:

$$\mathcal{F}(|x|^{-r}) = \hat{u}(\xi) = c_{n,r}|\xi|^{-n+r} \,. \tag{5.55}$$

We want to show this formula, but since we only know on beforehand that $v \in L_{2,\mathrm{loc}}$, an extra argument is needed. For example, one can reason as sketched in the following:

If a distribution f defined on a product set $Q_n = I^n \subset \mathbb{R}^n$ has $\partial_{x_1} f = \partial_{x_2} f = \cdots = \partial_{x_n} f = 0$ on Q_n, then it equals a constant, by Exercise 4.14. If a locally integrable function $g(x)$ is invariant under translations in the coordinate directions, we see from Exercise 3.13 that its first derivatives in the distribution sense are 0, so by the just mentioned result, it must equal a constant. We are almost in this situation with $v(\xi)$, except that v is invariant not under translations but under dilations and rotations (orthogonal transformations). But this can be carried over to the rectangular situation by a change of coordinates: Consider v on a conical neighborhood of a point $\xi_0 \neq 0$, say, and express it in terms of spherical coordinates there; then it is invariant under translation in the radial direction as well as in the directions orthogonal to this. Hence it must be constant there, in a neighborhood of every point, hence constant throughout.

Thus v equals a constant function c. Denoting the constant by $c_{n,r}$, we have obtained (5.55).

The constant $c_{n,r}$ is determined by suitable calculations (e.g. integration against $\exp(-|x|^2/2)$), and one finds that

$$c_{n,r} = (4\pi)^{n/2} 2^{-r} \frac{\Gamma(\frac{n-r}{2})}{\Gamma(\frac{r}{2})} \tag{5.56}$$

for $r \in]n/2, n[$ (we have this information from [R74], Exercise 8.6). It is seen directly that $c_{n,r}$ is real by observing that $\overline{\mathcal{F}}(|x|^{-r}) = \mathcal{F}(|-x|^{-r}) = \mathcal{F}(|x|^{-r})$.

When $r \in]0, n/2[$, then $r' = n - r$ lies in $]n/2, n[$, so we find from (5.55) using that $\mathcal{F}^{-1} = (2\pi)^{-n}\overline{\mathcal{F}}$:

$$|x|^{-n+r} = |x|^{-r'} = (2\pi)^{-n}\overline{\mathcal{F}}(c_{n,r'}|\xi|^{-n+r'}) = (2\pi)^{-n} c_{n,r'} \mathcal{F}(|\xi|^{-r});$$

this shows that (5.55) holds for $r \in]0, n/2[$, with

$$c_{n,r} = (2\pi)^n c_{n,n-r}^{-1} = (4\pi)^{n/2} 2^{-r} \frac{\Gamma(\frac{n-r}{2})}{\Gamma(\frac{r}{2})}. \tag{5.57}$$

So (5.56) is also valid here.

Since $c_{n,r}$ by (5.56) converges to $(2\pi)^{n/2}$ for $r \to n/2$, and $|x|^{-r}$ as well as $|x|^{-n+r}$ converge to $|x|^{-n/2}$ in $L_{1,\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'$ for $r \to n/2$, formula (5.55) is extended by passage to the limit to hold also for $r = n/2$. We have then obtained:

Theorem 5.22. *When $r \in]0, n[$, then*

$$\mathcal{F}(|x|^{-r}) = c_{n,r}|\xi|^{-n+r}\,, \tag{5.58}$$

where $c_{n,r}$ satisfies (5.57).

For $u(x) = |x|^{-r}$ with $r \geq n$, it is not evident how to interpret u as a distribution; there are special theories for this (see e.g. the definition of the "principal value" in [S50], and a general theory of homogeneous distributions in Section 3.2 of [H83]). See also Section 5.6.

Important special cases of Theorem 5.22 are the formulas

$$|\xi|^{-2} = \mathcal{F}\Big(\frac{1}{4\pi|x|}\Big) \text{ for } n=3\,; \quad |\xi|^{-2} = \mathcal{F}\Big(\frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}|x|^{n-2}}\Big) \text{ for } n\ge 3. \tag{5.59}$$

5.5. Application to the Laplace operator.

The preceding results make it possible to treat the equation for the Laplace operator

$$-\Delta u = f, \tag{5.60}$$

for a reasonable class of functions.

When u and f are temperate distributions, the equation gives by Fourier transformation:

$$|\xi|^2\hat{u} = \hat{f}\ .$$

A solution may then be written (provided that we can give it a meaning)

$$\hat{u}(\xi) = |\xi|^{-2}\hat{f}(\xi)\ . \tag{5.61}$$

If f is given in $\mathcal{S}$, we can use (5.34 iv) and (5.59) for $n \ge 3$, which gives:

$$\begin{aligned} u(x) &= \mathcal{F}^{-1}(|\xi|^{-2}) * f \\ &= \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}\int \frac{f(y)}{|x-y|^{n-2}}\,dy = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}\int \frac{f(x-y)}{|y|^{n-2}}\,dy\ , \end{aligned} \tag{5.62}$$

so this is a solution of (5.60). The solution u is a C^∞-function, since differentiation can be carried under the integral sign. There are many other solutions, namely all functions $u+w$ where w runs through the solutions of $\Delta w = 0$, the *harmonic functions* (which span an infinitely dimensional vector space; already the harmonic *polynomials* do so).

The function $\frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}|x|^{-n+2}$ is called the Newton potential. Once we have the formula (5.62), we can try to use it for more general f and thereby extend the applicability. For example, if we insert a continuous function with compact support as f, then we get a function u, which is not always two times differentiable in the classical sense, but still for bounded open sets Ω can be shown to belong to $H^2(\Omega)$ and solve (5.60) in the distribution sense (in fact it solves (5.60) as an H^2-function). See also Remark 6.11 later.

This solution method may in fact be extended to distributions $f \in \mathcal{E}'(\mathbb{R}^n)$, but this requires a generalization of the convolution operator which we refrain from including here.

The operator $-\Delta\colon u \mapsto f$ is *local*, in the sense that the shape of f in the neighborhood of a point depends only on the shape of u in a neighborhood of the point (this holds for all differential operators). On the other hand, the solution operator $T\colon f \mapsto u$ defined by (5.62) cannot be expected to be local (we see this explicitly from the expression for T as an integral operator).

Let Ω be a bounded open subset of $\mathbb{R}^n$. By use of T defined above, we define the operator T_Ω as the map that sends $\varphi \in C_0^\infty(\Omega)$ (extended by 0 in $\mathbb{R}^n \setminus \Omega$) into $(T\varphi)\big|_\Omega$, i.e.,

$$T_\Omega\colon \varphi \mapsto (T\varphi)\big|_\Omega \ \text{ for } \varphi \in C_0^\infty(\Omega)\ . \tag{5.63}$$

We here have that

$$(-\Delta T_\Omega \varphi)(x) = \varphi(x) \ \text{ for } x \in \Omega\ ,$$

because Δ is local. Thus T_Ω is a right inverse of $-\Delta$ on Ω. It is an integral operator

$$T_\Omega \varphi = \int_\Omega G(x,y)\varphi(y)\, dy\ , \tag{5.64}$$

with the kernel

$$G(x,y) = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}|x-y|^{-n+2},\ \text{ for } x,y \in \Omega\ .$$

An interesting question concerning this solution operator is whether it is a Hilbert-Schmidt operator (i.e., an integral operator whose kernel is in $L_2(\Omega\times\Omega)$). For this we calculate

$$\begin{aligned}\int_{\Omega\times\Omega} |G(x,y)|^2 dx\, dy &= c\int_{\Omega\times\Omega} |x-y|^{-2n+4} dx\, dy \\ &\le c' \int_{|z|,|w|\le R} |z|^{-2n+4} dz\, dw\ ,\end{aligned}$$

where we used the coordinate change $z = x-y$, $w = x+y$, and chose R so large that $\Omega \times \Omega \subset \{\, (x,y) \mid |x+y| \le R,\ |x-y| \le R \,\}$. The integral with respect to z in the last expression (and thereby the full integral) is finite if and only if $-2n+4 > -n$, i.e., $n < 4$. So T_Ω *is a Hilbert-Schmidt operator in* $L_2(\Omega)$, *when* $n = 3$ (in particular, a compact operator).

One can show more generally that $\overline{T}_\Omega$ for bounded Ω is a compact selfadjoint operator in $L_2(\Omega)$, for which the eigenvalue sequence $(\lambda_j(\overline{T}_\Omega))_{j\in\mathbb{N}}$ is in ℓ^p for $p > n/2$ ($\overline{T}_\Omega$ belongs to the p'th Schatten class; the Hilbert-Schmidt case is the case $p = 2$).

When Ω is unbounded, $\overline{T}_\Omega$ is in general not a compact operator in $L_2(\Omega)$ (unless Ω is very "thin").

5.6. Distributions associated with non-integrable functions.

We shall investigate some more types of distributions on $\mathbb{R}$ and their Fourier transforms. Theorem 5.20 treated homogeneous functions u of degree a, but the desire to have u and $\hat{u}$ in $L_{1,\mathrm{loc}}$ put essential restrictions on the values of a that could be covered. Now $n = 1$, so the calculations before Theorem 5.22 show how the cases $a \in\,]-1, 0[$ may be treated. This neither covers the case $a = 0$ (i.e., functions $u = c_1 H(x) + c_2 H(-x)$) nor the case $a = -1$ (where the functions $u = c_1 \dfrac{H(x)}{x} + c_2 \dfrac{H(-x)}{x}$ are not in $L_{1,\mathrm{loc}}$ in the neighborhood of 0 if $(c_1, c_2) \neq (0,0)$). We shall now consider these cases. One result is a description of the Fourier transform of the Heaviside function (which will allow us to treat the case $a = 0$), another result is that we give sense to a distribution which outside of 0 behaves like $\dfrac{1}{x}$.

When f is a function on $\mathbb{R}$ which is integrable on the intervals $]-\infty, -\varepsilon[$ and $[\varepsilon, \infty[$ for every $\varepsilon > 0$, then we define the *principal value integral* of f over $\mathbb{R}$ by

$$\mathrm{PV} \int_{\mathbb{R}} f(x) dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} f(x)\, dx \,, \tag{5.65}$$

when this limit exists. (It is important in the definition that the interval $[-\varepsilon, \varepsilon]$ is *symmetric* around 0; when $f \notin L_{1,\mathrm{loc}}(\mathbb{R})$ there is the risk of getting another limit by cutting out another interval like for example $[-\varepsilon, 2\varepsilon]$.) We now define the distributionen $\mathrm{PV}\,\dfrac{1}{x}$ by

$$\langle \mathrm{PV}\,\frac{1}{x}, \varphi \rangle = \mathrm{PV} \int_{\mathbb{R}} \frac{\varphi(x)}{x}\, dx \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}) \,. \tag{5.66}$$

(In some of the literature, this distribution is denoted $\mathrm{vp}\dfrac{1}{x}$, for "valeur principale".) We have to show that the functional in (5.66) is well-defined and continuous on $C_0^\infty(\mathbb{R})$. Here we use that by the Taylor formula,

$$\varphi(x) = \varphi(0) + x \cdot \varphi_1(x) \,, \tag{5.67}$$

where $\varphi_1(x) = \dfrac{\varphi(x) - \varphi(0)}{x}$ is in $C^\infty(\mathbb{R})$ (the reader should verify this). Moreover, we have for $x \in [-R, R]$, by the mean value theorem,

$$\sup_{|x| \le R} |\varphi_1(x)| = \sup_{|x| \le R} \Big| \frac{\varphi(x) - \varphi(0)}{x} \Big| = \sup_{|x| \le R} |\varphi'(\theta(x))| \le \sup_{|x| \le R} |\varphi'(x)| \,, \tag{5.68}$$

where $\theta(x)$ is a suitable point between 0 and x. This gives for $\mathrm{PV}\,\frac{1}{x}$, when $\operatorname{supp}\varphi \subset [-R,R]$,

$$\begin{aligned}\langle \mathrm{PV}\,\frac{1}{x},\varphi\rangle &= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x}\,dx \qquad (5.69)\\ &= \lim_{\varepsilon\to 0}\Big[\int_{[-R,-\varepsilon]\cup[\varepsilon,R]}\frac{\varphi(0)}{x}\,dx + \int_{[-R,-\varepsilon]\cup[\varepsilon,R]}\varphi_1(x)\,dx\Big]\\ &= \int_{-R}^{R}\varphi_1(x)\,dx\ ,\end{aligned}$$

since the first integral in the bracket is 0 because of the symmetry of $\frac{1}{x}$. Thus the functional $\mathrm{PV}\,\frac{1}{x}$ is well-defined, and we see from (5.68) that it is a distribution of order 1:

$$\begin{gathered}|\langle \mathrm{PV}\,\frac{1}{x},\varphi\rangle| \le 2R\sup_{|x|\le R}|\varphi_1(x)| \le 2R\sup_{|x|\le R}|\varphi'(x)|\,,\\ \text{when }\ \operatorname{supp}\varphi\subset[-R,R]\ .\end{gathered} \qquad (5.70)$$

Remark 5.23. One can also associate distributions with the other functions $\frac{1}{x^m}$, $m\in\mathbb{N}$; here one uses on one hand the principal value concept, on the other hand a modification of $\varphi(x)$ by a Taylor polynomial at 0; cf. Exercise 5.9. The resulting distributions are called $\mathrm{Pf}\,\frac{1}{x^m}$, where Pf stands for pseudo-function; for $m=1$ we have that $\mathrm{Pf}\,\frac{1}{x} = \mathrm{PV}\,\frac{1}{x}$.

Since $\mathrm{PV}\,\frac{1}{x} = \chi\,\mathrm{PV}\,\frac{1}{x} + (1-\chi)\frac{1}{x}$ has its first term in $\mathcal{S}'(\mathbb{R})$ and second term in $L_2(\mathbb{R})$, $v = \mathrm{PV}\,\frac{1}{x}$ belongs to $\mathcal{S}'$, hence has a Fourier transformed $\hat{v}$. We can find it in the following way: Observe that

$$x\cdot \mathrm{PV}\,\frac{1}{x} = 1 \qquad (5.71)$$

(using the definitions), so that $\hat{v}$ is a solution of the differential equation in $\mathcal{S}'$ (cf. (5.34) (ii) and (5.36))

$$i\partial_\xi \hat{v}(\xi) = 2\pi\delta\ . \qquad (5.72)$$

One solution of this equation is $-2\pi i H(\xi)$ (cf. (3.23)); all other solutions are of the form

$$-2\pi i H(\xi) + c, \qquad (5.72a)$$

where $c \in \mathbb{C}$, cf. Theorem 4.19. We then just have to determine the constant c. For this we observe that $\frac{1}{x}$ is an odd function and that $v = \mathrm{PV}\,\frac{1}{x}$ is an *odd distribution* (i.e., $\langle v, \varphi(-x)\rangle = -\langle v, \varphi(x)\rangle$ for all φ); then the Fourier transform $\hat{v}$ must likewise be odd.

Let us include a carefully elaborated explanation of what was just said: The reflection operator

$$S\colon \varphi(x) \mapsto \varphi(-x), \quad \varphi \in C_0^\infty(\mathbb{R}), \tag{5.73}$$

(for which we have also used the notation $\varphi \mapsto \check{\varphi}$) is a special case of a dilation (3.52), namely with $\lambda = -1$. Hence it carries over to distributions in the usual way (cf. (3.56)):

$$\langle Su, \varphi\rangle = \langle u, S\varphi\rangle \text{ for all } \varphi \in C_0^\infty(\mathbb{R}). \tag{5.74}$$

A function u is said to be *even* resp. *odd*, when $Su = u$ resp. $Su = -u$; this notation is now extended to distributions. For the connection with the Fourier transformation we observe that

$$\begin{aligned}(\mathcal{F}S\varphi)(\xi) &= \int e^{-ix\xi}\varphi(-x)dx \\ &= \int e^{iy\xi}\varphi(y)dy = (\overline{\mathcal{F}}\varphi)(\xi) = (\mathcal{F}\varphi)(-\xi) = (S\mathcal{F}\varphi)(\xi)\,,\end{aligned}$$

or in short:

$$\mathcal{F}S = \overline{\mathcal{F}} = S\mathcal{F}\,; \tag{5.75}$$

these formulas are carried over to distributions by use of (5.74) or (5.51). (The formula (5.33) could be written: $\mathcal{F}^2 = (2\pi)^n S$.) In particular, we see that $Sv = -v$ implies $S\hat{v} = -\hat{v}$.

The only odd function of the form (5.72a) is the one with $c = \pi i$, so:

$$\mathcal{F}[\mathrm{PV}\,\frac{1}{x}] = -2\pi i H(\xi) + \pi i = -\pi i \operatorname{sign}\xi; \tag{5.76}$$

cf. (3.25a) for $\operatorname{sign}\xi$. (It is possible to find $\hat{v}$ by direct calculations, but then one has to be very careful with the interpretation of convergences of the occurring integrals of functions not in L_1. We have avoided this by building up the Fourier transformation on $\mathcal{S}'$ by duality from the definition on $\mathcal{S}$.)

An application of $\frac{1}{2\pi}\overline{\mathcal{F}} = \frac{1}{2\pi}\mathcal{F}S$ (cf. (5.75)) to (5.76) gives

$$\begin{aligned}\mathrm{PV}\,\frac{1}{x} &= \frac{1}{2\pi}\mathcal{F}S(-2\pi i H(\xi) + \pi i) = \frac{1}{2\pi}\mathcal{F}(2\pi i H(\xi) - i\pi) \\ &= i\mathcal{F}H(\xi) - \frac{i}{2}\mathcal{F}[1] = i\mathcal{F}H - \pi i\delta\,.\end{aligned}$$

This leads to another interesting formula:

$$\mathcal{F}H(x) = -i\,\mathrm{PV}\,\frac{1}{\xi} + \pi\delta. \tag{5.77}$$

Using this formula, we can find the Fourier transforms of all homogeneous functions of degree 0.

Some further remarks: Corresponding to the decomposition

$$1 = H(x) + H(-x)$$

we now get the following decomposition of δ (which is used in theoretical physics)

$$\begin{aligned}\delta &= (2\pi)^{-1}\mathcal{F}[1] = (2\pi)^{-1}(\mathcal{F}H + \mathcal{F}SH) \qquad (5.78)\\ &= \Big(\frac{\delta}{2} + \frac{1}{2\pi i}\,\mathrm{PV}\,\frac{1}{x}\Big) + \Big(\frac{\delta}{2} - \frac{1}{2\pi i}\,\mathrm{PV}\,\frac{1}{x}\Big) = \delta_+ + \delta_- \ , \quad \text{where}\end{aligned}$$

$$\delta_\pm = \frac{\delta}{2} \pm \frac{1}{2\pi i}\,\mathrm{PV}\,\frac{1}{x} = \frac{1}{2\pi}\mathcal{F}[H(\pm x)]\ . \tag{5.79}$$

Observe also that since

$$H(x) = \lim_{a\to 0+} H(x)e^{-ax} \ \text{ in } \ \mathcal{S}'$$

one has that

$$\mathcal{F}H = \lim_{a\to 0+} \frac{1}{a+i\xi} \ \text{ in } \ \mathcal{S}' \tag{5.80}$$

(cf. Exercise 5.3), and then

$$\delta_+ = \frac{1}{2\pi}\lim_{a\to 0+}\frac{1}{a+ix} \ \text{ in } \mathcal{S}'\ . \tag{5.81}$$

Remark 5.24. To the non-integrable function $\dfrac{H(x)}{x}$ we associate the distribution $\mathrm{Pf}\,\dfrac{H(x)}{x}$, defined by

$$\langle \mathrm{Pf}\,\frac{H(x)}{x}, \varphi\rangle = \lim_{\varepsilon\to 0+}\Big[\int_\varepsilon^\infty \frac{\varphi(x)}{x}\,dx + \varphi(0)\log\varepsilon\Big]\ , \tag{5.82}$$

cf. [S61, Exercise II-14, p. 114-115]; note that there is a logarithmic correction. In this way, every function on $\mathbb{R}$ which is homogeneous of degree -1 is included in the distribution theory, namely as a distribution

$$c_1\,\mathrm{Pf}\,\frac{H(x)}{x} + c_2 S\,\mathrm{Pf}\,\frac{H(x)}{x}\ ,\ c_1 \ \text{ and } \ c_2 \in \mathbb{C}\ . \tag{5.83}$$

In particular, we define $\operatorname{Pf}\frac{1}{|x|}$ by

$$\operatorname{Pf}\frac{1}{|x|}=\operatorname{Pf}\frac{H(x)}{x}+S\operatorname{Pf}\frac{H(x)}{x}. \tag{5.84}$$

It is shown in Exercise 5.12 that

$$\mathcal{F}(\operatorname{Pf}\frac{1}{|x|})=-2\log|\xi|+C,$$

for some constant C.

Also for distributions on $\mathbb{R}^n$ there appear logarithmic terms, when one wants to include general homogeneous functions and their Fourier transforms in the theory. (A complete discussion of homogeneous distributions may be found in in [H83].)

Exercises for Chapter 5.

5.1. Let $n = 1$. Show that $e^x \notin \mathcal{S}'(\mathbb{R})$, whereas $e^x \cos(e^x) \in \mathcal{S}'(\mathbb{R})$. (*Hint.* Find an integral of $e^x \cos(e^x)$.)

5.2. Show the inequalities (5.2), and show that the systems of seminorms (5.3) and (5.4) define the same topology.

5.3. Let $a > 0$. With $H(t)$ denoting the Heaviside function, show that

$$\mathcal{F}[H(t)e^{-at}] = \frac{1}{a + i\xi} \,.$$

What is $\mathcal{F}[H(-t)e^{at}]$?

5.4. (a) Show that for $n = 1$,

$$\mathcal{F}^{-1}\Big[\frac{1}{1+\xi^2}\Big] = c\, e^{-|x|};$$

determine c. (One can use Exercise 5.3.)
(b) Show that for $n = 3$,

$$\mathcal{F}^{-1}\Big[\frac{1}{1+|\xi|^2}\Big] = \frac{c}{|x|}e^{-|x|},$$

with $c = \frac{1}{4\pi}$. (One may observe that the function is the unique solution v in $\mathcal{S}'$ of $(1 - \Delta)v = \delta$; or one can apply the rotation invariance directly.)

5.5. Let M and n be integers with $0 < 2M < n$. Find an integral operator T_M on $\mathcal{S}(\mathbb{R}^n)$ with the following properties:

(i) $\Delta^M T_M f = f$ for $f \in \mathcal{S}(\mathbb{R}^n)$.

(ii) When Ω is a bounded, open subset of $\mathbb{R}^n$, and $2M > n/2$, then the operator $(T_M)_\Omega$ (defined as in (5.63)) is a Hilbert-Schmidt operator in $L_2(\Omega)$.

5.6. Show that the differential equation on $\mathbb{R}^3$:

$$\frac{\partial^4 u}{\partial x_1^4} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2 \partial x_3} - \frac{\partial^2 u}{\partial x_3^2} + 3u = f$$

has one and only one solution $u \in \mathcal{S}'$ for each $f \in \mathcal{S}'$. Determine the values of $m \in \mathbb{N}_0$ for which u belongs to the Sobolev space $H^m(\mathbb{R}^3)$ when $f \in L_2(\mathbb{R}^3)$.

5.7. Let $a \in \mathbb{C}$, and show that the distribution $u = e^{-ax} H(x)$ is a solution of the differential equation

$$(\partial_x + a)u = \delta \quad \text{in } \mathcal{D}'(\mathbb{R}) \; .$$

Can we show this by Fourier transformation?

5.8. Show that the Cauchy-Riemann equation

$$\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) u(x,y) = f(x,y)$$

on $\mathbb{R}^2$ has a solution for each $f \in \mathcal{S}$; describe such a solution.

5.9. For $m \in \mathbb{N}$ and $\varphi \in C_0^\infty(\mathbb{R})$ we define the functional Λ_m by

$$\Lambda_m(\varphi) = \mathrm{PV} \int_{-\infty}^{\infty} \{ x^{-m}\varphi(x) - \sum_{p=0}^{m-2} \frac{x^{p-m}}{p!} \varphi^{(p)}(0) \} \, dx \; .$$

(a) Show that PV ... exists, so that $\Lambda_m(\varphi)$ is well-defined.
(b) Show that $\Lambda_m(\varphi') = m\Lambda_{m+1}(\varphi)$.
(c) Show that Λ_m is a distribution, and that

$$\Lambda_m = (-1)^{m-1}(m-1)\,! \, \frac{d^m}{dx^m} \log |x|.$$

Λ_m is also called $\mathrm{Pf}\, \dfrac{1}{x^m}$, where Pf stands for pseudo-function.

5.10. Let $I = \mathbb{R}$ or $I =]a, \infty[$. Show that when u and $D^m u \in L_2(I)$, then $u \in H^m(I)$. (One can show this for $I = \mathbb{R}$ by use of the Fourier transformation. Next, one can show it for $I =]a, \infty[$ by use of a cut-off function. This proves the assertion of Remark 4.21.)

5.11. Show that

$$\mathcal{F} \operatorname{sign} x = -2i \, \mathrm{PV} \, \frac{1}{\xi}.$$

5.12. Consider the locally integrable function $\log |x|$.
(a) Let $u = H(x) \log x$. Show that

$$\log |x| = u + Su.$$

(b) Show that

$$\frac{d}{dx}u = \operatorname{Pf}\frac{H(x)}{x}, \quad \frac{d}{dx}\log|x| = \operatorname{PV}\frac{1}{x}.$$

(c) Show that $x \operatorname{Pf}\frac{1}{|x|} = \operatorname{sign} x$, and that

$$\partial_\xi \mathcal{F}(\operatorname{Pf}\frac{1}{|x|}) = -2\operatorname{PV}\frac{1}{\xi};$$

cf. Exercise 5.11.

(d) Show that

$$\mathcal{F}(\operatorname{Pf}\frac{1}{|x|}) = -2\log|\xi| + C,$$

for some constant C.

(e) Show that

$$\mathcal{F}(\log|x|) = -\pi \operatorname{Pf}\frac{1}{|\xi|} + C_1\delta,$$

for some constant C_1.

(Information on the constant can be found in [S50, p. 258], [S61, Exercise V-10], where the Fourier transformation is normalized in a slightly different way.)