II. EXTENSIONS AND APPLICATIONS

§4. Realizations and Sobolev spaces

4.1. Realizations of differential operators.

There are various general methods to associate operators in Hilbert spaces to differential operators. We consider two types: the so-called "strong definitions" and the so-called "weak definitions." These definitions can be formulated without having distribution theory available, but in fact the weak definition is closely related to the ideas of distribution theory.

First we observe that nontrivial differential operators cannot be bounded in L^2-spaces. Just take the simplest example of $\frac{d}{dx}$ acting on functions on the interval $J = [0, 1]$. Let $f_n(x) = \frac{1}{n} \sin nx$, $n \in \mathbb{N}$. Then clearly $f_n(x) \to 0$ in $L_2(J)$ for $n \to \infty$, whereas the sequence $\frac{d}{dx} f_n(x) = \cos nx$ is not convergent in $L_2(J)$. Another example is $g_n(x) = x^n$, which goes to 0 in $L_2(J)$, whereas the L_2-norms of $g_n'(x) = nx^{n-1}$ go to ∞ for $n \to \infty$. So, at best, the differential operators can be viewed as suitable unbounded operators in L_2 spaces. A reader who is unfamiliar with unbounded operators in normed spaces should consult Chapter 12.

Let A be a differential operator of order m with C^∞ coefficients a_α on an open set $\Omega \subset \mathbb{R}^n$:

$$Au = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u \,. \tag{4.1}$$

When $u \in C^m(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, we find by integration by parts (using the notation (u, v) for $\int_\Omega u\bar{v}\,dx$ when one of the functions has compact support):

$$\begin{aligned}(Au, \varphi)_{L_2(\Omega)} &= \int_\Omega \sum_{|\alpha| \le m} (a_\alpha D^\alpha u)\, \overline{\varphi}\, dx \\ &= \int_\Omega u \sum_{|\alpha| \le m} \overline{D^\alpha(\overline{a}_\alpha \varphi)}\, dx = (u, A'\varphi)_{L_2(\Omega)},\end{aligned} \tag{4.2}$$

where we have defined A' by $A'v = \sum_{|\alpha| \le m} D^\alpha(\overline{a}_\alpha v)$. In (4.2), we used the second formula in (A.20) to carry all differentiations over to φ, using that the integral can be replaced by an integral over a bounded set Ω_1 with C^1-boundary chosen such that $\operatorname{supp}\varphi \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega$; then all contributions from the boundary are 0.

The operator A' is called *the formal adjoint* of A; it satisfies

$$A'v = \sum_{|\alpha|\le m} D^\alpha(\overline{a}_\alpha v) = \sum_{|\alpha|\le m} a'_\alpha(x) D^\alpha v, \tag{4.3}$$

for suitable functions $a'_\alpha(x)$ determined by the Leibniz formula (cf. (A.7)); note in particular that

$$a'_\alpha = \overline{a}_\alpha \quad \text{for } |\alpha| = m. \tag{4.4}$$

When $a_\alpha = a'_\alpha$ for all $|\alpha| \le m$, i.e., $A = A'$, we say that A is *formally selfadjoint.*

The formula $A = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ is regarded as a formal expression. In the following, we write $A|_M$ for an operator acting like A and defined on a set M of functions u for which Au has a meaning (that we specify in each case). As an elementary example, Au is defined classically as a function in $C_0^\infty(\Omega)$ when $u \in C_0^\infty(\Omega)$. We can then consider $A|_{C_0^\infty(\Omega)}$ as a densely defined operator in $L_2(\Omega)$. Similar operators can of course be defined for A'.

Now we introduce the following *realizations of* A *in* $L_2(\Omega)$:

Definition 4.1.

1° *The maximal realization* $A_{\max}$ *associated with* A *is defined as the operator acting like* A *in the distribution sense and having the domain*

$$D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}. \tag{4.5}$$

2° *The minimal realization* $A_{\min}$ *associated with* A *is defined as the operator*

$$A_{\min} = \textit{ the closure of } A|_{C_0^\infty(\Omega)} \textit{ as an operator in } L_2(\Omega)\ . \tag{4.6}$$

We shall justify point 2° in the following. Note that when $u \in L_2(\Omega)$, $u \in D(A_{\max})$ means that there exists a function $f \in L_2(\Omega)$ such that the distribution Au identifies with $f \in L_2(\Omega)$, i.e.,

$$\langle f, \overline{\varphi}\rangle = \langle u, \sum_{|\alpha|\le m} (-D)^\alpha(a_\alpha\overline{\varphi})\rangle, \text{ for all } \varphi \in C_0^\infty(\Omega). \tag{4.7}$$

Here $\sum_{|\alpha|\le m} (-D)^\alpha(a_\alpha\overline{\varphi}) = \overline{A'\varphi}$, so (4.7) can also be written:

$$(f, \varphi) = (u, A'\varphi) \text{ for all } \varphi \in C_0^\infty(\Omega). \tag{4.8}$$

This shows:

Lemma 4.2. *The operator* $A_{\max}$ *is the adjoint of the densely defined operator* $A'|_{C_0^\infty(\Omega)}$ *in* $L_2(\Omega)$*:*

$$A_{\max} = (A'|_{C_0^\infty(\Omega)})^*. \tag{4.9}$$

Note in particular that $A_{\max}$ is closed (cf. Lemma 12.4). We see from (4.2) that

$$A|_{C_0^\infty(\Omega)} \subset A_{\max};$$

so it follows that $A|_{C_0^\infty(\Omega)}$ is closable as an operator in $L_2(\Omega)$; hence its closure $A_{\min}$ introduced in Definition 4.1 2° is well-defined. Moreover,

$$A_{\min} \subset A_{\max}. \tag{4.10}$$

Since $A'_{\min}$ is the closure of $A'|_{C_0^\infty(\Omega)}$, and $A_{\max}$ is the Hilbert space adjoint of the latter operator, we see from the rules $T^* = \left(\overline{T}\right)^*$ and $T^{**} = \overline{T}$ (cf. Corollary 12.6):

Lemma 4.3. *The operators* $A_{\max}$ *and* $A'_{\min}$ *are the adjoints of one another, as unbounded operators in* $L_2(\Omega)$*.*

The definition of $A_{\max}$ is called a "weak" definition since it is based on duality. The definition of $A_{\min}$ is called a "strong" definition since the operator is obtained by closure of a classically defined operator.

Observe that $A_{\max}$ is the largest possible operator in $L_2(\Omega)$ associated to A by distribution theory. We have in particular that $D(A_{\max})$ is closed with respect to the graph norm $(\|u\|_{L_2}^2 + \|Au\|_{L_2}^2)^{\frac{1}{2}}$, and that $D(A_{\min})$ is the closure of $C_0^\infty(\Omega)$ in $D(A_{\max})$ with respect to the graph norm.

Whereas $A_{\max}$ is the largest operator in $L_2(\Omega)$ associated with A, $A_{\min}$ is the smallest closed restriction of $A_{\max}$ whose domain contains $C_0^\infty(\Omega)$. The operators $\widetilde{A}$ satisfying

$$A_{\min} \subset \widetilde{A} \subset A_{\max} \tag{4.11}$$

are called the *realizations of* A, here $A_{\min}$ and $A_{\max}$ are in themselves examples of realizations.

Note that if $A = A'$ (i.e., A is formally selfadjoint) *and* $D(A_{\max}) = D(A_{\min})$, then $A_{\max}$ is selfadjoint as an unbounded operator in $L_2(\Omega)$. This will often be the case when $\Omega = \mathbb{R}^n$.

In the treatment of differential equations involving A, one of the available tools is to apply results from functional analysis to suitable realizations of A. On one hand it is then important to find out when a realization can be constructed such that the functional analysis results apply, for example whether the operator is selfadjoint, or lower bounded, or variational (see

Chapter 12). On the other hand, one should at the same time keep track of how the realization corresponds to a concrete problem, for example whether it represents a specific boundary condition. This is an interesting interface between abstract functional analysis and concrete problems for differential operators.

Remark 4.4. One can also define other relevant "strong" realizations than $A_{\min}$. (The reader may skip this remark in a first reading.) For example, let

$$\begin{aligned} M_1 &= \{\, u \in C^m(\Omega) \mid u \text{ and } Au \in L_2(\Omega) \,\}, \\ M_2 = C^m_{L_2}(\Omega) &= \{\, u \in C^m(\Omega) \mid D^\alpha u \in L_2(\Omega) \text{ for all } |\alpha| \le m \,\}, \\ M_3 = C^m_{L_2}(\overline{\Omega}) &= \{\, u \in C^m(\overline{\Omega}) \mid D^\alpha u \in L_2(\Omega) \text{ for all } |\alpha| \le m \,\}. \end{aligned} \tag{4.12}$$

Then

$$A|_{C_0^\infty(\Omega)} \subset A|_{M_1} \subset A_{\max},$$

and, when the coefficient functions a_α are bounded,

$$A|_{C_0^\infty(\Omega)} \subset A|_{M_3} \subset A|_{M_2} \subset A|_{M_1} \subset A_{\max}.$$

Defining

$$A_{s_i} = \overline{A|_{M_i}}, \text{ for } i = 1, 2, 3,$$

we then have in general

$$A_{\min} \subset A_{s_1} \subset A_{\max}. \tag{4.13}$$

Furthermore,

$$A_{\min} \subset A_{s_3} \subset A_{s_2} \subset A_{s_1} \subset A_{\max}, \tag{4.14}$$

holds when the a_α's are bounded. For certain types of operators A and domains Ω one can show that some of these A_{s_i}'s coincide with each other or coincide with $A_{\max}$. This makes it possible to show properties of realizations by approximation from properties of classically defined operators.

In the one-dimensional case where Ω is an interval I, one will often find that $D(A_{\max})$ differs from $D(A_{\min})$ by a finite dimensional space (and that the three realizations A_{s_i} in the above remark coincide with $A_{\max}$). Then the various realizations represent various concrete boundary conditions which can be completely analyzed. (More about this in Section 4.3.) In the one-dimensional case, the weak definition of $\frac{d}{dx}$ is related to absolute continuity, as mentioned already in Chapter 1, and the large effort to introduce distributions is not strictly necessary.

But in higher dimensional cases ($n \ge 2$) where Ω is different from $\mathbb{R}^n$, there will usually be an infinite dimensional difference between $D(A_{\max})$ and

$D(A_{\min})$, so there is room for a lot of different realizations. A new difficult phenomenon here is that for a function $u \in D(A_{\max})$, the "intermediate" derivatives $D^\alpha u$ (with $|\alpha| \le m$) need not exist as functions on Ω, even though Au does so.

For example, the function f on $\Omega =]0,1[\times]0,1[\subset \mathbb{R}^2$ defined by

$$f(x,y) = \begin{cases} 1 & \text{for } x > y, \\ 0 & \text{for } x \le y, \end{cases} \tag{4.15}$$

is in $L_2(\Omega)$ and may be shown to belong to $D(A_{\max})$ for the second-order operator $A = \partial_x^2 - \partial_y^2$ considered on Ω (Exercise 4.3). But $\partial_x f$ and $\partial_y f$ do not have a good sense as L_2-functions (f does not belong to the domains of the maximal realizations of ∂_x or ∂_y, Exercise 4.4).

Also for the Laplace operator Δ one can give examples where $u \in D(A_{\max})$ but the first derivatives are not in $L_2(\Omega)$ (see Exercise 4.5).

It is here that we find great help in distribution theory, which gives a precise explanation of which sense we can give to these derivatives.

4.2. Sobolev spaces.

The domains of realizations are often described by the help of various *Sobolev spaces* that we shall now define.

Definition 4.5. *Let Ω be an open subset of $\mathbb{R}^n$. Let $m \in \mathbb{N}_0$.*

1° *The Sobolev space $H^m(\Omega)$ is defined by*

$$H^m(\Omega) = \{\, u \in L_2(\Omega) \mid D^\alpha u \in L_2(\Omega) \ \text{ for } |\alpha| \le m \,\}, \tag{4.16}$$

where D^α is applied in the distribution sense. $H^m(\Omega)$ is provided with the scalar product and norm (the m-norm)

$$(u,v)_m = \sum_{|\alpha|\le m} (D^\alpha u, D^\alpha v)_{L_2(\Omega)}, \quad \|u\|_m = (u,u)_m^{\frac12}. \tag{4.17}$$

2° *The Sobolev space $H_0^m(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$.*

It is clear that $(u,v)_m$ is a scalar product with associated norm $\|u\|_m$ (since $\|u\|_m \ge \|u\|_0 \equiv \|u\|_{L_2(\Omega)}$), so that $H^m(\Omega)$ is a pre-Hilbert space. That the space is complete, is easily obtained from distribution theory: Let $(u_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $H^m(\Omega)$. Then (u_k) is in particular a Cauchy sequence in $L_2(\Omega)$, hence has a limit u in $L_2(\Omega)$. The sequences $(D^\alpha u_k)_{k\in\mathbb{N}}$ with $|\alpha| \le m$ are also likewise Cauchy sequences in $L_2(\Omega)$ with limits u_α. Since $u_k \to u$ in $L_2(\Omega)$, we also have that $u_k \to u$ in $\mathcal{D}'(\Omega)$, so that $D^\alpha u_k \to D^\alpha u$ in $\mathcal{D}'(\Omega)$ (cf. Theorem 3.8 or 3.9). When we compare this with the fact that $D^\alpha u_k \to u_\alpha$ in $L_2(\Omega)$, we see that $u_\alpha = D^\alpha u$ for any $|\alpha| \le m$, and hence $u \in H^m(\Omega)$ and $u_k \to u$ in $H^m(\Omega)$. — The subspace $H_0^m(\Omega)$ is now also a Hilbert space, with the induced norm. We have shown:

Lemma 4.6. *$H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces.*

Note that we have continuous injections

$$C_0^\infty(\Omega) \subset H_0^m(\Omega) \subset H^m(\Omega) \subset L_2(\Omega) \subset \mathcal{D}'(\Omega) \;, \tag{4.18}$$

in particular, convergence in $H^m(\Omega)$ implies convergence in $\mathcal{D}'(\Omega)$. Note also that when A is an m'th order differential operator with bounded C^∞-coefficients, then

$$H_0^m(\Omega) \subset D(A_{\min}) \subset D(A_{\max}) \;, \tag{4.19}$$

since convergence in H^m implies convergence in the graph-norm. For *elliptic* operators of order m, the first inclusion in (4.19) can be shown to be an identity (cf. Theorem 6.26 for operators with constant coefficients in the principal part), while the second inclusion is not usually so, when $\Omega \neq \mathbb{R}^n$ (the case $\Omega \subset \mathbb{R}$ is treated later in this chapter). $D(A_{\max})$ is, for $n > 1$, usually strictly larger than $H^m(\Omega)$, cf. Exercises 4.3–4.5.

Remark 4.7. One can also define similar Sobolev spaces associated with L_p spaces for general $1 \le p \le \infty$; here one uses the norms written in (C.10) with $\partial^\alpha u$ taken in the distribution sense. These spaces are Banach spaces; they are often denoted $W_p^m(\Omega)$ (or $W^{m,p}(\Omega)$; the notation $H_p^m(\Omega)$ may also be used). They are useful for example in nonlinear problems where it may be advantageous to use several values of p at the same time. (For example, if the nonlinearity involves a power of u, one can use that $u \in L_p(\mathbb{R}^n)$ implies $u^a \in L_{p/a}(\mathbb{R}^n)$.)

The above definition of $H^m(\Omega)$ is a *weak* definition, in the sense that the derivatives are defined by the help of duality. For the sake of our applications, we shall compare this with various strong definitions.

First we show that $h_j * u$ is a good approximation to u in $H^m(\mathbb{R}^n)$:

Lemma 4.8. *Let $u \in H^m(\mathbb{R}^n)$. Then $h_j * u \in C^\infty \cap H^m(\mathbb{R}^n)$ with*

$$\begin{aligned} &D^\alpha(h_j * u) = h_j * D^\alpha u \text{ for } |\alpha| \le m; \\ &h_j * u \to u \text{ in } H^m(\mathbb{R}^n) \text{ for } j \to \infty, \end{aligned} \tag{4.20}$$

Proof. When $u \in H^m(\mathbb{R}^n)$, then $h_j * D^\alpha u \in L_2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ for each $|\alpha| \le m$, and $h_j * D^\alpha u \to D^\alpha u$ in $L_2(\mathbb{R}^n)$, by Theorem 2.10. Here

$$h_j * D^\alpha u = D^\alpha(h_j * u) \;, \tag{4.21}$$

according to Theorem 3.15, so it follows that $h_j * v \to v$ in $H^m(\mathbb{R}^n)$ for $j \to \infty$. $\square$

For general sets this can be used to prove:

Theorem 4.9. *Let m be integer ≥ 0 and let Ω be any open set in $\mathbb{R}^n$. Then $C^\infty(\Omega) \cap H^m(\Omega)$ is dense in $H^m(\Omega)$.*

Proof. We can assume that Ω is covered by a locally finite sequence of open sets V_j, $j \in \mathbb{N}_0$, with an associated partition of unity ψ_j as in Theorem 2.16 (cf. (2.4), (2.49)).

Let $u \in H^m(\Omega)$; we have to show that it can be approximated arbitrarily well in m-norm by functions in $C^\infty(\Omega) \cap H^m(\Omega)$. First we set $u_j = \psi_j u$, so that we can write $u = \sum_{j \in \mathbb{N}_0} u_j$, Here u_j has compact support in V_j. Let $\varepsilon > 0$ be given. Let $u_j' = h_{k_j} * u_j$, where k_j is taken so large that $\operatorname{supp} u_j' \subset V_j$ and $\|u_j' - u_j\|_{H^m(\Omega)} \leq \varepsilon 2^{-j}$. The first property of k_j can be obtained since $\operatorname{supp} u_j' \subset \operatorname{supp} u_j + \underline{B}(0, \frac{1}{k_j})$ and $\operatorname{supp} u_j$ has positive distance from ∂V_j (recall (2.34)); the second property can be obtained from Lemma 4.8 since the $H^m(\Omega)$ norms of u_j, u_j' and $u_j - u_j'$ are the same as the respective $H^m(\mathbb{R}^n)$ norms, where we identify the functions with their extensions by 0 outside the support.

Let $v = \sum_{j \in \mathbb{N}_0} u_j'$; it exists as a function on Ω since the cover $\{V_j\}_{j \in \mathbb{N}_0}$ is locally finite, and v is C^∞ on Ω. Let K be a compact subset of Ω, it meets a finite number of the sets V_j, say, for $j \leq j_0$. Then

$$\begin{aligned} \|u - v\|_{H^m(K^\circ)} &= \|\sum_{j \leq j_0} (u_j - u_j')\|_{H^m(K^\circ)} \\ &\leq \sum_{j \leq j_0} \|u_j - u_j'\|_{H^m(\Omega)} \leq \varepsilon \sum_{j \leq j_0} 2^{-j} \leq 2\varepsilon. \end{aligned} \tag{4.22}$$

Since this can be shown for any $K \subset \Omega$, it follows that $v \in H^m(\Omega)$ and $\|u - v\|_{H^m(\Omega)} \leq 2\varepsilon$. $\square$

The procedure of approximating a function u by smooth functions by convolution by h_j is in part of the literature called “mollifying”, and the operator $h_j *$ called “the Friedrichs mollifier” after K. O. Friedrichs, who introduced it in an important application.

Sometimes the above result is not informative enough for our purposes, since there is no control of how the approximating C^∞ functions behave near the boundary. When the boundary of Ω is sufficiently nice, we can show that C^∞ functions with a controlled behavior are dense.

We recall for the following theorem that for a smooth open set Ω (cf. Definition C.1) we set

$$C^\infty_{(0)}(\overline{\Omega}) = \{\, u \in C^\infty(\overline{\Omega}) \mid \operatorname{supp} u \text{ compact } \subset \overline{\Omega} \,\}. \tag{4.23}$$

When $\Omega = \mathbb{R}^n$ (and only then), $C_0^\infty(\mathbb{R}^n)$ and $C^\infty_{(0)}(\mathbb{R}^n)$ coincide.

Theorem 4.10. *Let $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$, or let Ω be a smooth open bounded set. Then $C^\infty_{(0)}(\overline{\Omega})$ is dense in $H^m(\Omega)$.*

Proof. 1°. *The case $\Omega = \mathbb{R}^n$.* We already know from Lemma 4.8 that $C^\infty(\mathbb{R}^n) \cap H^m(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$. Now let $u \in C^\infty(\mathbb{R}^n) \cap H^m(\mathbb{R}^n)$, it must be approximated by C_0^∞ functions. Here we apply the technique of "truncation". With $\chi(x)$ defined in (2.3), we clearly have that $\chi(x/N)u \to u$ in $L_2(\mathbb{R}^n)$ for $N \to \infty$. For the derivatives we have by the Leibniz formula:

$$\begin{aligned} D^\alpha(\chi(x/N)u) &= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta(\chi(x/N))D^{\alpha-\beta}u \qquad (4.24)\\ &= \chi(x/N)D^\alpha u + \sum_{\substack{\beta\le\alpha\\ \beta\ne 0}} \binom{\alpha}{\beta} D^\beta(\chi(x/N))D^{\alpha-\beta}u \,. \end{aligned}$$

The first term converges to $D^\alpha u$ in $L_2(\mathbb{R}^n)$ for $N \to \infty$. For the other terms we use that $\sup_{x\in\mathbb{R}^n} |D^\beta(\chi(x/N))|$ is $O(N^{-1})$ for $N \to \infty$, for each $\beta \ne 0$. Then the contribution from the sum over $\beta \ne 0$ goes to 0 in $L_2(\mathbb{R}^n)$ for $N \to \infty$. It follows that $\chi(x/N)u \to u$ in $H^m(\mathbb{R}^n)$; here $\chi(x/N)u \in C_0^\infty(\mathbb{R}^n)$. This proves 1°.

2°. *The case $\Omega = \mathbb{R}^n_+$.* We now combine the preceding methods (mollification and truncation) with a third technique: "translation". It is used here in a way where the truncated function is pushed a little *outwards*, across the boundary of the domain, before mollification, so that we are only using the mollified functions on a set where the convergence requirements can be verified.

First note that the truncation argument given under 1° works equally well, when $\mathbb{R}^n$ is replaced by an arbitrary open set Ω, so we can replace $u \in H^m(\mathbb{R}^n_+)$ by $v_N = \chi_N u$ having bounded support. (Using Theorem 4.9, we could even assume $u \in C^\infty(\mathbb{R}^n_+) \cap H^m(\mathbb{R}^n_+)$, but it has an interest to show a proof that departs from $u \in H^m(\mathbb{R}^n_+)$.)

Defining the translation operator τ_h by

$$\tau_h u(x) = u(x_1, \dots, x_n - h) \quad \text{for } h \in \mathbb{R}\,, \tag{4.25}$$

we have for $u \in L_2(\mathbb{R}^n)$ that

$$\int_M |u - \tau_h u|^2 dx \to 0 \quad \text{for } h \to 0\,, \tag{4.26}$$

when M is a measurable subset of $\mathbb{R}^n$. Indeed (as used in Theorem 2.10), for any ε there is a $u' \in C_0^0(\mathbb{R}^n)$ with $\|u - u'\|_{L^2(\mathbb{R}^n)} \le \varepsilon$, and this satisfies:

$\|u'-\tau_h u'\|_{L_2(\mathbb{R}^n)} \to 0$ in view of the uniform continuity and compact support. Taking h_0 so that $\|u'-\tau_h u'\|_{L_2(\mathbb{R}^n)} \le \varepsilon$ for $|h| \le h_0$, we have that

$$\begin{aligned}\|u-\tau_h u\|_{L^2(M)} &\le \|u-u'\|_{L^2(M)} + \|u'-\tau_h u'\|_{L^2(M)} + \|\tau_h u' - \tau_h u\|_{L^2(M)}\\ &\le 2\|u-u'\|_{L^2(\mathbb{R}^n)} + \|u'-\tau_h u'\|_{L^2(\mathbb{R}^n)} \le 3\varepsilon\end{aligned}$$

for $|h| \le h_0$.

Note that when $h \ge 0$, then τ_{-h} carries $H^m(\mathbb{R}^n_+)$ over to $H^m(\Omega_{-h})$, where $\Omega_{-h} = \{ x \in \mathbb{R}^n \mid x_n > -h \} \supset \mathbb{R}^n_+$. When $u \in L_2(\mathbb{R}^n_+)$, we denote its extension by 0 for $x_n < 0$ by e^+u. In particular, for $u \in H^m(\mathbb{R}^n_+)$, one can consider the functions $e^+D^\alpha u$ in $L_2(\mathbb{R}^n)$. Here $\tau_{-h}e^+D^\alpha u$ equals $\tau_{-h}D^\alpha u$ on Ω_{-h}, in particular on $\mathbb{R}^n_+$. Then the above consideration show that

$$\int_{\mathbb{R}^n_+} |D^\alpha u - \tau_{-h}D^\alpha u|^2 dx \to 0 \quad \text{for } \; h \to 0+\,. \tag{4.27}$$

We have moreover, for $\varphi \in C_0^\infty(\mathbb{R}^n_+)$ and $\Omega_h = \{x \mid x_n > h\}$ (still assuming $h \ge 0$),

$$\begin{aligned}\langle \tau_{-h}D^\alpha u, \varphi\rangle_{\mathbb{R}^n_+} &= \langle D^\alpha u, \tau_h\varphi\rangle_{\Omega_h} = \langle u, (-D)^\alpha \tau_h \varphi\rangle_{\Omega_h}\\ &= \langle u, \tau_h(-D)^\alpha\varphi\rangle_{\Omega_h} = \langle D^\alpha \tau_{-h}u, \varphi\rangle_{\mathbb{R}^n_+},\end{aligned}$$

so that D^α and τ_{-h} may be interchanged here. Then $(D^\alpha\tau_{-h}u)|_{\mathbb{R}^n_+}$ is in $L_2(\mathbb{R}^n_+)$ (for $|\alpha| \le m$), and $(\tau_{-h}u)|_{\mathbb{R}^n_+}$ conuerges to u in $H^m(\mathbb{R}^n_+)$ for $h \to 0+$.

Returning to our $v_N = \chi_N u$, we thus have that $w_{N,h} = (\tau_{-h}v_N)|_{\mathbb{R}^n_+}$ approximates v_N in $H^m(\mathbb{R}^n_+)$ for $h \to 0+$. We shall end the proof of this case by approximating $w_{N,h}$ in $H^m(\mathbb{R}^n_+)$ by $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$-functions. This can be done because $\tau_{-h}v_N$ is in fact in $H^m(\Omega_{-h})$. Recall from Lemma 2.12 that for $j > \frac{1}{\varepsilon}$, $(h_j * \tau_{-h}v_N)(x)$ is a well-defined C^∞ function on the set of x with distance $> \varepsilon$ from the boundary, and there is L_2-convergence to $\tau_{-h}v_N$ on this set, for $j \to \infty$. Let us take $\varepsilon = \frac{h}{2}$; then the set equals $\Omega_{-\frac{h}{2}}$ and contains $\mathbb{R}^n_+$. Clearly also $(h_j * D^\alpha\tau_{-h}v_N)(x)$ converges to $D^\alpha\tau_{-h}v_N$ in L_2 on $\Omega_{-\frac{h}{2}}$, hence on $\mathbb{R}^n_+$, when $|\alpha| \le m$. Now we show that for $j > \frac{2}{h}$, $D^\alpha(h_j * \tau_{-h}v_N) = h_j * D^\alpha(\tau_{-h}v_N)$ on $\mathbb{R}^n_+$ (it even holds on $\Omega_{-\frac{h}{2}}$): For any $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$,

$$\begin{aligned}\langle D^\alpha(h_j * \tau_{-h}v_N), \varphi\rangle_{\mathbb{R}^n_+} &= \langle h_j * \tau_{-h}v_N, (-D)^\alpha\varphi\rangle_{\mathbb{R}^n_+}\\ &= \int_{x\in\mathbb{R}^n_+}\int_{y\in B(0,\frac{1}{j})} h_j(x-y)(\tau_{-h}v_N)(y)(-D)^\alpha\varphi(x)\,dydx\\ &= \int_{y\in\mathbb{R}^n_+ + B(0,\frac{1}{j})} (\tau_{-h}v_N)(y)(\check{h}_j * (-D)^\alpha\varphi)(y)\,dy\end{aligned}$$

$$= \int_{y\in\mathbb{R}^n_+ + B(0,\frac{1}{j})} (\tau_{-h}v_N)(y)(-D)^\alpha(\check{h}_j * \varphi)(y)\,dy$$

$$= \int_{y\in\mathbb{R}^n_+ + B(0,\frac{1}{j})} D^\alpha(\tau_{-h}v_N)(y)(\check{h}_j * \varphi)(y)\,dy = \langle h_j * D^\alpha(\tau_{-h}v_N), \varphi\rangle_{\mathbb{R}^n_+}.$$

Thus also $D^\alpha(h_j * \tau_{-h}v_N)$ converges to $D^\alpha\tau_{-h}v_N$ in $L_2(\mathbb{R}^n_+)$ for each $|\alpha| \le m$; so $(h_j * \tau_{-h}v_N)|_{\mathbb{R}^n_+}$ converges to $(\tau_{-h}v_N)|_{\mathbb{R}^n_+} = w_{N,h}$ in $H^m(\mathbb{R}^n_+)$ for $j \to \infty$. Since $(h_j * \tau_{-h}v_N)|_{\mathbb{R}^n_+}$ is in $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, this ends the proof of 2°.

3°. *The case where Ω is smooth, open and bounded.* Here we moreover include the technique of "localization." We cover $\overline{\Omega}$ by open sets Ω_0, Ω_1, …, Ω_N, where $\Omega_1, \dots, \Omega_N$ are of the type U described in Definition C.1, and $\overline{\Omega}_0 \subset \Omega$. Let $\psi_0, \dots, \psi_N$ be a partition of unity with $\psi_l \in C_0^\infty(\Omega_l)$ and $\psi_0 + \cdots + \psi_N = 1$ on $\overline{\Omega}$ (cf. Theorem 2.17). The function $\psi_0 u$ has compact support in Ω and gives by extension by 0 outside of Ω a function $\widetilde{\psi_0 u}$ in $H^m(\mathbb{R}^n)$ (since $D^\alpha(\psi_0 u) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta\psi_0 D^{\alpha-\beta}u$ has support in $\operatorname{supp}\psi_0$, so that $(D^\alpha(\psi_0 u))^\sim$ equals $D^\alpha(\widetilde{\psi_0 u})$). Now $\widetilde{\psi_0 u}$ can be approximated according to 1°, and restriction to Ω then gives the desired approximation of $\psi_0 u$. The functions $\psi_l u$, $l = 1, \dots, N$, are by the diffeomorphisms associated with each Ω_l carried into functions v_l in $H^m(\mathbb{R}^n_+)$ (with support in $B(0,1)$),[1] which is approximated in $H^m(\mathbb{R}^n_+)$ according to 2°. (Since $\operatorname{supp} v_l$ is compact $\subset B(0,1)$, the translated function stays supported in the ball for sufficiently small h.) Transforming this back to Ω_l, we get an approximation of $\psi_l u$, for each l. The sum of the approximations of the $\psi_l u, l = 0, \dots, N$, approximates u. ☐

The result in 3° holds also under weaker regularity requirements on Ω, where the idea of translation across the boundary can still be used.

Corollary 4.11. 1° *$H^m(\mathbb{R}^n) = H_0^m(\mathbb{R}^n)$ for all $m \in \mathbb{N}_0$; i.e., $C_0^\infty(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$ for all $m \in \mathbb{N}_0$.*

2° *For Ω smooth open and bounded, $C^\infty(\overline{\Omega})$ is dense in $H^m(\Omega)$, for all $m \in \mathbb{N}_0$.*

Proof. 1° follows from Theorem 4.10 and the fact that $C^\infty_{(0)}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$, cf. Definition 4.5 2°. 2° follows from Theorem 4.10 and the fact that for Ω smooth, open and bounded, $C^\infty_{(0)}(\overline{\Omega}) = C^\infty(\overline{\Omega})$. ☐

Of course we always have that $H^0(\Omega) = L_2(\Omega) = H_0^0(\Omega)$ for arbitrary Ω (since $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$). But for $m > 0$, $H^m(\Omega) \neq H_0^m(\Omega)$ when $\mathbb{R}^n \setminus \overline{\Omega} \neq \emptyset$.

We can now also show an extension theorem.

---

[1] Here we use the chain rule for distributions, cf. (3.42). Since κ, κ^{-1} and their derivatives are C^∞ functions, the property $u \in H^m$ is invariant under diffeomorphisms κ where κ, κ^{-1} and their derivatives are bounded.

Theorem 4.12. *Let* $m \in \mathbb{N}$*, and let* Ω *be smooth open and bounded, or equal to* $\mathbb{R}^n_+$*. There is a continuous linear operator* $E\colon H^m(\Omega) \to H^m(\mathbb{R}^n)$ *so that* $u = (Eu)|_\Omega$ *for* $u \in H^m(\Omega)$*.*

Proof. 1°. *The case* $\Omega = \mathbb{R}^n_+$. Choose in an arbitrary way a set of $m+1$ different positive numbers $\lambda_0, \dots, \lambda_m$. Let $\{\alpha_0, \dots, \alpha_m\}$ be the solution of the system of equations

$$\begin{aligned} \textstyle\sum_{k=0}^m \alpha_k &= 1\ , \\ \textstyle\sum_{k=0}^m \lambda_k \alpha_k &= -1\ , \\ &\ \vdots \\ \textstyle\sum_{k=0}^m \lambda_k^m \alpha_k &= (-1)^m\ . \end{aligned} \tag{4.28}$$

The solution exists and is uniquely determined, since the determinant of the system is the Vandermonde determinant,

$$\det \begin{pmatrix} 1 & \dots & 1 \\ \lambda_0 & \dots & \lambda_m \\ \vdots & & \vdots \\ \lambda_0^m & \dots & \lambda_m^m \end{pmatrix} = \prod_{0 \le i < j \le m} (\lambda_j - \lambda_i) \neq 0\ . \tag{4.29}$$

Now when $u \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, we define Eu by

$$(Eu)(x) = \begin{cases} u(x) & \text{for } x_n \ge 0\ , \\ \sum_{k=0}^m \alpha_k u(x', -\lambda_k x_n) & \text{for } x_n < 0\ . \end{cases} \tag{4.30}$$

Because of (4.28), we have that $D^\alpha Eu$ is continuous on $\mathbb{R}^n$ for all $|\alpha| \le m$, so that $Eu \in H^m(\mathbb{R}^n)$; it is easy to verify by use of (4.30) that

$$\|Eu\|_{H^m(\mathbb{R}^n)} \le c\|u\|_{H^m(\mathbb{R}^n_+)} \tag{4.31}$$

for some constant c. Since the operator $E\colon u \mapsto Eu$ is defined linearly on the dense subset $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ of $H^m(\mathbb{R}^n_+)$ and by (4.31) is continuous in m-norm, it extends by continuity to a continuous map of $H^m(\mathbb{R}^n_+)$ into $H^m(\mathbb{R}^n)$ with the desired properties.

2°. *The case where* Ω *is smooth, open and bounded.* This is reduced to an application of the preceding case by use of a covering $\bigcup_{l=0}^N \Omega_l$ of $\overline{\Omega}$ as in the proof of Theorem 4.10 3° (with $\overline{\Omega}_0 \subset \Omega$) and associated diffeomorphisms $\kappa_{(l)} : \Omega_l \to B(0,1)$ for $l > 0$, together with a partition of unity $\psi_0, \dots, \psi_n$ with $\psi_l \in C_0^\infty(\Omega_l)$ and $\psi_0 + \dots + \psi_N = 1$ on $\overline{\Omega}$. Let $u \in C^\infty(\overline{\Omega})$ $(= C^\infty_{(0)}(\overline{\Omega}))$. For

$\psi_0 u$, we take $E(\psi_0 u) = \widetilde{\psi_0 u}$ (extension by 0 outside Ω). For each $l > 0$, $\psi_l u$ is by the diffeomorphism $\kappa_{(l)}$ carried over to a function $v_l \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ supported in $B(0,1) \cap \overline{\mathbb{R}}^n_+$. Here we use the extension operator $E_{\mathbb{R}^n_+}$ constructed in 1°, choosing λ_k to be > 1 for each k (e.g., $\lambda_k = k+2$), so that the support of $E_{\mathbb{R}^n_+} v_l$ is a compact subset of $B(0,1)$. With the notation

$$(T_l v)(x) = v(\kappa_{(l)}(x)), \quad l = 1, \dots, N \ ,$$

we set

$$E_\Omega u = \widetilde{\psi_0 u} + \sum_{l=1}^{n} T_l(E_{\mathbb{R}^n_+} v_l) \ .$$

Defining E_Ω in this way for smooth functions, we get by extension by continuity a map E_Ω having the desired properties. $\square$

Part 1° of the proof could actually have been shown using only m numbers $\lambda_0, \dots, \lambda_{m-1}$ and m equations (4.28) (with m replaced by $m-1$); for we could then appeal to Theorem 3.20 in the proof that the m'th order derivatives of Eu were in L_2.

There exist constructions where the extension operator does not at all depend on m, and where the boundary is allowed to be considerably less smooth, cf. e.g. [EE87, Th. V 4.11–12].

The analysis of Sobolev spaces will be continued in connection with studies of boundary value problems and in connection with the Fourier transformation. A standard reference for the use of Sobolev spaces in the treatment of boundary value problems is the books of J.-L. Lions and E. Magenes [LM68].

Theorem 4.9 and 4.10 show the equivalence of weak and strong definitions of differential operators in the particular case where we consider the whole family of differential operators $\{D^\alpha\}_{|\alpha| \le m}$ taken together. For a general operator $A = \sum_{|\alpha| \le m} a_\alpha D^\alpha$, such properties are harder to show (and need not hold when there is a boundary). For example, in an application of Friedrichs' mollifier, one will have to treat $h_j * (\sum_\alpha a_\alpha D^\alpha u) - \sum_\alpha a_\alpha D^\alpha (h_j * u)$, which requires further techniques when the a_α depend on x.

For operators in one variable one can often take recourse to absolute continuity (explained below). For some operators on $\mathbb{R}^n$ (for example the Laplace operator), the Fourier transformation is extremely useful, as we shall see in Chapters 5 and 6.

4.3. The one-dimensional case.

In the special case where Ω is an interval of $\mathbb{R}$, we use the notation $u', u'', u^{(k)}$ along with $\partial u, \partial^2 u, \partial^k u$, also for distribution derivatives. Here the derivative defined on Sobolev spaces is closely related to the derivative of absolutely continuous functions.

Traditionally, a function f is said to be absolutely continuous on $\overline{I}$, $I =]\alpha, \beta[$, when f has the property that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for every set $J_1, \dots, J_N$ of disjoint subintervals $J_k = [\alpha_k, \beta_k]$ of $\overline{I}$ with total length $\leq \delta$ (i.e., with $\sum_{k=1,\dots,N}(\beta_k - \alpha_k) \leq \delta$),

$$\sum_{k=1,\dots,N} |f(\beta_k) - f(\alpha_k)| \leq \varepsilon.$$

It is known from measure theory that this property is equivalent with the existence of an integrable function g on I and a number k such that

$$f(x) = \int_\alpha^x g(s) ds + k, \text{ for } x \in \overline{I}. \tag{4.32}$$

We shall show that $H^1(I)$ consists precisely of such functions with $g \in L_2(I)$.

Theorem 4.13. *Let $I =]\alpha, \beta[$ for some $\alpha < \beta$. Then*

$$C^0(\overline{I}) \supset H^1(I) \supset C^1(\overline{I}); \text{ with} \tag{4.33}$$

$$\|u\|_{L_\infty(I)} \leq c_1 \|u\|_{H^1(I)} \leq c_2(\|u\|_{L_\infty(I)} + \|\partial u\|_{L_\infty(I)}), \tag{4.34}$$

for some constants c_1 and $c_2 > 0$. Moreover,

$$H^1(I) = \{ f \mid f(x) = \textstyle\int_\alpha^x g(s) ds + k,\ g \in L_2(I),\ k \in \mathbb{C}\}. \tag{4.35}$$

Proof. It is obvious that $C^1(\overline{I}) \subset H^1(I)$ (the second inclusion in (4.33)), with the inequality

$$\|u\|^2_{H^1(I)} = \int_\alpha^\beta (|u(x)|^2 + |u'(x)|^2)\, dx \leq (\beta - \alpha)(\|u\|^2_{L_\infty(I)} + \|u'\|^2_{L_\infty(I)});$$

this implies the second inequality in (4.34).

Now let $u \in C^1(\overline{I})$ and set $v(x) = \int_\alpha^x u'(s)\, ds$, so that

$$u(x) = u(\alpha) + \int_\alpha^x u'(s)\, ds = u(\alpha) + v(x). \tag{4.36}$$

Clearly,

$$\int_\alpha^\beta |v(x)|^2\, dx \leq (\beta - \alpha) \|v\|^2_{L_\infty(I)}.$$

Moreover, by the Cauchy-Schwarz inequality,

$$|v(x)|^2 = |\int_\alpha^x u'(s)\,ds|^2 \le \int_\alpha^x 1\,ds \int_\alpha^x |u'(s)|^2\,ds \le (\beta-\alpha)\int_\alpha^\beta |u'(s)|^2\,ds;$$

we collect the inequalities in:

$$(\beta-\alpha)^{-1}\|v\|^2_{L_2(I)} \le \|v\|^2_{L_\infty(I)} \le (\beta-\alpha)\|u'\|^2_{L_2(I)}. \tag{4.37}$$

It follows (cf. (4.36)) that

$$\begin{aligned}|u(\alpha)| &= \big((\beta-\alpha)^{-1}\textstyle\int_\alpha^\beta |u(\alpha)|^2\,dx\big)^{\frac12} = (\beta-\alpha)^{-\frac12}\|u-v\|_{L_2(I)}\\ &\le (\beta-\alpha)^{-\frac12}\|u\|_{L_2(I)} + (\beta-\alpha)^{-\frac12}\|v\|_{L_2(I)}\\ &\le (\beta-\alpha)^{-\frac12}\|u\|_{L_2(I)} + (\beta-\alpha)^{\frac12}\|u'\|_{L_2(I)};\end{aligned}$$

and then furthermore:

$$\begin{aligned}|u(x)| &= |v(x)+u(\alpha)| \le \|v\|_{L_\infty(I)} + |u(\alpha)|\\ &\le (\beta-\alpha)^{-\frac12}\|u\|_{L_2(I)} + 2(\beta-\alpha)^{\frac12}\|u'\|_{L_2(I)}, \text{ for all } x\in\overline{I}.\end{aligned}$$

This shows the first inequality in (4.34) for $u \in C^1(\overline{I})$. For a general $u \in H^1(I)$, let $u_n \in C^\infty(\overline{I})$ be a sequence converging to u in $H^1(I)$ for $n \to \infty$. Then u_n is a Cauchy sequence in $L_\infty(I)$, and hence since the u_n are continuous, has a continuous limit u_0 in sup-norm. Since $u_n \to u_0$ also in $L_2(I)$, u_0 is a continuous representative of u; we use this representative in the following, denoting it u again. We have hereby shown the first inclusion in (4.33), and the first inequality in (4.34) extends to $u \in H^1(I)$.

Now let us show (4.35). Let $u \in H^1(I)$ and let $u_n \in C^\infty(\overline{I})$, $u_n \to u$ in $H^1(I)$. Then $u_n' \to u'$ in $L_2(I)$, and $u_n(x) \to u(x)$ uniformly for $x \in \overline{I}$, as we have just shown. Then

$$u_n(x) = \int_\alpha^x u_n'(s)\,ds + u_n(\alpha) \text{ implies } u(x) = \int_\alpha^x u'(s)\,ds + u(\alpha),$$

by passage to the limit, so u belongs to the right-hand side of (4.35).

Conversely, let f satisfy

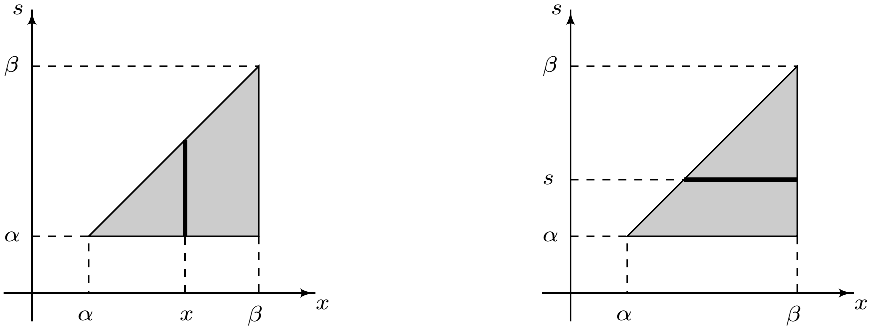
$$f(x) = \int_\alpha^x g(s)\,ds + k,$$

for some $g \in L_2(I)$, $k \in \mathbb{C}$. Clearly, $f \in C^0(\overline{I})$ with $f(\alpha) = k$. We shall show that ∂f, taken in the distribution sense, equals g; this will imply $f \in H^1(I)$.

We have for any $\varphi \in C_0^\infty(I)$, using the Fubini theorem and the fact that $\varphi(\alpha) = \varphi(\beta) = 0$ (see the figure):

$$\begin{aligned}\langle \partial f, \varphi \rangle = -\langle f, \partial\varphi \rangle &= -\int_\alpha^\beta f(x)\varphi'(x)\,dx \\ &= -\int_\alpha^\beta \int_\alpha^x g(s)\varphi'(x)\,dsdx - \int_\alpha^\beta k\varphi'(x)\,dx \\ &= -\int_\alpha^\beta \int_s^\beta g(s)\varphi'(x)\,dxds - k(\varphi(\beta) - \varphi(\alpha)) \\ &= -\int_\alpha^\beta g(s)(\varphi(\beta) - \varphi(s))\,ds \\ &= \int_\alpha^\beta g(s)\varphi(s)\,ds = \langle g, \varphi \rangle.\end{aligned}$$

Thus $\partial f = g$ in $\mathcal{D}'(I)$, and the proof is complete. $\square$



We have furthermore:

Theorem 4.14. *When u and $v \in H^1(I)$, then also $uv \in H^1(I)$, and*

$$\partial(uv) = (\partial u)v + u(\partial v); \quad \textit{with} \tag{4.38}$$

$$\|uv\|_{H^1(I)} \le 5^{\frac{1}{2}} c_1 \|u\|_{H^1(I)} \|v\|_{H^1(I)}. \tag{4.39}$$

Moreover,

$$(\partial u, v)_{L_2(I)} + (u, \partial v)_{L_2(I)} = u(\beta)\bar{v}(\beta) - u(\alpha)\bar{v}(\alpha), \tag{4.40}$$

$$(Du, v)_{L_2(I)} - (u, Dv)_{L_2(I)} = -iu(\beta)\bar{v}(\beta) + iu(\alpha)\bar{v}(\alpha). \tag{4.43}$$

Proof. Let $u, v \in H^1(I)$ and let $u_n, v_n \in C^1(\overline{I})$ with $u_n \to u$ and $v_n \to v$ in $H^1(I)$ for $n \to \infty$. By Theorem 4.13, the convergences hold in $C^0(\overline{I})$, so $u_n v_n \to uv$ in $C^0(\overline{I})$, hence also in $\mathcal{D}'(I)$. Moreover, $(\partial u_n)v_n \to (\partial u)v$ and $u_n(\partial v_n) \to u(\partial v)$ in $L_2(I)$, hence in $\mathcal{D}'(I)$, so the formula

$$\partial(u_n v_n) = (\partial u_n)v_n + u_n(\partial v_n),$$

valid for each n, implies

$$\partial(uv) = (\partial u)v + u(\partial v) \text{ in } \mathcal{D}'(I),$$

by Theorem 3.9. This shows (4.38), and (4.39) follows since

$$\begin{aligned} \|uv\|_{L_2(I)} &\le \|u\|_{L_\infty(I)} \|v\|_{L_2(I)} \le c_1 \|u\|_{H^1(I)} \|v\|_{H^1(I)}, \\ \|\partial(uv)\|_{L_2(I)} &= \|(\partial u)v + u(\partial v)\|_{L_2(I)} \\ &\le \|\partial u\|_{L_2(I)} \|v\|_{L_\infty(I)} + \|u\|_{L_\infty(I)} \|\partial v\|_{L_2(I)} \\ &\le 2c_1 \|u\|_{H^1(I)} \|v\|_{H^1(I)}, \end{aligned}$$

by (4.34). The formula (4.40) is shown by an application of (4.35) to $u\bar{v}$:

$$\begin{aligned} u(\beta)\bar{v}(\beta) - u(\alpha)\bar{v}(\alpha) &= \int_\alpha^\beta \partial(u(s)\bar{v}(s))\, ds \\ &= \int_\alpha^\beta (\partial u(s)\, \bar{v}(s) + u(s)\partial \bar{v}(s))\, ds, \end{aligned}$$

and (4.43) follows by multiplication by $-i$. $\square$

The subspace $H_0^1(I)$ is characterized as follows:

Theorem 4.15. *Let $I =]\alpha, \beta[$. The subspace $H_0^1(I)$ of $H^1(I)$ (the closure of $C_0^\infty(I)$ in $H^1(I)$) .satisfies:*

$$\begin{aligned} H_0^1(I) &= \{\, u \in H^1(I) \mid u(\alpha) = u(\beta) = 0 \,\} \\ &= \{\, u(x) = \textstyle\int_\alpha^x g(s)\, ds \mid g \in L_2(I),\ (g, 1)_{L_2(I)} = 0 \}. \end{aligned} \tag{4.41}$$

Proof. When $u = \int_\alpha^x g(s)\, ds + k$, then $u(\alpha) = 0$ if and only if $k = 0$, and when this holds, $u(\beta) = 0$ if and only if $\int_\alpha^\beta g(s)\, ds = 0$. In view of (4.35), this proves the second equality in (4.41).

Let $u \in H_0^1(I)$, then it is the limit in $H^1(I)$ of a sequence of functions $u_n \in C_0^\infty(I)$. Since $u_n(\alpha) = u_n(\beta) = 0$ and the convergence holds in $C^0(\overline{I})$ (cf. (4.33)–(4.34)), $u(\alpha) = u(\beta) = 0$.

Conversely, let $u \in H^1(I)$ with $u(\alpha) = u(\beta) = 0$. Then $\widetilde{u}$, the extension by 0 outside $[\alpha, \beta]$, is in $H^1(\mathbb{R})$, since $\widetilde{u}(x) = \int_\alpha^x \widetilde{u'}\, ds$ for any $x \in \mathbb{R}$, where $\widetilde{u'}$ is the extension of u' by 0 outside $[\alpha, \beta]$. Let us "shrink" $\widetilde{u}$ by defining, for $0 < \delta \leq \frac{1}{2}$,

$$\widetilde{v}_\delta = \widetilde{u}(\tfrac{1}{1-\delta}(x - \tfrac{\alpha+\beta}{2}));$$

since $\widetilde{u}$ vanishes for $|x - \frac{\alpha+\beta}{2}| \geq \frac{\beta-\alpha}{2}$, $\widetilde{v}_\delta$ vanishes for $|x - \frac{\alpha+\beta}{2}| \geq (1-\delta)\frac{\beta-\alpha}{2}$, i.e., is supported in the interval

$$[\alpha + \delta\tfrac{\beta-\alpha}{2}, \beta - \delta\tfrac{\beta-\alpha}{2}] \subset]\alpha, \beta[\,.$$

Clearly, $\widetilde{v}_\delta \to \widetilde{u}$ in $H^1(\mathbb{R})$ for $\delta \to 0$. Mollifying $\widetilde{v}_\delta$ to $h_j * \widetilde{v}_\delta$ for $\frac{1}{j} < \delta\frac{\beta-\alpha}{2}$, we get a C^∞-function with compact support in I, such that $h_j * \widetilde{v}_\delta \to \widetilde{v}_\delta$ in $H^1(I)$ for $j \to \infty$. Thus we can find a $C_0^\infty(I)$-function $(h_j * \widetilde{v}_\delta)|_I$ arbitrarily close to u in $H^1(I)$ by choosing first δ small enough and then j large enough. This completes the proof of the first equality in (4.41). $\square$

Now let us consider realizations of the basic differential operator $A = D = \frac{1}{i}\partial$. By definition (see Lemma 4.2 ff.),

$$\begin{aligned} D(A_{\max}) &= H^1(I), \\ D(A_{\min}) &= H_0^1(I). \end{aligned} \tag{4.42}$$

Equation (4.43) implies

$$(A_{\min}u, v) - (u, A_{\min}v) = 0,$$

so $A_{\min}$ is symmetric, and $A_{\max} = {A_{\min}}^*$, cf. (4.9). Here $A_{\min}$ is too small and $A_{\max}$ is too large to be selfadjoint; in fact we shall show that the realization $A_\#$ defined by

$$D(A_\#) = \{\, u \in H^1(I) \mid u(\alpha) = u(\beta) \,\} \tag{4.44}$$

(the "periodic boundary condition") is selfadjoint. To see this, note that $A_\#$ is symmetric since the right hand side of (4.43) vanishes if u and $v \in D(A_\#)$, so $A_\# \subset A_\#^*$. Since $A_{\min} \subset A_\#$, $A_\#^* \subset A_{\max}$. Then we show $A_\#^* \subset A_\#$ as follows: Let $u \in D(A_\#^*)$, then for any $v \in D(A_\#)$ with $v(\alpha) = v(\beta) = k$,

$$0 = (A_\#^* u, v) - (u, A_\# v) = (Du, v) - (u, Dv) = -iu(\beta)\bar{k} + iu(\alpha)\bar{k}.$$

Since k can be arbitrary, this implies that $u(\beta) = u(\alpha)$, hence $u \in D(A_\#)$.

We have shown:

Theorem 4.16. *Consider the realizations* $A_{\max}$*,* $A_{\min}$ *and* $A_\#$ *of* $A = D$ *with domains described in* (4.42) *and* (4.44)*. Then*

(1) $A_{\min}$ *is symmetric,*
(2) $A_{\min}$ *and* $A_{\max}$ *are adjoints of one another,*
(3) $A_\#$ *is selfadjoint.*

For general m, the characterizations of the Sobolev spaces extend as follows:

Theorem 4.17. *Let* $I =]\alpha, \beta[$ *.*

1° $H^m(I)$ *consists of the functions* $u \in C^{m-1}(\overline{I})$ *such that* $u^{(m-1)} \in H^1(I)$*. The inequality*

$$\sum_{j\le m-1} \|\partial^j u\|^2_{L_\infty(I)} \le C \sum_{k\le m} \|\partial^k u\|^2_{L_2(I)} \tag{4.45}$$

holds for all $u \in H^m(I)$*, with some constant* $C > 0$*.*

2° *The subspace* $H_0^m(I)$ *(the closure of* $C_0^\infty(I)$ *in* $H^m(I)$*) satisfies:*

$$H_0^m(I) = \{\, u \in H^m(I) \mid u^{(j)}(\alpha) = u^{(j)}(\beta) = 0 \text{ for } j = 0, \dots, m-1 \,\}. \tag{4.46}$$

Proof. We can assume $m > 1$.

1°. It is clear from the definition that a function u is in $H^m(I)$ if and only if $u, u', \dots, u^{(m-1)}$ (defined in the distribution sense) belong to $H^1(I)$. This holds in particular if $u \in C^{m-1}(\overline{I})$ with $u^{(m-1)} \in H^1(I)$. To show that an arbitrary function $u \in H^m(I)$ is in $C^{m-1}(\overline{I})$, note that by Theorem 4.13,

$$\begin{aligned} &u^{(m-1)} \in C^0(\overline{I}),\\ &u^{(j)}(x) = \int_\alpha^x u^{(j+1)}(s)\,ds + u^{(j)}(\alpha) \text{ for } j < m; \end{aligned}$$

the latter gives by successive application for $j = m-1, m-2, \dots, 0$ that $u \in C^{m-1}(\overline{I})$. The inequality (4.45) follows by applying (4.34) in each step.

2°. When $u \in C_0^\infty(I)$, then all derivatives at α and β are 0, so by passage to the limit in m-norm we find that $u^{(j)}(\alpha) = u^{(j)}(\beta) = 0$ for $j \le m-1$ in view of (4.45). Conversely, if u belongs to the right-hand side of (4.46), one can check that the extension $\widetilde{u}$ by zero outside $[\alpha, \beta]$ is in $H^m(\mathbb{R})$ and proceed as in the proof of Theorem 4.15. □

The "integration by parts" formulas (4.40) and (4.43) have the following generalization to mth order operators on $I =]\alpha, \beta[$, often called the *Lagrange* *formula:*

$$
\begin{aligned}
(D^m u, v)_{L_2(I)} - (u, D^m v)_{L_2(I)} &= \sum_{k=0}^{m-1} \big[(D^{m-k}u, D^k v) - (D^{m-k-1}u, D^{k+1}v)\big] \\
&= -i \sum_{k=0}^{m-1} \big[D^{m-1-k}u(\beta)\overline{D^k v(\beta)} - D^{m-1-k}u(\alpha)\overline{D^k v(\alpha)}\big], \\
&\qquad \text{for } u, v \in H^m(I). \quad (4.47)
\end{aligned}
$$

Unbounded intervals are included in the analysis by the following theorem:

Theorem 4.18. 1° *An inequality* (4.45) *holds also when I is an unbounded interval of $\mathbb{R}$.*

2° *For $I = \mathbb{R}$, $H^m(I) = H_0^m(I)$. For $I =]\alpha, \infty[$,*

$$H_0^m(I) = \left\{ u \in H^m(I) \mid u^{(j)}(\alpha) = 0 \quad \textit{for } \; j = 0, \dots, m-1 \right\}. \qquad (4.48)$$

There is a similar characterization of $H_0^m(\,]-\infty, \beta[\,)$.

Proof. 1°. We already have (4.45) for bounded intervals I', with constants $C(I')$. Since derivatives commute with translation,

$$\max_{x \in \overline{I'} + a} \|\partial^j u(x - a)\| = \max_{x \in \overline{I'}} \|\partial^j u(x)\|, \quad \|\partial^j u(\cdot - a)\|_{L_2(I'+a)} = \|\partial^j u(\cdot)\|_{L_2(I')};$$

then the constant $C(I')$ can be taken to be invariant under translation, depending only on m and the length of I'. When I is unbounded, every point $x \in \overline{I}$ lies in an interval $\overline{I_x} \subset \overline{I}$ of length 1, and then

$$\sum_{0 \le j \le m-1} |\partial^j u(x)|^2 \le C(1) \sum_{0 \le j \le m} \|\partial^j u\|^2_{L_2(I_x)} \le C(1) \sum_{0 \le j \le m} \|\partial^j u\|^2_{L_2(I)},$$

where $C(1)$ is the constant used for intervals of length 1.

2°. The first statement is a special case of Corollary 4.11.

For the second statement, the inclusion '$\subset$' is an obvious consequence of 1°. For the inclusion '$\supset$' let $u \in H^m(I)$ and consider the decomposition

$$u = \chi_N u + (1 - \chi_N) u\,,$$

where $\chi_N(x) = \chi(x/N)$ with N taken $\ge 2|\alpha|$ (cf. (2.3)). We see as in Theorem 4.10 that $\chi_N u \to u$ in $H^m(I)$ for $N \to \infty$. Here, if $u^{(k)}(\alpha) = 0$ for $k = 0, 1, \dots, m-1$, then $\chi_N u$ lies in $H_0^m(\,]\alpha, 2N[\,)$ according to (4.46), hence can be approximated in m-norm by functions in $C_0^\infty(\,]\alpha, 2N[\,)$. So, by taking first N sufficiently large and next choosing $\varphi \in C_0^\infty(\,]\alpha, 2N[\,)$ close to $\chi_N u$, one can approximate u in m-norm by functions in $C_0^\infty(\,]\alpha,\infty[\,)$. $\square$

The functions in $H^m(\mathbb{R})$ satisfy $u^{(j)}(x) \to 0$ for $x \to \pm\infty$, $j < m$ (Exercise 4.20); and the formulas (4.40) and (4.43) take the form

$$(\partial u, v)_{L_2(I)} + (u, \partial v)_{L_2(I)} = -u(\alpha)\bar{v}(\alpha), \tag{4.40a}$$

$$(Du, v)_{L_2(I)} - (u, Dv)_{L_2(I)} = iu(\alpha)\bar{v}(\alpha), \tag{4.43a}$$

for $u, v \in H^1(I)$, $I = \,]\alpha,\infty[\,$. We also have the Lagrange formula

$$(D^m u, v)_{L_2(I)} - (u, D^m v)_{L_2(I)} = i \sum_{k=0}^{m-1} D^{m-1-k} u(\alpha) \overline{D^k v(\alpha)}, \tag{4.47a}$$

for $u, v \in H^m(I)$, $I = \,]\alpha,\infty[\,$.

To prepare for the analysis of realizations of D^m, we shall show an interesting "uniqueness theorem".

Theorem 4.19. *Let I be an open interval of $\mathbb{R}$, let $k \geq 1$ and let $u \in \mathcal{D}'(I)$. If $Du = 0$, then u equals a constant. If $D^k u = 0$, then u is a polynomial of degree $\leq k-1$.*

Proof. The theorem is shown by induction in k. For the case $k = 1$, one has when $Du = 0$ that

$$0 = \langle Du, \varphi\rangle = -\langle u, D\varphi\rangle \quad \text{for all } \ \varphi \in C_0^\infty(I)\ .$$

Choose a function $h \in C_0^\infty(I)$ with $\langle 1, h\rangle = 1$ and define, for $\varphi \in C_0^\infty(I)$,

$$\psi(x) = i \int_{-\infty}^{x} [\varphi(s) - \langle 1, \varphi\rangle h(s)] ds.$$

Here $D\psi = \varphi - \langle 1, \varphi\rangle h$; and $\psi \in C_0^\infty(I)$ since

$$\int_{-\infty}^{x} [\varphi(s) - \langle 1, \varphi\rangle h(s)] ds = \langle 1, \varphi\rangle - \langle 1, \varphi\rangle\langle 1, h\rangle = 0$$

when $x > \sup(\operatorname{supp}\varphi \cup \operatorname{supp} h)$. Hence φ can be written as

$$\varphi = \varphi - \langle 1, \varphi\rangle h + \langle 1, \varphi\rangle h = D\psi + \langle 1, \varphi\rangle h\ , \tag{4.49}$$

whereby

$$\langle u, \varphi\rangle = \langle u, D\psi\rangle + \langle 1, \varphi\rangle\langle u, h\rangle = \langle\langle u, h\rangle 1, \varphi\rangle = \langle c_u, \varphi\rangle,$$

for all $\varphi \in C_0^\infty(I)$; here c_u is the constant $\langle u, h\rangle$. This shows that $u = c_u$.

The induction step goes as follows: Assume that the theorem has been proved up to the index k. If $D^{k+1}u = 0$, then $D^k Du = 0$ (think over why!), so Du is a polynomial $p(x)$ of degree $\leq k-1$. Let $P(x)$ be an integral of $ip(x)$; then $D(u - P(x)) = 0$ and hence $u = P(x) + c$. $\square$

Theorem 4.20. *Let I be an open interval of $\mathbb{R}$. Let $m \geq 1$. If $u \in \mathcal{D}'(I)$ and $D^m u \in L_2(I)$, then $u \in H^m(I')$ for each bounded subinterval I' of I.*

Proof. It suffices to show the result for a bounded interval $I =]\alpha, \beta[$. By successive integration of $D^m u$ we obtain a function

$$v(t) = i^m \int_\alpha^t ds_1 \int_\alpha^{s_1} ds_2 \int_\alpha^{s_2} \cdots \int_\alpha^{s_{m-1}} D^m u(s_m) ds_m \,, \tag{4.50}$$

which belongs to $H^m(I)$. Now $D^m v = D^m u$, hence $u - v$ is a polynomial p (of degree $\leq m - 1$) according to Theorem 4.19, hence a C^∞-function. In particular, $u = v + p \in H^m(I)$. □

Remark 4.21. For unbounded intervals, the property $D^m u \in L_2(I)$ is not sufficient for the conclusion that $u \in H^m(I)$ *globally.* But here one can show that when *both u and $D^m u$* are in $L_2(I)$, then u is in $H^m(I)$. It is easily shown by use of the Fourier transformation studied later on, so we save the proof till then. (See Exercise 5.10.)

Remark 4.22. Most of the above theorems can easily be generalized to Sobolev spaces $W_p^m(\Omega)$ (cf. Remark 4.7), with certain modifications for the case $p = \infty$. The L_1 case is of particular interest, for example Theorem 4.19 carries over immediately to $W_1^m(I)$.

The differential operator D^m can now be treated in a similar way as D:

Theorem 4.23. *Let $I =]\alpha, \beta[$ and let $m > 0$. Let $A = D^m$. Then*

$$\begin{aligned} D(A_{\max}) &= H^m(I), \\ D(A_{\min}) &= H_0^m(I). \end{aligned} \tag{4.51}$$

There exists a constant $C_1 > 0$ so that

$$(\|u\|_0^2 + \|D^m u\|_0^2)^{\frac{1}{2}} \leq \|u\|_m \leq C_1(\|u\|_0^2 + \|D^m u\|_0^2)^{\frac{1}{2}} \tag{4.52}$$

for $u \in D(A_{\max})$.

Proof. Clearly, $H^m(I) \subset D(A_{\max})$. The opposite inclusion follows from Theorem 4.20. The first inequality in (4.52) follows from the definition of the m-norm; thus the injection of $D(A_{\max})$ (which is a Hilbert space with respect to the graph-norm since $A_{\max}$ is closed) into $H^m(I)$ is continuous. We have just shown that this injection is surjective, then it follows from the open mapping principle (cf. Theorem B.14) that it is a homeomorphism, so also the second inequality holds.

Now consider $A_{\min}$. Since D^m is formally selfadjoint, $A_{\min} = {A_{\max}}^*$ and $A_{\min} \subset A_{\max}$. Since $D(A_{\min})$ is the closure of $C_0^\infty(I)$ in the graph-norm, and the graph-norm on $D(A_{\max})$ is equivalent with the m-norm, $D(A_{\min})$ equals the closure of $C_0^\infty(I)$ in the m-norm, hence by definition equals $H_0^m(I)$. $\square$

Also here, the minimal operator is too small and the maximal operator too large to be selfadjoint, and one can find intermediate realizations that are selfadjoint. The operators D^m are quite simple since they have constant coefficients; questions for operators with x-dependent coefficients are harder to deal with.

As a concrete example of a differential operator in one variable with variable coefficients, we shall briefly consider a *regular Sturm-Liouville operator:* Let $I =]\alpha, \beta[$, and let L be defined by

$$(Lu)(x) = -\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) + q(x)u(x) \; ; \tag{4.53}$$

where $p \in C^\infty(I) \cap C^1(\overline{I})$ and $q \in C^\infty(I) \cap C^0(\overline{I})$, and

$$p(x) \geq c > 0 \,, \quad q(x) \geq 0 \quad \text{on } \overline{I} \,. \tag{4.54}$$

(Sturm-Liouville operators are often given with a positive factor $1/\varrho(x)$ on the whole expression; this is left out here for simplicity. When the factor is present, the realizations of L should be considered in $L_2(I, \varrho(x)dx)$.)

In an analysis laid out in Exercises 4.11–4.13 one shows that $D(L_{\max}) = H^2(I)$ and $D(L_{\min}) = H_0^2(I)$, and that the realizations of L are characterized by boundary conditions (linear conditions on the boundary values $u(\alpha), u'(\alpha), u(\beta), u'(\beta)$); moreover, symmetric, selfadjoint and semibounded realizations are discussed.

4.4. Boundary value problems in higher dimensions.

Consider an open subset Ω of $\mathbb{R}^n$, where $n \geq 1$. For $H^1(\Omega)$ and $H_0^1(\Omega)$ defined in Definition 4.5 one may expect that, similarly to the case $n = 1$, $\Omega =]\alpha, \beta[$, the *boundary value* $u|_{\partial\Omega}$ has a good meaning when $u \in H^1(\Omega)$ and that the boundary value *is* 0 *exactly when* $u \in H_0^1(\Omega)$. But the functions in $H^1(\Omega)$ need not even be continuous when $n > 1$. For the cases $n \geq 3$ we have the example: $x_1/(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$, which is in $H^1(B(0,1))$; it is bounded but discontinuous (Exercises 3.7 and 4.23). An example in the case $n = 2$ is the function $\log|\log(x_1^2 + x_2^2)^{\frac{1}{2}}|$, which is in $H^1(B(0, \frac{1}{2}))$ and is unbounded at 0 (Exercise 4.15).

However, the concept of boundary value can here be introduced in a more sophisticated way. We shall now show a result which can be considered an easy generalization of (4.45). (The information about which space the boundary value belongs to may be further sharpened: There is a continuous extension of γ_0 as a mapping from $H^1(\overline{\mathbb{R}}^n_+)$ *onto* $H^{\frac{1}{2}}(\mathbb{R}^{n-1})$, see Chapter 9. Sobolev spaces of noninteger order will be introduced in Chapter 6.)

Theorem 4.24. *The map* $\gamma_0\colon u(x',x_n)\mapsto u(x',0)$ *that sends* $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ *into* $C^\infty_0(\mathbb{R}^{n-1})$ *extends by continuity to a continuous map (also denoted* γ_0*) from* $H^1(\mathbb{R}^n_+)$ *into* $L_2(\mathbb{R}^{n-1})$*.*

Proof. As earlier, we denote $(x_1,\dots,x_{n-1})=x'$. For $u\in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, we have the inequality (using that $|2\operatorname{Re}ab|\le |a|^2+|b|^2$ for $a,b\in\mathbb{C}$):

$$\begin{aligned}|u(x',0)|^2&=-\int_0^\infty \partial_n(u(x',x_n)\bar u(x',x_n))\,dx_n\\&=-\int_0^\infty 2\operatorname{Re}(\partial_n u(x',x_n)\bar u(x',x_n))\,dx_n\\&\le\int_0^\infty(|u(x',x_n)|^2+|\partial_n u(x',x_n)|^2)\,dx_n.\end{aligned}$$

(Inequalities of this kind occur already in (4.45), Theorem 4.18.) Integrating with respect to x', we find that

$$\|\gamma_0 u\|^2_{L_2(\mathbb{R}^{n-1})}\le\|u\|^2_{L_2(\mathbb{R}^n_+)}+\|\partial_n u\|^2_{L_2(\mathbb{R}^n_+)}\le\|u\|^2_{H^1(\mathbb{R}^n_+)}.$$

Hence the map γ_0, considered on $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, is bounded with respect to the mentioned norms; and since $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ is dense in $H^1(\mathbb{R}^n_+)$ according to Theorem 4.10, the map γ_0 extends by closure to a continuous map of $H^1(\mathbb{R}^n_+)$ into $L_2(\mathbb{R}^{n-1})$. ☐

Since $\gamma_0 u$ is 0 for $u\in C^\infty_0(\mathbb{R}^n_+)$, it follows from the continuity that $\gamma_0 u=0$ for $u\in H^1_0(\mathbb{R}^n_+)$. The converse also holds:

Theorem 4.25. *The following identity holds:*

$$H^1_0(\mathbb{R}^n_+)=\{\,u\in H^1(\mathbb{R}^n_+)\mid \gamma_0 u=0\,\}.\tag{4.55}$$

Proof. As already noted, the inclusion '$\subset$' follows immediately from the fact that $C^\infty_0(\mathbb{R}^n_+)$ is dense in $H^1_0(\mathbb{R}^n_+)$.

The converse demands more effort. Let $u\in H^1(\mathbb{R}^n_+)$ be such that $\gamma_0 u=0$, then we shall show how u may be approximated by functions in $C^\infty_0(\mathbb{R}^n_+)$.

According to the density shown in Theorem 4.10, there is a sequence of functions $v_k\in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ such that $v_k\to u$ in $H^1(\mathbb{R}^n_+)$ for $k\to\infty$. Since $\gamma_0 u=0$, it follows from Theorem 4.24 that $\gamma_0 v_k\to 0$ in $L_2(\mathbb{R}^{n-1})$ for $k\to\infty$. From the inequality

$$|v_k(x',x_n)|\le|v_k(x',0)|+\int_0^{x_n}|\partial_n v_k(x',y_n)|\,dy_n,\ \text{for } x_n>0,$$

follows, by the Cauchy-Schwarz inequality,

$$\begin{aligned}|v_k(x',x_n)|^2 &\le 2|v_k(x',0)|^2 + 2\big(\int_0^{x_n} |\partial_n v_k(x',y_n)|\,dy_n\big)^2\\ &\le 2|v_k(x',0)|^2 + 2\int_0^{x_n} 1\,dy_n \int_0^{x_n} |\partial_n v_k(x',y_n)|^2\,dy_n\\ &= 2|v_k(x',0)|^2 + 2x_n\int_0^{x_n} |\partial_n v_k(x',y_n)|^2\,dy_n.\end{aligned}$$

Integration with respect to $x'\in\mathbb{R}^{n-1}$ and $x_n\in\,]0,a[$ gives:

$$\begin{aligned}&\int_0^a\int_{\mathbb{R}^{n-1}} |v_k(x',x_n)|^2\,dx'dx_n\\ &\le 2a\int_{\mathbb{R}^{n-1}} |v_k(x',0)|^2\,dx' + 2\int_0^a x_n\int_0^{x_n}\int_{\mathbb{R}^{n-1}} |\partial_n v_k(x',y_n)|^2\,dx'dy_ndx_n\\ &\le 2a\int_{\mathbb{R}^{n-1}} |v_k(x',0)|^2\,dx' + 2\int_0^a x_n\,dx_n\int_0^a\int_{\mathbb{R}^{n-1}} |\partial_n v_k(x',y_n)|^2\,dx'dy_n\\ &= 2a\int_{\mathbb{R}^{n-1}} |v_k(x',0)|^2\,dx' + a^2\int_0^a\int_{\mathbb{R}^{n-1}} |\partial_n v_k(x',y_n)|^2\,dx'dy_n.\end{aligned}$$

For $k\to\infty$ we then find, since $\gamma_0 v_k\to 0$ in $L_2(\mathbb{R}^{n-1})$,

$$\int_0^a\int_{\mathbb{R}^{n-1}} |u(x',x_n)|^2\,dx'dx_n \le a^2\int_0^a\int_{\mathbb{R}^{n-1}} |\partial_n u(x',y_n)|^2\,dy_ndx', \tag{4.56}$$

for $a>0$. For $\varepsilon>0$, consider

$$u_\varepsilon(x',x_n) = (1-\chi(x_n/\varepsilon))u(x',x_n),$$

with χ defined in (2.3). Clearly,

$$\begin{aligned}&u_\varepsilon\to u \text{ in } L_2(\mathbb{R}^n_+) \text{ for } \varepsilon\to 0,\\ &\partial_j u_\varepsilon = (1-\chi(x_n/\varepsilon))\partial_j u\to\partial_j u \text{ for } \varepsilon\to 0 \text{ when } j<n,\\ &\partial_n u_\varepsilon = (1-\chi(x_n/\varepsilon))\partial_n u - \tfrac{1}{\varepsilon}\chi'(x_n/\varepsilon)u, \text{ where}\\ &(1-\chi(x_n/\varepsilon))\partial_n u\to\partial_n u \text{ for } \varepsilon\to 0.\end{aligned}$$

For the remaining term, we have, denoting $\sup|\chi'| = C_1$ and using (4.56),

$$\begin{aligned}\|\partial_n u_\varepsilon - (1-\chi(x_n/\varepsilon))\partial_n u\|^2_{L_2(\mathbb{R}^n_+)} &= \|\tfrac{1}{\varepsilon}\chi'(x_n/\varepsilon)u\|^2_{L_2(\mathbb{R}^n_+)}\\ &\le \varepsilon^{-2}C_1^2\int_0^{2\varepsilon}\int_{\mathbb{R}^{n-1}} |u(x',x_n)|^2\,dx'dx_n\\ &\le 4C_1^2\int_0^{2\varepsilon}\int_{\mathbb{R}^{n-1}} |\partial_n u(x',y_n)|^2\,dx'dy_n\\ &\to 0 \text{ for } \varepsilon\to 0, \text{ since } \partial_n u\in L_2(\mathbb{R}^n_+).\end{aligned}$$

So we may conclude that $u_\varepsilon \to u$ in $H^1(\mathbb{R}^n_+)$ for $\varepsilon \to 0$. Since the support of u_ε is inside $\mathbb{R}^n_+$, with distance $\geq \varepsilon$ from the boundary $\{x_n = 0\}$, u_ε may by truncation and mollification as in the proofs of Lemma 4.8 and Theorem 4.10 be approximated in 1-norm by functions in $C_0^\infty(\mathbb{R}^n_+)$. $\square$

There are other proofs in the literature. For example, one can approximate u by a sequence of continuous functions in H^1 with boundary value 0, and combine truncation and mollification with a translation of the approximating sequence *into* $\mathbb{R}^n_+$, in order to get an approximationg sequence in $C_0^\infty(\mathbb{R}^n_+)$.

Remark 4.26. On the basis of Theorem 4.25 one can also introduce a boundary map γ_0 from $H^1(\Omega)$ to $L_2(\partial\Omega)$ for smooth open bounded sets Ω, by working in local coordinates as in Definition C.1. We omit details here, but will just mention that one has again:

$$H_0^1(\Omega) = \{\, u \in H^1(\Omega) \mid \gamma_0 u = 0 \,\}, \tag{4.57}$$

like for $\Omega = \mathbb{R}^n_+$. A systematic presentation is given e.g. in [LM68].

A very important partial differential operator is the Laplace operator Δ,

$$\Delta u = \partial_{x_1}^2 u + \cdots + \partial_{x_n}^2 u. \tag{4.58}$$

We shall now consider some realizations of $A = -\Delta$ on an open subset Ω of $\mathbb{R}^n$. Introduce first the auxiliary operator S in $L_2(\Omega)$ with domain $D(S) = C_0^\infty(\Omega)$ and action $Su = -\Delta u$. The minimal operator $A_{\min}$ equals $\overline{S}$. Since Δ is clearly formally selfadjoint, S is symmetric; moreover, it is ≥ 0:

$$\begin{aligned}(Su, v) &= \int_\Omega (-\partial_1^2 u - \cdots - \partial_n^2 u)\overline{v}\, dx \qquad (4.59)\\ &= \int_\Omega (\partial_1 u \overline{\partial_1 v} + \cdots + \partial_n u \overline{\partial_n v}) dx = (u, Sv), \text{ for } u, v \in C_0^\infty(\Omega);\end{aligned}$$

where the third expression shows that $(Su, u) \geq 0$ for all $u \in D(S)$. It follows that the closure $A_{\min}$ is likewise symmetric and ≥ 0. Then, moreover, $A_{\max} = A_{\min}{}^*$.

It is clear that $H^2(\Omega) \subset D(A_{\max})$. The inclusion is strict (unless $n = 1$), cf. Exercise 4.5, and Exercise 6.2 later.

For $A_{\min}$ it can be shown that $D(A_{\min}) = H_0^2(\Omega)$, cf. Theorem 6.20 later.

As usual, $A_{\min}$ is too small and $A_{\max}$ is too large for being selfadjoint, but we can use general results for unbounded operators in Hilbert space, as presented in Chapter 12, to construct various selfadjoint realizations.

In fact, Theorem 12.23 shows that S (or $A_{\min}$) has an interesting selfadjoint extension T, namely the Friedrichs extension. It has the same lower bound as S; $m(T) = m(S)$. Let us find out what the associated space V and sesquilinear form $s(u,v)$ are. Following the notation in Chapter 12, we see that s is an extension of

$$s_0(u,v) = \sum_{k=1}^{n} (\partial_k u, \partial_k v),$$

defined on $D(s_0) = D(S) = C_0^\infty(\Omega)$. Since $m(S) \geq 0$, $s_0(u,v) + \alpha \cdot (u,v)$ is a scalar product on $D(s_0)$ for any $\alpha > 0$. Clearly, the associated norm is (for any fixed $\alpha > 0$) *equivalent with the* 1*-norm on* Ω. But then

$$\begin{aligned} V = &\text{ the completion of } C_0^\infty(\Omega) \text{ in } H^1\text{-norm } = H_0^1(\Omega), \\ &s(u,v) = \sum_{j=1}^{n} (\partial_k u, \partial_k v) \text{ on } H_0^1(\Omega). \end{aligned} \tag{4.60}$$

This also explains how T arises from the Lax-Milgram construction (cf. Theorem 12.18); it is the variational operator associated with $(L_2(\Omega), H_0^1(\Omega), s)$, with s defined in (4.60). We can now formulate:

Theorem 4.27. *The Friedrichs extension T of $S = -\Delta|_{C_0^\infty(\Omega)}$ is a selfadjoint realization of $-\Delta$. Its lower bound $m(T)$ is ≥ 0 in general. T is the variational operator determined from the triple (H, V, s) with $H = L_2(\Omega)$, $V = H_0^1(\Omega)$, $s(u,v) = \sum_{k=1}^{n} (\partial_k u, \partial_k v)$.*

T is the unique lower bounded selfadjoint realization of $-\Delta$ with domain contained in $H_0^1(\Omega)$. The domain equals

$$D(T) = D(A_{\max}) \cap H_0^1(\Omega). \tag{4.61}$$

In this sense, T represents the boundary condition

$$\gamma_0 u = 0, \tag{4.62}$$

i.e., the Dirichlet condition.

Proof. It remains to account for the second paragraph. The uniqueness follows from Theorem 12.24. In formula (4.61), the inclusion '$\subset$' is clear since $T \subset A_{\max}$. Conversely, if $u \in D(A_{\max}) \cap H_0^1(\Omega)$, then for any $\varphi \in C_0^\infty(\Omega)$,

$$s(u,\varphi) = \sum_{k=1}^{n} (\partial_k u, \partial_k \varphi) = -\langle u, \sum_{k=1}^{n} \overline{\partial_k^2 \varphi} \rangle = \langle -\Delta u, \overline{\varphi} \rangle = (A_{\max} u, \varphi),$$

where the equality of the first and last expression extends to $v \in H_0^1(\Omega)$ by closure:

$$s(u,v) = (A_{\max} u, v), \text{ for } v \in H_0^1(\Omega). \tag{4.63}$$

Then by definition (cf. (12.43)), u is in the domain of the variational operator defined from (H, V, s), and it acts like $A_{\max}$ on u. $\square$

It is shown further below that $m(T)$ is positive when Ω has finite width.

We note that this theorem assures that the first derivatives are well-defined on $D(T)$ as L_2-functions. It is a deeper fact that is much harder to show, that when Ω is smooth and bounded, also the second derivatives are well-defined as L_2-functions on $D(T)$ (i.e., $D(T) \subset H^2(\Omega)$). Some special cases are treated in Chapter 9, and the general result is shown at the end of Chapter 11.

The Lax-Milgram lemma can also be used to define other realizations of $-\Delta$ which are not reached by the Friedrichs construction departing from S.

Let

$$(H, V_1, s_1(u,v)) = \big(L_2(\Omega), H^1(\Omega), \sum_{k=1}^n (\partial_k u, \partial_k v)\big); \tag{4.64}$$

here s_1 is V_1-coercive and ≥ 0, but not V_1-elliptic, since for example $u = 1$ gives $s_1(u,u) = 0$. Let T_1 be the associated operator in H defined by Corollary 12.19; it is ≥ 0, and selfadjoint since s_1 is symmetric.

Since $C_0^\infty(\Omega) \subset V_1$ and we have for $\varphi \in C_0^\infty(\Omega)$:

$$(T_1 u, \varphi) = s_1(u, \varphi) = \langle u, -\Delta\varphi \rangle = \langle -\Delta u, \varphi \rangle, \text{ when } u \in D(T_1),$$

we see that T_1 *acts like* $-\Delta$ (in the distribution sense); hence $T_1 \subset A_{\max}$. Moreover, we clearly have that $C_0^\infty(\Omega) \subset D(T_1)$, so since T_1 is closed, $T_1 \supset A_{\min}$ and hence T_1 is a realization of $-\Delta$.

The domain $D(T_1)$ consists of the $u \in H^1(\Omega) \cap D(A_{\max})$ for which

$$(-\Delta u, v) = s_1(u,v), \text{ for all } v \in H^1(\Omega), \tag{4.65}$$

cf. Definition 12.15. Let us investigate those elements $u \in D(T_1)$ which moreover belong to $C^2(\overline{\Omega})$, and let us assume that Ω is smooth. For functions in $H^1(\Omega) \cap C^2(\overline{\Omega})$, we have using (A.20):

$$(-\Delta u, v) - s_1(u,v) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \overline{v} \, d\sigma, \text{ when } v \in C^1(\overline{\Omega}) \cap H^1(\Omega). \tag{4.66}$$

Then if $u \in D(T_1)$, the right-hand side of (4.66) must be 0 for all $v \in C^1(\overline{\Omega}) \cap H^1(\Omega)$, so we conclude that

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{4.67}$$

Conversely, if $u \in C^2(\overline{\Omega}) \cap H^1(\Omega)$ and satisfies (4.67), then (4.66) shows that $u \in D(T_1)$. So we see that

$$D(T_1) \cap C^2(\overline{\Omega}) = \{u \in C^2(\overline{\Omega}) \cap H^1(\Omega) \mid \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}. \tag{4.68}$$

In this sense, T_1 represents the *Neumann condition* (4.67). One can in fact show the validity of (4.67) for general elements of $D(T_1)$ using generalizations of (A.20) to Sobolev spaces. Like for the Dirichlet problem there is a deep result showing that $D(T_1) \subset H^2(\Omega)$; here both $-\Delta$, $\gamma_0 u$ and $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ have a good sense. See Sections 9.3 and 11.3.

We have then obtained:

Theorem 4.28. *The variational operator T_1 defined from the triple* (4.64) *is a realization of $-\Delta$ with $D(T_1)$ contained in $H^1(\Omega) \cap D(A_{\max})$ and dense in $H^1(\Omega)$; it satisfies* (4.68)*, and represents the Neumann problem for $-\Delta$ (with* (4.67)*) in a generalized sense.*

One can show a sharpening of the lower bound of S for suitably limited domains. For this, we first show:

Theorem 4.29 (Poincaré). *Let $b > 0$. When φ is in $C_0^\infty(\mathbb{R}^n)$ with* $\operatorname{supp}\varphi$ *contained in a "slab" $\Omega_b = \{\, x \in \mathbb{R}^n \mid 0 \le x_j \le b \,\}$ for a j between 1 and n, then*

$$\|\varphi\|_0 \le \frac{b}{\sqrt{2}} \|\partial_j \varphi\|_0 \,. \tag{4.69}$$

The inequality extends to $u \in H_0^1(\Omega_b)$.

Proof. We can assume that $j = n$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with support in the slab. Since $\varphi(x', 0) = 0$, we find for $x_n \in [0, b]$, using the Cauchy-Schwarz inequality,

$$\begin{aligned} |\varphi(x', x_n)|^2 &= |\int_0^{x_n} \partial_{x_n} \varphi(x', t)\, dt|^2 \le x_n \int_0^{x_n} |\partial_{x_n} \varphi(x', t)|^2\, dt \\ &\le x_n \int_0^b |\partial_n \varphi(x', t)|^2\, dt, \quad \text{and hence} \end{aligned}$$

$$\begin{aligned} \int_0^b |\varphi(x', x_n)|^2\, dx_n &\le \int_0^b x_n\, dx_n \int_0^b |\partial_n \varphi(x', t)|^2\, dt \\ &= \tfrac{1}{2} b^2 \int_0^b |\partial_n \varphi(x', t)|^2\, dt. \end{aligned}$$

This implies by integration with respect to x' (since $\varphi(x)$ is 0 for $x_n \notin [0, b]$):

$$\begin{aligned} \|\varphi(x', x_n)\|^2_{L_2(\mathbb{R}^n)} &\le \tfrac{1}{2} b^2 \int_{\mathbb{R}^{n-1}} \int_0^b |\partial_n \varphi(x', t)|^2\, dt dx' \\ &= \tfrac{1}{2} b^2 \|\partial_n \varphi\|^2_{L_2(\mathbb{R}^n)}. \end{aligned}$$

The inequality extends to functions in $H^1_0(\Omega_R)$ by approximation in H^1-norm by functions in $C_0^\infty(\Omega_b)$.

(This proof has some ingredients in common with the proof of Theorem 4.25.) $\square$

When φ is as in the lemma, we have:

$$(S\varphi, \varphi) = \sum_{k=1}^n \|\partial_k \varphi\|_0^2 \ge \frac{2}{b^2}\|\varphi\|_0^2. \tag{4.70}$$

The resulting inequality extends to $A_{\min}$ by closure. The inequality of course also holds for $\varphi \in C_0^\infty(\Omega)$ when Ω is contained in a translated slab

$$\Omega \subset \{\, x \mid a \le x_j \le a + b \,\}, \tag{4.71}$$

for some $a \in \mathbb{R}$, $b > 0$ and $j \in \{1, \dots, n\}$. It holds in particular when Ω is bounded with diameter $\le b$. The inequality moreover holds when Ω is contained in a slab with thickness b in an arbitrary direction, since this situation by an orthogonal change of coordinates can be carried over into a position as in (4.71), and $-\Delta$ as well as the 1-norm are invariant under orthogonal coordinate changes. So we have, recalling the notation for the lower bound from (12.22):

Corollary 4.30. *When Ω is bounded with diameter $\le b$, or is just contained in a slab with thickness b, then*

$$m(S) = m(A_{\min}) \ge 2b^{-2}. \tag{4.72}$$

It follows that the lower bound of the Friedrichs extension, alias the Dirichlet realization, is positive in this case;

$$m(T) \ge 2b^{-2}. \tag{4.73}$$

It is seen from the analysis in Chapter 13 that there exist many other selfadjoint realizations of $-\Delta$ than the ones we have considered here.

One can also replace $-\Delta$ by more general operators A such as

$$Au = -\sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_k u), \quad a_{jk} \in C^\infty(\overline{\Omega}) \cap L_\infty(\Omega), \tag{4.74}$$

with $\operatorname{Re}\sum_{j,k=1}^n a_{jk}(x)\eta_k\bar{\eta}_j \ge c|\eta|^2$ for $\eta \in \mathbb{C}^n$, some $c > 0$. It is found here that the Lax-Milgram lemma with $H = L_2(\Omega)$, $V = H^1_0(\Omega)$ and sesquilinear form

$$a(u, v) = \sum_{j,k=1}^n (a_{jk}\partial_k u, \partial_j v) \tag{4.75}$$

leads to a realization of A with *Dirichlet boundary condition*, whereas the choice $V = H^1(\Omega)$ and the same sesquilinear form gives a realization of A with a first-order boundary condition (where $\nu = (\nu_1, \dots, \nu_n)$ is the normal to the boundary, cf. Chapter 1):

$$\sum_{j,k=1}^{n} \nu_j a_{jk} \partial_k u = 0 \text{ on } \partial\Omega, \tag{4.76}$$

often called *the oblique Neumann condition* belonging to A.

Note that these variational operators are not in general selfadjoint; they are ususally not normal (cf. Exercise 12.20) either.

Exercises for Chapter 4.

4.1. Let Ω be open $\subset \mathbb{R}^n$ and let $m \in \mathbb{N}$. Show that $H^m(\Omega)$ has a denumerable orthonormal basis, i.e., is separable. (*Hint.* One can use that $H^m(\Omega)$ may be identified with a subspace of $\prod_{|\alpha|\le m} L_2(\Omega)$.)

4.2. Let $A = \sum_{i=1}^n a_i D_i$ be a first-order differential operator on $\mathbb{R}^n$ with constant coefficients. Show that $A_{\max} = A_{\min}$.
(*Hint.* Some ingredients from the proof of Theorem 4.10 may be of use.)

4.3. Let $A = \partial_x^2 - \partial_y^2$ on $\Omega =]0,1[\times]0,1[$ in $\mathbb{R}^2$. Show that $A' = \partial_x^2 - \partial_y^2$. Show that $f(x,y)$ defined by (4.15) belongs to $D(A_{\max})$, with $A_{\max}f = 0$.
(*Hint.* One has to carry out some integrations by part, where it may make the arguments easier to begin by changing to new coordinates $s = x + y$, $t = x - y$.)

4.4. Let $B = \partial_x$ on $\Omega =]0,1[\times]0,1[$ in $\mathbb{R}^2$. Show that $B' = -\partial_x$. Show that f defined by (4.15) does not belong to $D(B_{\max})$.
(*Hint.* If there exists a function $g \in L_2(\Omega)$ such that

$$(f, -\partial_x\varphi)_{L_2(\Omega)} = (g, \varphi)_{L_2(\Omega)} \text{ for all } \varphi \in C_0^\infty(\Omega),$$

then g must be 0 a.e. on $\Omega \setminus \{x = y\}$ (try with $\varphi \in C_0^\infty(\Omega \setminus \{x = y\})$). But then $g = 0$ as an element of $L_2(\Omega)$, and hence $(f, -\partial_x\varphi) = 0$ for all φ. Show by counterexamples that this cannot be true.)

4.5. Let $A = -\Delta$ on $\Omega = B(0,1)$ in $\mathbb{R}^2$. Let $u(x,y)$ be a function given on Ω by

$$u(x,y) = c_0 + \sum_{k\in\mathbb{N}} (c_k(x+iy)^k + c_{-k}(x-iy)^k),$$

i.e., $u(r,\theta) = \sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{in\theta}$ in polar coordinates; assume that $\sup_n |c_n| < \infty$. It is known from elementary PDE (separation of variables methods) that $u \in C^\infty(\Omega)$ with $\Delta u = 0$ on Ω, and that if furthermore $\sum_{n\in\mathbb{Z}} |c_n| < \infty$, then $u \in C^0(\overline{\Omega})$ with boundary value $\sum_{n\in\mathbb{Z}} c_n e^{in\theta} = \varphi(\theta)$.

(a) Show that if merely $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$ (corresponding to $\varphi \in L_2([-\pi,\pi])$), then $u \in D(A_{\max})$.

(b) Give an example where $u \in D(A_{\max})$ but $\partial_x u \notin L_2(\Omega)$.
(*Hint.* One can show by integration in polar coordinates that the expressions

$$s_{M,N}(r,\theta) = \sum_{M\le|n|\le N} c_n r^{|n|} e^{in\theta}$$

satisfy

$$\|s_{M,N}\|^2_{L_2(\Omega)} = C \sum_{M\le |n|\le N} \frac{|c_n|^2}{|n|+1},$$

for some constant C.)

4.6. Let $I =]\alpha, \beta[$.

(a) Let $x_0 \in [\alpha, \beta[$ and let $\zeta \in C^\infty(\overline{I})$ with $\zeta(x_0) = 1$ and $\zeta(\beta) = 0$. Show that for $u \in H^1(I)$,

$$|u(x_0)| = |\int_{x_0}^{\beta} \frac{\partial}{\partial x}(\zeta u)\, dx| \le \|\zeta'\|_{L_2(I)}\|u\|_{L_2(I)} + \|\zeta\|_{L_2(I)}\|u'\|_{L_2(I)}.$$

Deduce (4.45) from this for $m = 1$, by a suitable choice of ζ.

(b) Show that for each $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ so that for $u \in H^1(I)$,

$$|u(x)| \le \varepsilon\|u\|_1 + C(\varepsilon)\|u\|_0 \text{ , for all } x \in \overline{I}.$$

(*Hint*: Choose ζ in (a) of the form $\chi_{\delta,2\delta}(x - x_0)$ for a suitably small δ.)

4.7. Are there other selfadjoint realizations $\widetilde{A}$ than $A_\#$, for $A = D$ on $I =]\alpha, \beta[$?

4.8. Find a selfadjoint realization of $A = -\frac{d^4}{dx^4}$ on $I =]\alpha, \beta[$, and show which boundary condition it represents.
(*Hint.* One can for example use Theorem 12.11.)

4.9. Let $I =]a, b[$. For $A = D$, show that $A_{\max}A_{\min}$ is the Friedrichs extension of $-\frac{d^2}{dx^2}|_{C_0^\infty(I)}$.

4.10. Let $I =]\alpha, \beta[$ and let $m \in \mathbb{N}$. Let A be the differential operator

$$A = D^m + p_1(x)D^{m-1} + \cdots + p_m(x)$$

with coefficients $p_j(x) \in C^\infty(\overline{I})$. Show that $D(A_{\max}) = H^m(I)$ and that $D(A_{\min}) = H_0^m(I)$.
(*Hint.* For the proof that $D(A_{\max}) = H^m(I)$ one can use that Au may be written in the form $D^m u + D^{m-1}(q_1 u) + \cdots + q_m u$, with coefficients $q_j \in C^\infty(\overline{I})$.)

4.11. Let $I =]\alpha, \beta[$, let L be the regular Sturm-Liouville operator defined in (4.53)–(4.54), and let $c > 0$ and $C > 0$ be chosen such that

$$c \le p(x) \le C, \quad |p'(x)| \le C, \quad 0 \le q(x) \le C \quad \text{on } \overline{I}\,.$$

For $u \in H^2(I)$, ϱu denotes the set of four boundary values

$$\varrho u = \{u(\alpha), u(\beta), u'(\alpha), u'(\beta)\} \in \mathbb{C}^4.$$

(a) Show that L is formally selfadjoint.

(b) Show that

$$\varrho : H^2(I) \to \mathbb{C}^4$$

is a continuous linear map, which is surjective.

(c) Show that $D(L_{\max}) = H^2(I)$ and $D(L_{\min}) = H_0^2(I)$.

(d) Show that the realizations of L, i.e., the operators $\widetilde{L}$ with

$$L_{\min} \subset \widetilde{L} \subset L_{\max},$$

are described by boundary conditions:

$$D(\widetilde{L}) = \{\, u \in H^2(I) \mid \varrho u \in W \,\},$$

where W is a subspace of $\mathbb{C}^4$, such that each subspace W corresponds to exactly one realization $\widetilde{L}$.

4.12. Hypotheses and notation as in Exercise 4.11.

(a) Show that one has for u and $v \in H^2(I)$:

$$(Lu, v)_{L_2(I)} - (u, Lv)_{L_2(I)} = (\mathcal{B}\varrho u, \varrho v)_{\mathbb{C}^4}\,,$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & p(\alpha) & 0 \\ 0 & 0 & 0 & -p(\beta) \\ -p(\alpha) & 0 & 0 & 0 \\ 0 & p(\beta) & 0 & 0 \end{pmatrix}.$$

(b) Let $\widetilde{L}$ correspond to W as in Exercise 4.11 (d). Show that $\widetilde{L}$ is symmetric if and only if $W \subset (\mathcal{B}W)^\perp$, and that $\widetilde{L}$ then is selfadjoint precisely when, in addition, $\dim W = 2$; in this case, $W = (\mathcal{B}W)^\perp$.

(c) Find out how the realization is in the cases

(1) $W = \{z \in \mathbb{C}^4 \mid az_1 + bz_3 = 0,\ cz_2 + dz_4 = 0\,\}$ where (a, b) and $(c, d) \in \mathbb{R}^2 \setminus \{(0, 0)\}$;

(2) $W = \{z \in \mathbb{C}^4 \mid z_1 = z_2,\ az_3 + bz_4 = 0\,\}$, where $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$;

(3) $W = \{z \in \mathbb{C}^4 \mid \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = F \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\}$, where F is a complex 2×2-matrix.

(*Comment.* Since $\mathcal{B}$ is skew-selfadjoint ($\mathcal{B}^* = -\mathcal{B}$) and invertible, the sesquilinear form $b(z,w) = (\mathcal{B}z, w)_{\mathbb{C}^4}$ is what is called a non-degenerate symplectic form. The spaces W such that $W = (\mathcal{B}W)^\perp$ are maximal with respect to the vanishing of $b(w,w)$ on them; they are called Lagrangian. See e.g. Everitt and Markus [EM99] for a general study of boundary conditions for operators like the Sturm-Liouville operator, using symplectic forms and decompositions as in Exercise 12.19.)

4.13. Hypotheses and notation as in Exercises 4.11–4.12. Let $l(u,v)$ be the sesquilinear form

$$l(u,v) = \int_\alpha^\beta [pu'\overline{v}' + qu\overline{v}]dx$$

defined on $H^1(I)$.

(a) Show that one has for $u \in H^2(I)$, $v \in H^1(I)$:

$$(Lu,v) = -p(\beta)u'(\beta)\overline{v}(\beta) + p(\alpha)u'(\alpha)\overline{v}(\alpha) + l(u,v).$$

(b) Show that the triple $(H, V, l(u,v))$ with $H = L_2(I)$, $V = H_0^1(I)$ and $l_0(u,v) = l(u,v)$ on V, by the Lax-Milgram lemma defines the realization of L determined by the boundary condition $u(\alpha) = u(\beta) = 0$.

(c) Show that when $V = H_0^1(I)$ in (b) is replaced by $V = H^1(I)$, one gets the realization defined by the boundary condition $u'(\alpha) = u'(\beta) = 0$.

(d) Let T_1 be the operator determined by the Lax-Milgram lemma from the triple $(H, V, l_1(u,v))$ with $H = L_2(I)$, $V = H^1(I)$ and

$$l_1(u,v) = l(u,v) + u(\beta)\overline{v}(\beta).$$

Show that T_1 is a realization of L and find the boundary condition it represents.

4.14. Let I be an open interval of $\mathbb{R}$ (bounded or unbounded) and let Q_n denote the product set

$$Q_n = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_j \in I \ \text{ for } \ j = 1, \dots, n \} .$$

Let $h \in C_0^\infty(I)$ with $\int_I h(t)\,dt = 1$, and let $\tilde{h}(x) = h(x_1)\dots h(x_n)$; it is a function in $C_0^\infty(Q_n)$ with $\langle 1, \tilde{h} \rangle = 1$.

(a) Show that every function $\varphi \in C_0^\infty(Q_n)$ can be written in the form

$$\varphi = \partial_{x_1}\psi_1 + \dots + \partial_{x_n}\psi_n + \langle 1, \varphi \rangle \tilde{h} \ , \qquad (*)$$

where $\psi_1, \dots, \psi_n$ belong to $C_0^\infty(Q_n)$.
(*Hint.* One can for example obtain the formula successively as follows: Let $\varphi_1(x_2, \dots, x_n) = \int_I \varphi(x_1, x_2, \dots, x_n)\, dx_1$, and put

$$\zeta_1(x_1, \dots, x_n) = \int_{-\infty}^{x_1} [\varphi(s, x_2, \dots, x_n) - h(s)\varphi_1(x_2, \dots, x_n)]\, ds\ ;$$

show that $\zeta_1 \in C_0^\infty(Q_n)$ and

$$\varphi = \partial_{x_1}\zeta_1 + h(x_1)\varphi_1(x_2, \dots, x_n)\ .$$

Perform the analogous construction for $\varphi_1 \in C_0^\infty(Q_{n-1})$ and insert in the formula for φ; continue until $(*)$ has been obtained.)
(b) Show that if $v \in \mathcal{D}'(Q_n)$ satisfies

$$\partial_{x_1} v = \partial_{x_2} v = \cdots = \partial_{x_n} v = 0\ ,$$

then v equals a constant (namely the constant $c = \langle v, \tilde{h} \rangle$).

4.15. (a) Consider the function

$$f(x) = \log|\log(x_1^2 + x_2^2)^{\frac{1}{2}}|$$

on the set $M = \{\, x \in \mathbb{R}^2 \mid 0 < |x| < \frac{1}{2}\,\}$. Show that f, $\partial_1 f$ and $\partial_2 f$ are in $L_2(M)$.
(b) Now consider f as an element of $L_2(B(0, \frac{1}{2}))$. Show that the distribution derivatives of f of order 1 are $L_2(B(0, \frac{1}{2}))$ functions equal to the functions $\partial_1 f$ and $\partial_2 f$ defined above on M.
(c) Show that $f \in H^1(B(0, \frac{1}{2})) \setminus C^0(B(0, \frac{1}{2}))$.

4.16. We denote by $L_2(\mathbb{T})$ the space of L_2-functions on $[-\pi, \pi]$ provided with the scalar product and norm

$$(f, g)_{L_2(\mathbb{T})} = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\bar{g}(\theta)\, d\theta, \quad \|f\|_{L_2(\mathbb{T})} = \big(\tfrac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta\big)^{\frac{1}{2}},$$

it identifies with the space of locally square integrable functions on $\mathbb{R}$ with period 2π. (There is the usual convention of identification of functions that are equal almost everywhere.) It is known from the theory of Fourier series that the system of functions $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L_2(\mathbb{T})$ such that when

$$f = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}, \text{ then } \|f\|_{L_2(\mathbb{T})} = \big(\sum_{n\in\mathbb{Z}} |c_n|^2\big)^{\frac{1}{2}}.$$

For m integer ≥ 0, we denote by $C^m(\mathbb{T})$ the space of C^m-functions on $\mathbb{R}$ with period 2π; it identifies with the subspace of $C^m([-\pi,\pi])$ consisting of the functions f with $f^{(j)}(-\pi) = f^{(j)}(\pi)$ for $0 \leq j \leq m$.
(a) Define $H^1(\mathbb{T})$ as the completion of $C^1(\mathbb{T})$ in the norm

$$\|f\|_1 = (\|f\|^2_{L_2(\mathbb{T})} + \|f'\|^2_{L_2(\mathbb{T})})^{\frac{1}{2}}.$$

Show that it identifies with the subspace of $L_2(\mathbb{T})$ consisting of the functions f whose Fourier series satisfy

$$\sum_{n\in\mathbb{Z}} n^2|c_n|^2 < \infty.$$

(b) Show that

$$H^1(\mathbb{T}) = \{\, f(\theta) = \int_{-\pi}^{\theta} g(s)\,ds + k \mid g \in L_2(\mathbb{T}) \text{ with } (g,1)_{L_2(\mathbb{T})} = 0,\ k \in \mathbb{C}\,\}.$$

(c) Also for higher m, one can define $H^m(\mathbb{T})$ as the completion of $C^m(\mathbb{T})$ in the norm

$$\|f\|_m = (\sum_{0\leq j\leq m} \|f^{(j)}\|^2_{L_2(\mathbb{T})})^{\frac{1}{2}}.$$

and show that $f \in H^m(\mathbb{T})$ if and only if $\sum_{n\in\mathbb{Z}} n^{2m}|c_n|^2 < \infty$. Moreover, $H^m(\mathbb{T}) \subset C^{m-1}(\mathbb{T})$.

4.17. Let J be the closed interval $[\alpha,\beta]$ and let $\sigma \in]0,1[$.
(a) By $C^\sigma(J)$ we denote the space of Hölder continuous functions of order σ on J, i.e., functions u for which there exists a constant C (depending on u) so that

$$|u(x) - u(y)| \leq C|x-y|^\sigma, \text{ for } x, y \in J.$$

Show that $C^\sigma(J)$ is a Banach space with the norm

$$\|u\|_{C^\sigma} = \sup_{x,y\in J, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\sigma} + \sup_{x\in J} |u(x)|.$$

(b) By $C^\sigma(\mathbb{T})$ we denote the space of Hölder continuous 2π-periodic functions on $\mathbb{R}$, i.e., functions on $\mathbb{R}$ with period 2π which satisfy the inequality with J replaced by $\mathbb{R}$. Show that it is a Banach space with a norm as in (a) with J replaced by $\mathbb{R}$.

4.18. For $I =]\alpha, \beta[$, show that $H^1(I) \subset C^{\frac{1}{2}}(\overline{I})$. (Notation as in Exercise 4.17.)

4.19. We consider functions u on $\mathbb{R}$ with period 2π, written as trigonometric Fourier series of the type

$$u(x) \sim \sum_{j \in \mathbb{N}} a_j e^{i2^j x}$$

(also called lacunary trigonometric series).

(a) Show that $u \in H^1(\mathbb{T})$ if and only if $\sum_{j\in\mathbb{N}} 2^{2j}|a_j|^2 < \infty$ (cf. Exercise 4.16).

(b) Let $\sigma \in]0,1[$. Show that if $\sum_{j\in\mathbb{N}} 2^{\sigma j}|a_j| < \infty$, then $u \in C^\sigma(\mathbb{T})$ (cf. Exercise 4.17).

(*Hint.* For each term $u_j(x) = a_j e^{i2^j x}$, we have that $|u_j(x)| = |a_j|$ and $|u_j'(x)| = 2^j|a_j|$ for all x, and therefore

$$\begin{aligned} |u_j(x) - u_j(y)| &= |u_j(x) - u_j(y)|^{1-\sigma}|u_j(x) - u_j(y)|^\sigma \\ &\leq |2a_j|^{1-\sigma}(2^j|a_j|)^\sigma|x-y|^\sigma, \end{aligned}$$

by the mean value theorem.)

(c) Show that for each $\sigma \in]0,1[$, $C^\sigma(\mathbb{T}) \setminus H^1(\mathbb{T}) \neq \emptyset$. (*Hint.* Let $a_j = 2^{-j}$.)

4.20. (a) Show that when $u \in H^1(\mathbb{R}_+)$, then $u(x) \to 0$ for $x \to \infty$.
(*Hint.* Observe that $|u(x)|^2 \leq \int_x^\infty (|u(y)|^2 + |u'(y)|^2)\,dy$.)

(b) Show (4.40a), (4.43a) and (4.47a).

4.21. Show that the mapping γ_0 defined in Theorem 4.24 sends $H^m(\mathbb{R}^n_+)$ into $H^{m-1}(\mathbb{R}^{n-1})$ for any $m \in \mathbb{N}$, with $D^{\alpha'}\gamma_0 u = \gamma_0 D^{\alpha'}u$ for $u \in H^m(\mathbb{R}^n_+)$ when $\alpha' = \{\alpha_1, \dots, \alpha_{n-1}\}$ is of length $\leq m-1$.

4.22. Let $a(u,v)$ be the sesquilinear form on $H^1(\mathbb{R}^n_+)$ defined as the scalar product in $H^1(\mathbb{R}^n_+)$:

$$a(u,v) = (u,v)_{L_2(\mathbb{R}^n_+)} + \sum_{j=1}^n (D_j u, D_j v)_{L_2(\mathbb{R}^n_+)},$$

and let $a_0(u,v)$ denote its restriction to $H^1_0(\mathbb{R}^n_+)$.

a) With $H = L_2(\mathbb{R}^n_+)$, $V_0 = H^1_0(\mathbb{R}^n_+)$, let A_γ be the variational operator defined from the triple (H, V_0, a_0). Show that A_γ is the realization of $A = I - \Delta$ with domain

$$D(A_\gamma) = D(A_{\max}) \cap H^1_0(\mathbb{R}^n_+).$$

b) With $V = H^1(\mathbb{R}^n_+)$, let A_ν be the variational operator defined from the triple (H, V, a). Show that A_ν is the realization of $A = I - \Delta$ with domain

$$D(A_\nu) = \{u \in D(A_{\max}) \cap H^1(\mathbb{R}^n_+) \mid (Au, v) = a(u, v) \text{ for all } v \in H^1(\mathbb{R}^n_+)\},$$

and that the $C^2(\overline{\mathbb{R}}^n_+)$-functions in $D(A_\nu)$ satisfy

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}.$$

(*Comment.* A_γ and A_ν represent the Dirichlet resp. Neumann conditions on A. More on these operators in Sections 9.2 and 9.3, where regularity is shown, and a full justification of the Neumann condition is given.)

4.23. Show that the function $x_1/|x|$ is in $H^1(B(0,1))$ when $n \geq 3$.
(*Hint.* One can use that it is known from Exercise 3.7 that the first distribution derivatives are functions in $L_{1,\mathrm{loc}}(B(0,1))$ that coincide with the derivatives defined outside of 0, when $n \geq 2$.)