§3. Distributions. Examples and rules of calculus

3.1. Distributions.

The space $C_0^\infty(\Omega)$ is often denoted $\mathcal{D}(\Omega)$ in the literature. The distributions are simply the elements of the dual space:

Definition 3.1. *A distribution on Ω is a continuous linear functional on $C_0^\infty(\Omega)$. The vector space of distributions on Ω is denoted $\mathcal{D}'(\Omega)$. When $\Lambda \in \mathcal{D}'(\Omega)$, we denote the value of Λ on $\varphi \in C_0^\infty(\Omega)$ by $\Lambda(\varphi)$ or $\langle \Lambda, \varphi \rangle$.*

The tradition is here to take linear (rather than conjugate linear) functionals. But it is easy to change to conjugate linear functionals if needed, for $\varphi \mapsto \Lambda(\varphi)$ is a linear functional on $C_0^\infty(\Omega)$ if and only if $\varphi \mapsto \Lambda(\overline{\varphi})$ is a conjugate linear functional.

See Theorem 2.5 (d) for how the continuity of a functional on $C_0^\infty(\Omega)$ is checked.

The space $\mathcal{D}'(\Omega)$ itself is provided with the weak*-topology, i.e. the topology defined by the system of seminorms p_φ on $\mathcal{D}'(\Omega)$:

$$p_\varphi \colon u \mapsto |\langle u, \varphi \rangle|, \tag{3.1}$$

where φ runs through $C_0^\infty(\Omega)$. We here use Theorem B.5, noting that the family of seminorms is separating (since $u \neq 0$ in $\mathcal{D}'(\Omega)$ means that $\langle u, \varphi \rangle \neq 0$ for some φ).

Let us consider some *examples.* When $f \in L_{1,\mathrm{loc}}(\Omega)$, then the map

$$\Lambda_f \colon \varphi \mapsto \int_\Omega f(x)\varphi(x)\,dx, \tag{3.2}$$

is a distribution. For we have on every K_j (cf. (2.4)), when $\varphi \in C_{K_j}^\infty(\Omega)$,

$$|\Lambda_f(\varphi)| = \Big| \int_{K_j} f(x)\varphi(x) dx \Big| \le \sup |\varphi(x)| \int_{K_j} |f(x)| dx, \tag{3.3}$$

so (2.15) is satisfied with $N_j = 0$ and $c_j = \|f\|_{L_1(K_j)}$. Here one can in fact *identify* Λ_f *with* f, in view of the following fact:

Lemma 3.2. *When $f \in L_{1,\mathrm{loc}}(\Omega)$ with $\int f(x)\varphi(x)dx = 0$ for all $\varphi \in C_0^\infty(\Omega)$, then $f = 0$.*

Proof. Let $\varepsilon > 0$ and consider $v_j(x) = (h_j * f)(x)$ for $j > 1/\varepsilon$ as in Lemma 2.12. When $x \in \Omega_\varepsilon$, then $h_j(x - y) \in C_0^\infty(\Omega)$, so that $v_j(x) = 0$ in Ω_ε. From (2.46) we conclude that $f = 0$ in $\Omega_\varepsilon \cap B(0, R)$. Since ε and R can take all values in $\mathbb{R}_+$, it follows that $f = 0$ in Ω. □

The lemma (and variants of it) is sometimes called “the fundamental lemma of the calculus of variations” or “Du Bois-Reymond’s lemma”.

The lemma implies that when the *distribution* Λ_f defined from $f \in L_{1,\mathrm{loc}}(\Omega)$ by (3.2) gives 0 on all test functions, then the *function* f is equal to 0 as an element of $L_{1,\mathrm{loc}}(\Omega)$. Then the map $f \mapsto \Lambda_f$ is *injective* from $L_{1,\mathrm{loc}}(\Omega)$ to $\mathcal{D}'(\Omega)$, so that we may identify f with Λ_f and write

$$L_{1,\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega). \tag{3.4}$$

The element 0 of $\mathcal{D}'(\Omega)$ will from now on be identified with the function 0 (where we as usual take the continuous representative).

Since $L_{p,\mathrm{loc}}(\Omega) \subset L_{1,\mathrm{loc}}(\Omega)$ for $p > 1$, these space are also naturally injected in $\mathcal{D}'(\Omega)$.

Remark 3.3. Let us also mention how Radon measures fit in here. The space $C_0^0(\Omega)$ of continuous functions with compact support in Ω is defined in (C.7). In topological measure theory it is shown how the vector space $\mathcal{M}(\Omega)$ of complex *Radon measures* μ on Ω can be identified with the space of continuous linear functionals Λ_μ on $C_0^0(\Omega)$ in such a way that

$$\Lambda_\mu(\varphi) = \int_{\operatorname{supp}\varphi} \varphi \, d\mu \quad \text{for} \quad \varphi \in C_0^0(\Omega).$$

Since one has that

$$|\Lambda_\mu(\varphi)| \le |\mu|(\operatorname{supp}\varphi) \cdot \sup|\varphi(x)|, \tag{3.5}$$

Λ_μ is continuous on $C_0^\infty(\Omega)$, hence defines a distribution $\Lambda'_\mu \in \mathcal{D}'(\Omega)$. Since $C_0^\infty(\Omega)$ *is dense in* $C_0^0(\Omega)$ (cf. Theorem 2.15 1°), the map $\Lambda_\mu \mapsto \Lambda'_\mu$ is *injective.* Then the space of complex Radon measures identifies with a subset of $\mathcal{D}'(\Omega)$:

$$\mathcal{M}(\Omega) \subset \mathcal{D}'(\Omega). \tag{3.6}$$

The inclusions (3.4) and (3.6) place $L_{1,\mathrm{loc}}(\Omega)$ and $\mathcal{M}(\Omega)$ as subspaces of $\mathcal{D}'(\Omega)$. They are consistent with the usual injection of $L_{1,\mathrm{loc}}(\Omega)$ in $\mathcal{M}(\Omega)$, where a function $f \in L_{1,\mathrm{loc}}(\Omega)$ defines the Radon measure μ_f by the formula

$$\mu_f(K) = \int_K f \, dx \quad \text{for} \quad K \quad \text{compact} \quad \subset \Omega. \tag{3.7}$$

For, it is known from measure theory that

$$\int f\varphi \, dx = \int \varphi \, d\mu_f \quad \text{for all } \varphi \in C_0^0(\Omega) \tag{3.8}$$

(hence in particular for $\varphi \in C_0^\infty(\Omega)$), so the distributions Λ_f and Λ_{μ_f} coincide.

When $f \in L_{1,\mathrm{loc}}(\Omega)$, we shall usually write f instead of Λ_f; then we also write

$$\Lambda_f(\varphi) = \langle \Lambda_f, \varphi \rangle = \langle f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx. \tag{3.9}$$

Moreover, one often writes μ instead of Λ_μ when $\mu \in \mathcal{M}(\Omega)$. In the following we shall even use the notation f or u (resembling a function) to indicate an arbitrary distribution!

In the systematical theory we will in particular be concerned with the inclusions

$$C_0^\infty(\Omega) \subset L_2(\Omega) \subset \mathcal{D}'(\Omega) \tag{3.10}$$

(and other L_2-inclusions of importance in Hilbert space theory). We shall show how the large gaps between $C_0^\infty(\Omega)$ and $L_2(\Omega)$, and between $L_2(\Omega)$ and $\mathcal{D}'(\Omega)$, are filled out by Sobolev spaces.

Here is another important *example.*

Let x_0 be a point in Ω. The map

$$\delta_{x_0} \colon \varphi \mapsto \varphi(x_0) \tag{3.11}$$

sending a testfunction into its *value at* x_0 is a distribution, for it is clearly a linear map from $C_0^\infty(\Omega)$ to $\mathbb{C}$, and one has for any j, when $\operatorname{supp}\varphi \subset K_j$ (where K_j is as in (2.4)),

$$|\langle \delta_{x_0}, \varphi \rangle| = |\varphi(x_0)| \le \sup\{\, |\varphi(x)| \mid x \in K_j \,\} \tag{3.12}$$

(note that $\varphi(x_0) = 0$ when $x_0 \notin K_j$). Here (2.15) is satisfied with $c_j = 1$, $N_j = 0$, for all j. In a similar way one finds that the maps

$$\Lambda_\alpha \colon \varphi \mapsto (D^\alpha \varphi)(x_0) \tag{3.13}$$

are distributions, now with $c_j = 1$ and $N_j = |\alpha|$ for each j. The distribution (3.11) is the famous "Dirac's δ-function" or "δ-measure ". The notation *measure* is correct, for we can write

$$\langle \delta_{x_0}, \varphi \rangle = \int \varphi \, d\mu_{x_0}, \tag{3.14}$$

where μ_{x_0} is the point measure that has the value 1 on the set $\{x_0\}$ and the value 0 on compact sets disjoint from x_0. The notation δ-*function* is a wild "abuse of notation" (see also (3.22)ff. later). Maybe it has survived because it is so bad that the motivation for introducing the concept of distributions becomes clear.

The distribution δ_0 is often just denoted δ.

Still other distributions are obtained in the following way: Let $f \in L_{1,\mathrm{loc}}(\Omega)$ and let $\alpha \in \mathbb{N}_0^n$. Then the map

$$\Lambda_{f,\alpha} \colon \varphi \mapsto \int f(x)(D^\alpha \varphi)(x)dx \ , \ \varphi \in C_0^\infty(\Omega), \tag{3.15}$$

is a distribution, since we have for any $\varphi \in C_{K_j}^\infty(\Omega)$:

$$|\langle \Lambda_{f,\alpha}, \varphi \rangle| = \Big| \int_{K_j} f \ D^\alpha \varphi \, dx \Big| \le \int_{K_j} |f(x)| dx \cdot \sup_{x \in K_j} |D^\alpha \varphi(x)|; \tag{3.16}$$

here (2.15) is satisfied with $c_j = \|f\|_{L_1(K_j)}$ and $N_j = |\alpha|$ for each j.

One can show that the most general distributions are not much worse than this last example. One has in fact that when Λ is an arbitrary distribution, then for any fixed compact set $K \subset \Omega$ there is an N (depending on K) and a system of functions $f_\alpha \in C^0(\Omega)$ for $|\alpha| \le N$ such that

$$\langle \Lambda, \varphi \rangle = \sum_{|\alpha| \le N} \langle f_\alpha, D^\alpha \varphi \rangle \ \text{ for } \ \varphi \in C_K^\infty(\Omega) \tag{3.17}$$

(the *Structure Theorem*). We shall show this later in connection with the theorem of Sobolev in Chapter 6.

In the fulfillment of (2.15) one cannot always find an N that works for *all* $K_j \subset \Omega$ (only one N_j for each K_j); another way of expressing this is to say that a distribution does not necessarily have a finite *order*, where the concept of order is defined as follows:

Definition 3.4. *We say that* $\Lambda \in \mathcal{D}'(\Omega)$ *is* **of order** $N \in \mathbb{N}_0$ *when the inequalities* (2.15) *hold for* Λ *with* $N_j \le N$ *for all* j *(but the constants* c_j *may very well depend on* j*).* Λ *is said to be of* **infinite order** *if it is not of order* N *for any* N*; otherwise it is said to be of finite order.* **The** *order of* Λ *is the smallest* N *that can be used, resp.* ∞.

In all the examples we have given, the order is finite. Namely, $L_{1,\mathrm{loc}}(\Omega)$ and $\mathcal{M}(\Omega)$ define distributions of order 0 (cf. (3.3), (3.5) and (3.12)), whereas Λ_α and $\Lambda_{f,\alpha}$ in (3.13) and (3.15) are of order $|\alpha|$. To see an example of a distribution of infinte order we consider the distribution $\Lambda \in \mathcal{D}'(\mathbb{R})$ defined by

$$\langle \Lambda, \varphi \rangle = \sum_{N=1}^\infty \langle 1_{[N,2N]}, \varphi^{(N)}(x) \rangle, \tag{3.18}$$

cf. (A.27). (As soon as we have defined the notion of *support* of a distribution it will be clear that when a distribution has *compact* support in Ω, its order is finite, cf. Theorem 3.12 below.)

The theory of distributions was introduced systematically by L. Schwartz; his monograph [S 1950] is still a principal reference in the literature on distributions.

3.2. Rules of calculus for distributions.

When T is a continuous linear operator in $C_0^\infty(\Omega)$, and $\Lambda \in \mathcal{D}'(\Omega)$, then the composition defines another element $\Lambda T \in \mathcal{D}'(\Omega)$, namely the functional

$$(\Lambda T)(\varphi) = \langle \Lambda, T\varphi \rangle.$$

The map $T^\times \colon \Lambda \mapsto \Lambda T$ in $\mathcal{D}'(\Omega)$ is simply the *adjoint map* of the map $\varphi \mapsto T\varphi$. (We write $T^\times$ to avoid conflict with the notation for taking adjoints of operators in complex Hilbert spaces, where a certain conjugate linearity has to be taken into account. The notation T' may also be used, but the prime could be misunderstood as differentiation.)

As shown in Theorem 2.6, the following simple maps are continuous in $C_0^\infty(\Omega)$:

$$\begin{aligned} &M_f \colon \varphi \mapsto f\varphi\,, \quad \text{when } f \in C^\infty(\Omega),\\ &D^\alpha \colon \varphi \mapsto D^\alpha \varphi\,. \end{aligned}$$

They induce two maps in $\mathcal{D}'(\Omega)$ that we shall temporarily denote $M_f^\times$ and $(D^\alpha)^\times$:

$$\begin{aligned} \langle M_f^\times \Lambda, \varphi \rangle &= \langle \Lambda, f\varphi \rangle\\ \langle (D^\alpha)^\times \Lambda, \varphi \rangle &= \langle \Lambda, D^\alpha \varphi \rangle \end{aligned}$$

for $\Lambda \in \mathcal{D}'(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$.

How do these new maps look when Λ itself is a function? If $\Lambda = v \in L_{1,\mathrm{loc}}(\Omega)$, then

$$\langle M_f^\times v, \varphi \rangle = \langle v, f\varphi \rangle = \int v(x) f(x) \varphi(x)\,dx = \langle fv, \varphi \rangle;$$

hence

$$M_f^\times v = fv\,, \quad \text{when } v \in L_{1,\mathrm{loc}}(\Omega). \tag{3.19}$$

When $v \in C^\infty(\Omega)$,

$$\begin{aligned} \langle (D^\alpha)^\times v, \varphi \rangle &= \langle v, D^\alpha \varphi \rangle = \int v(x)(D^\alpha \varphi)(x)\,dx\\ &= (-1)^{|\alpha|} \int (D^\alpha v)(x) \varphi(x)\,dx = \langle (-1)^{|\alpha|} D^\alpha v, \varphi \rangle, \end{aligned}$$

so that

$$(-1)^{|\alpha|}(D^\alpha)^\times v = D^\alpha v\ , \quad \text{when } \ v \in C^\infty(\Omega). \tag{3.20}$$

These formulas motivate the following definition.

Definition 3.5. 1° *When $f \in C^\infty(\Omega)$, we define the operator M_f in $\mathcal{D}'(\Omega)$ by*

$$\langle M_f \Lambda, \varphi\rangle = \langle \Lambda, f\varphi\rangle \quad \textit{for} \ \ \varphi \in C_0^\infty(\Omega).$$

Instead of M_f we often just write f.

2° *For any $\alpha \in \mathbb{N}_0^n$, the operator D^α in $\mathcal{D}'(\Omega)$ is defined by*

$$\langle D^\alpha \Lambda, \varphi\rangle = \langle \Lambda, (-1)^{|\alpha|} D^\alpha \varphi\rangle \ \ \textit{for} \ \ \varphi \in C_0^\infty(\Omega).$$

Similarly, we define the operator ∂^α in $\mathcal{D}'(\Omega)$ by

$$\langle \partial^\alpha \Lambda, \varphi\rangle = \langle \Lambda, (-1)^{|\alpha|} \partial^\alpha \varphi\rangle \ \ \textit{for} \ \ \varphi \in C_0^\infty(\Omega).$$

In particular, these extensions still satisfy: $D^\alpha \Lambda = (-i)^{|\alpha|}\partial^\alpha \Lambda$.

The definition really just says that we denote the adjoint of $M_f \colon \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ by M_f again (usually abbreviated to f), and that we denote the adjoint of $(-1)^{|\alpha|}D^\alpha \colon \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ by D^α; the motivation for this "abuse of notation" lies in the consistency with classical formulas shown in (3.19) and (3.20). As a matter of fact, the abuse is not very grave, since one can show that $C^\infty(\Omega)$ is a *dense* subset of $\mathcal{D}'(\Omega)$, when the latter is provided with the weak*-topology, cf. Theorem 3.18 below, so that the extension of the operators f and D^α from elements $v \in C^\infty(\Omega)$ to $\Lambda \in \mathcal{D}'(\Omega)$ is *uniquely* determined.

Observe also that when $v \in C^k(\Omega)$, the distribution derivatives $D^\alpha v$ coincide with the usual partial derivatives for $|\alpha| \le k$, because of the usual formulas for integration by parts. We may write $(-1)^{|\alpha|}D^\alpha$ as $(-D)^\alpha$.

The exciting aspect of Definition 3.5 is that we can now define *derivatives of distributions* — hence in particular derivatives of functions in $L_{1,\mathrm{loc}}$ which were not differentiable in the original sense.

Note that Λ_α and $\Lambda_{f,\alpha}$ defined in (3.13) and (3.15) satisfy

$$\langle \Lambda_\alpha, \varphi\rangle = \langle (-D)^\alpha \delta_{x_0}, \varphi\rangle\ , \quad \langle \Lambda_{f,\alpha}, \varphi\rangle = \langle (-D)^\alpha f, \varphi\rangle \tag{3.21}$$

for $\varphi \in C_0^\infty(\Omega)$. Let us consider an important example (already mentioned in Chapter 1):

By $H(x)$ we denote the function on $\mathbb{R}$ defined by

$$H(x) = 1_{\{x>0\}} \tag{3.22}$$

(cf. (A.27)); it is called the Heaviside function. Since $H \in L_{1,\mathrm{loc}}(\mathbb{R})$, we have that $H \in \mathcal{D}'(\mathbb{R})$. The derivative in $\mathcal{D}'(\mathbb{R})$ is found as follows:

$$\begin{aligned}\langle \frac{d}{dx} H, \varphi \rangle &= \langle H, -\frac{d}{dx}\varphi \rangle = -\int_0^\infty \varphi'(x)dx \\ &= \varphi(0) = \langle \delta_0, \varphi \rangle \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}).\end{aligned}$$

We see that

$$\frac{d}{dx} H = \delta_0, \tag{3.23}$$

the delta-measure at 0! H and $\frac{d}{dx}H$ are distributions of order 0, while the higher derivatives $\frac{d^k}{dx^k}H$ are of order $k-1$. As shown already in Example 1.1, there is no $L_{1,\mathrm{loc}}(\mathbb{R})$-function that identifies with δ_0.

There is a similar calculation in higher dimensions, based on the Gauss formula (A.18). Let Ω be an open subset of $\mathbb{R}^n$ with C^1-boundary. The function 1_Ω (cf. (A.27)) has distribution derivatives described as follows: For $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\langle \partial_j 1_\Omega, \varphi \rangle \equiv -\int_\Omega \partial_j \varphi \, dx = \int_{\partial\Omega} \nu_j(x)\varphi(x)\, d\sigma. \tag{3.24}$$

Since

$$\Big| \int_{\partial\Omega} \nu_j(x)\varphi(x)\, d\sigma \Big| \le \int_{\partial\Omega\cap K} 1\, d\sigma \cdot \sup_{x \in \partial\Omega\cap K} |\varphi(x)|,$$

when K is a compact set containing $\operatorname{supp}\varphi$, $\partial_j 1_\Omega$ is a distribution in $\mathcal{D}'(\mathbb{R}^n)$ of order 0; (3.24) shows precisely how it acts.

Another important aspect is that the distributions theory allows us to define derivatives of functions which only to a mild degree lack classical derivatives. Recall that the classical concept of differentiation for functions of several variables only works really well when the partial derivatives are *continuous*, for then we can exchange the order of differentiation. More precisely, $\partial_1\partial_2 u = \partial_2\partial_1 u$ holds when u is C^2, whereas the rule often fails for more general functions (e.g. for $u(x_1, x_2) = |x_1|$, where $\partial_1\partial_2 u$ but not $\partial_2\partial_1 u$ has a classical meaning on $\mathbb{R}^2$.)

The new concept of derivative is *insensitive* to the order of differentiation. In fact, $\partial_1\partial_2$ and $\partial_2\partial_1$ define the same operator in $\mathcal{D}'$, since they are carried over to C_0^∞ where they have the same effect:

$$\langle \partial_1\partial_2 u, \varphi \rangle = \langle u, (-\partial_2)(-\partial_1)\varphi \rangle = \langle u, \partial_2\partial_1\varphi \rangle = \langle u, \partial_1\partial_2\varphi \rangle = \langle \partial_2\partial_1 u, \varphi \rangle.$$

In the next lemma, we consider a useful special case of how the distribution definition works for a function that lacks classical derivatives on part of the domain.

Lemma 3.6. *Let $R > 0$, and let $\Omega = B(0,R)$ in $\mathbb{R}^n$; define also $\Omega_\pm = \Omega \cap \mathbb{R}^n_\pm$. Let $k > 0$, and let $u \in C^{k-1}(\overline{\Omega})$ with k'th derivatives defined in Ω_+ and Ω_- in such a way that they extend to continuous functions on $\overline{\Omega}_+$ resp. $\overline{\Omega}_-$ (so u is piecewise C^k). For $|\alpha| = k$, the α'th derivative in the distribution sense is then equal to the function $v \in L_1(\Omega)$ defined by*

$$v = \begin{cases} \partial^\alpha u & \text{on } \Omega_+, \\ \partial^\alpha u & \text{on } \Omega_-. \end{cases} \tag{3.25}$$

Proof. Let $|\alpha| = k$, and write $\partial^\alpha = \partial_j \partial^\beta$, where $|\beta| = k-1$. When $\varphi \in C_0^\infty(\Omega)$, we have if $j = n$ (using the notation $x' = (x_1, \dots, x_{n-1})$):

$$\begin{aligned}
\langle \partial^\alpha u, \varphi \rangle &= -\langle \partial^\beta u, \partial_n \varphi \rangle = -\int_{\Omega_-} \partial^\beta u \partial_n \varphi \, dx - \int_{\Omega_+} \partial^\beta u \partial_n \varphi \, dx \\
&= \int_{\Omega_-} [(\partial_n \partial^\beta u)\varphi - \partial_n(\partial^\beta u \varphi)] dx + \int_{\Omega_+} [(\partial_n \partial^\beta u)\varphi - \partial_n(\partial^\beta u \varphi)] dx \\
&= \int_{\Omega_-} \partial^\alpha u \varphi dx - \int_{|x'|<R} (\lim_{x_n \to 0-} \partial^\beta u \varphi - \lim_{x_n \to 0+} \partial^\beta u \varphi) dx' + \int_{\Omega_+} \partial^\alpha u \varphi dx \\
&= \int_\Omega v \varphi dx;
\end{aligned}$$

we use here that the two contributions from $\{x_n = 0\}$ cancel each other since $\partial^\beta u$ is continuous on Ω. If $j < n$, we get more simply that

$$-\int_{\Omega_\pm} \partial^\beta u \partial_j \varphi \, dx = \int_{\Omega_\pm} \partial^\alpha u \varphi \, dx,$$

using that integration by parts in the x_j-direction gives no boundary contributions since $\operatorname{supp} \varphi \subset \Omega$. It follows that the distribution $\partial^\alpha u$ equals v. $\square$

We note, as a special case of the lemma, that the derivative of the function $|x|$ on the interval $]-1,1[$ is what it should be, namely the discontinuous (but integrable) function

$$\operatorname{sign} x = \begin{cases} 1 \text{ for} & x > 0, \\ -1 \text{ for} & x < 0. \end{cases} \tag{3.25a}$$

The operations multiplication by a smooth function and differentiation are combined in the following rule of calculus:

Lemma 3.7 (The Leibniz formula). *When $u \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega)$, and $\alpha \in \mathbb{N}_0^n$, then*

$$\begin{aligned} \partial^\alpha(f\,u) &= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} u, \\ D^\alpha(f\,u) &= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} u. \end{aligned} \tag{3.26}$$

Proof. When f and u are C^∞-functions, the first formula is obtained by induction from the simplest case

$$\partial_j(f\,u) = (\partial_j f)u + f\partial_j u. \tag{3.27}$$

The same induction works in the distribution case, if we can only show (3.27) in that case. This is done by use of the definitions: For $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \langle \partial_j(fu), \varphi\rangle &= \langle fu, -\partial_j\varphi\rangle = \langle u, -f\partial_j\varphi\rangle = \langle u, -\partial_j(f\varphi) + (\partial_j f)\varphi\rangle \\ &= \langle \partial_j u, f\varphi\rangle + \langle (\partial_j f)u, \varphi\rangle = \langle f\partial_j u + (\partial_j f)u, \varphi\rangle. \end{aligned}$$

The second formula is an immediate consequence. □

Recall that the space $\mathcal{D}'(\Omega)$ is provided with the weak*-topology, i.e. the topology defined by the system of seminorms (3.1)ff.

Theorem 3.8. *Let T be a continuous linear operator in $\mathcal{D}(\Omega)$. Then the adjoint operator in $\mathcal{D}'(\Omega)$, defined by:*

$$\langle T^\times u, \varphi\rangle = \langle u, T\varphi\rangle \text{ for } u \in \mathcal{D}'(\Omega),\ \varphi \in \mathcal{D}(\Omega), \tag{3.28}$$

is a continuous linear operator in $\mathcal{D}'(\Omega)$.

In particular, when $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}_0^n$, then the operators M_f and D^α introduced in Definition 3.5 are continuous in $\mathcal{D}'(\Omega)$.

Proof. Let W be a neighborhood of 0 in $\mathcal{D}'(\Omega)$. Then W contains a neighborhood W_0 of 0 of the form

$$\begin{aligned} W_0 &= W(\varphi_1, \dots, \varphi_N, \varepsilon) \\ &\equiv \{ v \in \mathcal{D}'(\Omega) \mid |\langle v, \varphi_1\rangle| < \varepsilon, \dots, |\langle v, \varphi_N\rangle| < \varepsilon \}, \end{aligned} \tag{3.29}$$

where $\varphi_1, \dots, \varphi_N \in C_0^\infty(\Omega)$. Since $T\varphi_1, \dots, T\varphi_N$ belong to $C_0^\infty(\Omega)$, we can define the neighborhood

$$V = W(T\varphi_1, \dots, T\varphi_N, \varepsilon).$$

Since $\langle T^\times u, \varphi_j \rangle = \langle u, T\varphi_j \rangle$ for each φ_j, we see that $T^\times$ sends V into W_0. This shows the continuity of $T^\times$, and it follows for the operators M_f and D^α in $\mathcal{D}'$, since they are defined as adjoints of continuous operators in $\mathcal{D}(\Omega)$. $\square$

(Further discussions of the topology of $\mathcal{D}'$ are found in Section 3.5 below.)

The topology in $L_{1,\mathrm{loc}}(\Omega)$ is clearly stronger than the topology induced from $\mathcal{D}'(\Omega)$. One has in general that convergence in $C_0^\infty(\Omega)$, $L_p(\Omega)$ or $L_{p,\mathrm{loc}}(\Omega)$ ($p \in [1,\infty]$) implies convergence in $\mathcal{D}'(\Omega)$.

By use of the Banach-Steinhaus theorem (as applied in Appendix B) one obtains the following fundamental property of $\mathcal{D}'(\Omega)$:

Theorem 3.9 (The limit theorem). *A sequence of distributions $u_k \in \mathcal{D}'(\Omega)$ ($k \in \mathbb{N}$) is convergent in $\mathcal{D}'(\Omega)$ for $k \to \infty$ if and only if the sequence $\langle u_k, \varphi \rangle$ is convergent in $\mathbb{C}$ for all $\varphi \in C_0^\infty(\Omega)$. The limit of u_k in $\mathcal{D}'(\Omega)$ is then the functional u defined by*

$$\langle u, \varphi \rangle = \lim_{k\to\infty} \langle u_k, \varphi \rangle \,, \quad \textit{for} \;\; \varphi \in C_0^\infty(\Omega). \tag{3.30}$$

Then also $fD^\alpha u_k \to fD^\alpha u$ in $\mathcal{D}'(\Omega)$ for all $f \in C^\infty(\Omega)$, all $\alpha \in \mathbb{N}_0^n$.

Proof. When the topology is defined by the seminorms (3.28) (cf. Theorem B.5), then $u_k \to v$ in $\mathcal{D}'(\Omega)$ if and only if

$$\langle u_k - v, \varphi \rangle \to 0 \;\; \text{for} \;\; k \to \infty$$

holds for all $\varphi \in C_0^\infty(\Omega)$.

We will show that when we just know that the sequences $\langle u_k, \varphi \rangle$ converge, then there is a distribution $u \in \mathcal{D}'(\Omega)$ so that $\langle u_k - u, \varphi \rangle \to 0$ for all φ. Here we use Corollary B.13 and Theorem 2.5. Define the functional Λ by

$$\Lambda(\varphi) = \lim_{k\to\infty} \langle u_k, \varphi \rangle \quad \text{for} \;\; \varphi \in C_0^\infty(\Omega).$$

According to Theorem 2.5 (c), Λ is continuous from $C_0^\infty(\Omega)$ to $\mathbb{C}$ if and only if Λ defines continuous maps from $C_{K_j}^\infty(\Omega)$ to $\mathbb{C}$ for each K_j. Since $C_{K_j}^\infty(\Omega)$ is a Fréchet space, we can apply Corollary B.13 to the map $\Lambda\colon C_{K_j}^\infty(\Omega) \to \mathbb{C}$, as the limit for $k \to \infty$ of the functionals $u_k\colon C_{K_j}^\infty(\Omega) \to \mathbb{C}$; this gives the desired continuity. The last assertion now follows immediately from Theorem 3.8. $\square$

One has for example that $h_j \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ for $j \to \infty$. (The reader is encouraged to verify this.) Also more general convergence concepts (for nets) can be allowed, by use of Theorem B.12.

3.3. Distributions with compact support.

In the following we often use a convention of "extension by zero" as mentioned for test functions in Section 2.1, namely that a function f defined on a subset ω of Ω is identified with the function on Ω that equals f on ω and equals 0 on $\Omega \setminus \omega$.

Definition 3.10. *Let $u \in \mathcal{D}'(\Omega)$.*

1° *We say that u is 0 on the open subset $\omega \subset \Omega$ when*

$$\langle u, \varphi \rangle = 0 \quad \textit{for all} \;\; \varphi \in C_0^\infty(\omega). \tag{3.31}$$

2° *The* **support of** u *is defined as the set*

$$\operatorname{supp} u = \Omega \setminus \Big(\bigcup \{\omega \mid \omega \textit{ open } \subset \Omega, \; u \textit{ is } 0 \textit{ on } \omega\} \Big). \tag{3.32}$$

Observe for example that the support of the nontrivial distribution $\partial_j 1_\Omega$ defined in (3.24) is contained in $\partial\Omega$ (a deeper analysis will show that $\operatorname{supp} \partial_j 1_\Omega = \partial\Omega$). Since the support of $\partial_j 1_\Omega$ is a null-set in $\mathbb{R}^n$, and 0 is the only $L_{1,\mathrm{loc}}$-function with support in a null-set, $\partial_j 1_\Omega$ cannot be a function in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ (see also the discussion after Lemma 3.2).

Lemma 3.11. *Let $(\omega_\lambda)_{\lambda \in \Lambda}$ be a family of open subsets of Ω. If $u \in \mathcal{D}'(\Omega)$ is 0 on ω_λ for each $\lambda \in \Lambda$, then u is 0 on the union $\bigcup_{\lambda \in \Lambda} \omega_\lambda$.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ with support $K \subset \bigcup_{\lambda \in \Lambda} \omega_\lambda$; we must show that $\langle u, \varphi \rangle = 0$. The compact set K is covered by a finite system of the ω_λ's, say $\omega_1, \dots, \omega_N$. According to Theorem 2.17, there exist $\psi_1, \dots, \psi_N \in C_0^\infty(\Omega)$ with $\psi_1 + \cdots + \psi_N = 1$ on K and $\operatorname{supp} \psi_j \subset \omega_j$ for each j. Now let $\varphi_j = \psi_j \varphi$, then $\varphi = \sum_{j=1}^N \varphi_j$, and $\langle u, \varphi \rangle = \sum_{j=1}^N \langle u, \varphi_j \rangle = 0$ by assumption. □

Because of this lemma, we can also describe the support as *the complement of the largest open set where* u is 0.

An interesting subset of $\mathcal{D}'(\Omega)$ is the set of *distributions with compact support in* Ω. It is usually denoted $\mathcal{E}'(\Omega)$,

$$\mathcal{E}'(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid \operatorname{supp} u \text{ is compact } \subset \Omega\}. \tag{3.33}$$

When $u \in \mathcal{E}'(\Omega)$, there is a j such that $\operatorname{supp} u \subset K_{j-1} \subset K_j^\circ$ (cf. (2.4)). Since $u \in \mathcal{D}'(\Omega)$, there exist c_j and N_j so that

$$|\langle u, \psi \rangle| \le c_j \sup \{ |D^\alpha \psi(x)| \mid x \in K_j \, , \; |\alpha| \le N_j \} \, ,$$

for all ψ with support in K_j. Choose a function $\eta \in C_0^\infty(\Omega)$ which is 1 on a neighborhood of K_{j-1} and has support in K_j° (cf. Corollary 2.14). An *arbitrary test function* $\varphi \in C_0^\infty(\Omega)$ can then be written as

$$\varphi = \eta\varphi + (1-\eta)\varphi,$$

where $\operatorname{supp} \eta\varphi \subset K_j^\circ$ and $\operatorname{supp}(1-\eta)\varphi \subset \complement K_{j-1}$. Since u is 0 on $\complement K_{j-1}$, $\langle u, (1-\eta)\varphi\rangle = 0$, so that

$$\begin{aligned} |\langle u, \varphi\rangle| = |\langle u, \eta\varphi\rangle| &\le c_j \sup\{\, |D^\alpha(\eta(x)\varphi(x))| \mid x \in K_j \ ,\ |\alpha| \le N_j \,\} \\ &\le c' \sup\{\, |D^\alpha\varphi(x)| \mid x \in \operatorname{supp}\varphi \ ,\ |\alpha| \le N_j \,\} \ , \end{aligned} \tag{3.34}$$

where c' depends on the derivatives of η up to order N_j (by the Leibniz formula, cf. also (2.18)). Since φ was arbitrary, this shows that u *has order* N_j (it shows even more: that we can use the same constant c' on all compact sets $K_m \subset \Omega$). We have shown:

Theorem 3.12. *When $u \in \mathcal{E}'(\Omega)$, there is an $N \in \mathbb{N}_0$ so that u has order N.*

Let us also observe that when $u \in \mathcal{D}'(\Omega)$ has compact support, then $\langle u, \varphi\rangle$ can be given a sense also for $\varphi \in C^\infty(\Omega)$ (since it is only the behavior of φ on a neighborhood of the support of u that in reality enters in the expression). The space $\mathcal{E}'(\Omega)$ may in fact be *identified with* the space of continuous functionals on $C^\infty(\Omega)$ (which is sometimes denoted $\mathcal{E}(\Omega)$; this explains the terminology $\mathcal{E}'(\Omega)$ for the dual space). See Exercise 3.11.

Remark 3.13. When Ω' is an open subset of Ω with $\overline{\Omega'}$ compact $\subset \Omega$, and K is compact with $\overline{\Omega'} \subset K^\circ \subset K \subset \Omega$, then an arbitrary distribution $u \in \mathcal{D}'(\Omega)$ can be written as the sum of a distribution supported in K and a distribution which is 0 on Ω':

$$u = \zeta u + (1-\zeta)u, \tag{3.35}$$

where $\zeta \in C_0^\infty(K^\circ)$ is chosen to be 1 on $\overline{\Omega'}$ (such functions exist according to Corollary 2.14). The distribution ζu has support in K since $\zeta\varphi = 0$ for $\operatorname{supp}\varphi \subset \Omega \setminus K$; and $(1-\zeta)u$ is 0 on Ω' since $(1-\zeta)\varphi = 0$ for $\operatorname{supp}\varphi \subset \Omega'$.

In this connection we shal also consider *restrictions* of distributions, and describe how distributions are *glued together*.

When $u \in \mathcal{D}'(\Omega)$ and Ω' is an open subset of Ω, we define the *restriction* of u to Ω' as the element $u|_{\Omega'} \in \mathcal{D}'(\Omega')$ defined by

$$\langle u|_{\Omega'}, \varphi\rangle_{\Omega'} = \langle u, \varphi\rangle_\Omega \quad \text{for} \ \ \varphi \in C_0^\infty(\Omega'). \tag{3.36}$$

(For the sake of precision, we here indicate the duality between $\mathcal{D}'(\omega)$ and $C_0^\infty(\omega)$ by $\langle\ ,\ \rangle_\omega$, when ω is an open set.)

When $u_1 \in \mathcal{D}'(\Omega_1)$ and $u_2 \in \mathcal{D}'(\Omega_2)$, and ω is an open subset of $\Omega_1 \cap \Omega_2$, we say that $u_1 = u_2$ *on* ω, *when*

$$u_1|_\omega - u_2|_\omega = 0 \quad \textit{as an element of } \mathcal{D}'(\omega). \tag{3.37}$$

The following theorem is well-known for continuous functions and for $L_{1,\mathrm{loc}}$-functions.

Theorem 3.14 (Gluing distributions together). *Let $(\omega_\lambda)_{\lambda\in\Lambda}$ be an arbitrary system of open sets in $\mathbb{R}^n$ and let $\Omega = \bigcup_{\lambda\in\Lambda} \omega_\lambda$. Assume that there is given a system of distributions $u_\lambda \in \mathcal{D}'(\omega_\lambda)$ with the property that u_λ equals u_μ on $\omega_\lambda \cap \omega_\mu$, for each pair of indices $\lambda, \mu \in \Lambda$. Then there exists one and only one distribution $u \in \mathcal{D}'(\Omega)$ such that $u|_{\omega_\lambda} = u_\lambda$ for all $\lambda \in \Lambda$.*

Proof. Observe to begin with that there is at most one solution u. For if u and v are solutions, then $(u-v)|_{\omega_\lambda} = 0$ for all λ. This implies that $u - v = 0$, by Lemma 3.11.

We construct u as follows: Let $(K_l)_{l\in\mathbb{N}}$ be a sequence of compact sets as in (2.4) and consider a fixed l. Since K_l is compact, it is covered by a finite subfamily $(\Omega_j)_{j=1,\dots,N}$ of the sets $(\omega_\lambda)_{\lambda\in\Lambda}$; we denote u_j the associated distributions given in $\mathcal{D}'(\Omega_j)$, respectively. By Theorem 2.17 there is a partition of unity $\psi_1, \dots, \psi_N$ consisting of functions $\psi_j \in C_0^\infty(\Omega_j)$ satisfying $\psi_1 + \dots + \psi_N = 1$ on K_l. For $\varphi \in C_{K_l}^\infty(\Omega)$ we set

$$\langle u, \varphi\rangle_\Omega = \langle u, \sum_{j=1}^N \psi_j\varphi\rangle_\Omega = \sum_{j=1}^N \langle u_j, \psi_j\varphi\rangle_{\Omega_j}. \tag{3.38}$$

In this way, we have given $\langle u, \varphi\rangle$ a value which apparently depends on a lot of choices (of l, of the subfamily $(\Omega_j)_{j=1,\dots,N}$ and of the partition of unity $\{\psi_j\}$). But if $(\Omega'_k)_{k=1,\dots,M}$ is another subfamily covering K_l, and $\psi'_1, \dots, \psi'_M$ is an associated partition of unity, we have, with u'_k denoting the distribution given on Ω'_k:

$$\begin{aligned}\sum_{j=1}^N \langle u_j, \psi_j\varphi\rangle_{\Omega_j} &= \sum_{j=1}^N \sum_{k=1}^M \langle u_j, \psi'_k\psi_j\varphi\rangle_{\Omega_j} = \sum_{j=1}^N \sum_{k=1}^M \langle u_j, \psi'_k\psi_j\varphi\rangle_{\Omega_j\cap\Omega'_k}\\ &= \sum_{j=1}^N \sum_{k=1}^M \langle u'_k, \psi'_k\psi_j\varphi\rangle_{\Omega'_k} = \sum_{k=1}^M \langle u'_k, \psi'_k\varphi\rangle_{\Omega'_k}\,,\end{aligned}$$

since $u_j = u'_k$ on $\Omega_j \cap \Omega'_k$. This shows that u has been defined for $\varphi \in C_{K_l}^\infty(\Omega)$ independently of the choice of finite subcovering of K_l and associated partition of unity. If we use such a definition for each K_l, $l = 1, 2, \dots$, we find moreover that these definitions are consistent with each other. Indeed, for both K_l and K_{l+1} one can use one cover and partition of unity chosen for K_{l+1}. (In a similar way one finds that u does not depend on the choice of the sequence $(K_l)_{l\in\mathbb{N}}$.) This defines u as an element of $\mathcal{D}'(\Omega)$.

Now we check the consistency of u with each u_λ as follows: Let $\lambda \in \Lambda$. For each $\varphi \in C_0^\infty(\omega_\lambda)$ there is an l such that $\varphi \in C_{K_l}^\infty(\Omega)$. Then $\langle u, \varphi \rangle$ can be defined by (3.38). Here

$$\begin{aligned}\langle u, \varphi\rangle_\Omega &= \langle u, \sum_{j=1}^N \psi_j \varphi\rangle_\Omega = \sum_{j=1}^N \langle u_j, \psi_j\varphi\rangle_{\Omega_j} \\ &= \sum_{j=1}^N \langle u_j, \psi_j\varphi\rangle_{\Omega_j\cap\omega_\lambda} = \sum_{j=1}^N \langle u_\lambda, \psi_j\varphi\rangle_{\Omega_j\cap\omega_\lambda} = \langle u_\lambda, \varphi\rangle_{\omega_\lambda},\end{aligned}$$

which shows that $u|_{\omega_\lambda} = u_\lambda$. □

In the French literature the procedure is called “recollement des morceaux” (gluing the pieces together).

A very simple example is the case where $u \in \mathcal{E}'(\Omega)$ is glued together with the 0-distribution on a neighborhood of $\mathbb{R}^n \setminus \Omega$. In other words, u is “extended by 0” to a distribution in $\mathcal{E}'(\mathbb{R}^n)$. Such an extension is often tacitly understood.

3.4. Convolutions and coordinate changes.

We here give two other useful applications of Theorem 3.8, namely an extension to $\mathcal{D}'(\mathbb{R}^n)$ of the definition of *convolutions with* φ, and a generalization of *coordinate changes*. First we consider convolutions:

When φ and ψ are in $C_0^\infty(\mathbb{R}^n)$, then $\varphi * \psi$ (recall (2.26)) is in $C_0^\infty(\mathbb{R}^n)$ and satisfies $\partial^\alpha(\varphi * \psi) = \varphi * \partial^\alpha\psi$ for each α. Note here that $\varphi * \psi(x)$ is 0 except if $x - y \in \operatorname{supp}\varphi$ for some $y \in \operatorname{supp}\psi$; the latter means that $x \in \operatorname{supp}\varphi + y$ for some $y \in \operatorname{supp}\psi$, i.e., $x \in \operatorname{supp}\varphi + \operatorname{supp}\psi$. Thus

$$\operatorname{supp}\varphi * \psi \subset \operatorname{supp}\varphi + \operatorname{supp}\psi. \tag{3.39}$$

The map $\psi \mapsto \varphi*\psi$ is continuous, for if K is an arbitrary subset of Ω, then the application of $\varphi*$ to $C_K^\infty(\mathbb{R}^n)$ gives a continuous map into $C_{K+\operatorname{supp}\varphi}^\infty(\mathbb{R}^n)$, since one has for $k \in \mathbb{N}_0$:

$$\begin{aligned}&\sup\{|\partial^\alpha(\varphi * \psi)(x)| \mid x \in \mathbb{R}^n,\, |\alpha| \le k\} \\ &\quad = \sup\{|\varphi * \partial^\alpha\psi(x)| \mid x \in \mathbb{R}^n,\, |\alpha| \le k\} \\ &\quad \le \|\varphi\|_{L_1} \cdot \sup\{|\partial^\alpha\psi(x)| \mid x \in K,\, |\alpha| \le k\} \text{ for } \psi \text{ in } C_K^\infty(\mathbb{R}^n)\,.\end{aligned}$$

One has for φ and χ in $C_0^\infty(\mathbb{R}^n)$, $u \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, denoting $\varphi(-x)$ by $\check{\varphi}(x)$, that

$$\begin{aligned}\langle \varphi * u, \chi\rangle = \int_{\mathbb{R}^n} (\varphi * u)(y)\chi(y)\,dy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(x)u(y-x)\chi(y)\,dxdy \\ &= \int_{\mathbb{R}^n} u(x)(\check{\varphi} * \chi)(x)\,dx = \langle u, \check{\varphi} * \chi\rangle,\end{aligned}$$

by the Fubini theorem. So we see that the adjoint $T^\times$ of $T = \check{\varphi} *\colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ acts like $\varphi *$ on functions in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$. Therefore we define the operator $\varphi *$ on distributions as the adjoint of the operator $\check{\varphi} *$ on test functions:

$$\langle \varphi * u, \chi\rangle = \langle u, \check{\varphi} * \chi\rangle,\ u \in \mathcal{D}'(\mathbb{R}^n)\,,\ \varphi, \chi \in C_0^\infty(\mathbb{R}^n); \tag{3.40}$$

this makes $u \mapsto \varphi * u$ a continuous operator on $\mathcal{D}'(\mathbb{R}^n)$ by Theorem 3.8. The rule

$$\partial^\alpha(\varphi * u) = (\partial^\alpha\varphi) * u = \varphi * (\partial^\alpha u),\ \text{for } \varphi \in C_0^\infty(\mathbb{R}^n),\ u \in \mathcal{D}'(\mathbb{R}^n), \tag{3.41}$$

follows by use of the defining formulas and calculations on test functions:

$$\begin{aligned}\langle \partial^\alpha(\varphi * u), \chi\rangle &= \langle \varphi * u, (-\partial)^\alpha\chi\rangle = \langle u, \check{\varphi} * (-\partial)^\alpha\chi\rangle \\ &= \langle u, (-\partial)^\alpha(\check{\varphi} * \chi)\rangle = \langle \partial^\alpha u, \check{\varphi} * \chi\rangle = \langle \varphi * \partial^\alpha u, \chi\rangle,\ \text{also} \\ &= \langle u, (-\partial)^\alpha\check{\varphi} * \chi\rangle = \langle u, (\partial^\alpha\varphi)^\vee * \chi\rangle = \langle \partial^\alpha\varphi * u, \chi\rangle.\end{aligned}$$

In a similar way one verifies the rule

$$(\varphi * \psi) * u = \varphi * (\psi * u),\ \text{for } \varphi, \psi \in C_0^\infty(\mathbb{R}^n),\ u \in \mathcal{D}'(\mathbb{R}^n). \tag{3.42}$$

We have then obtained:

Theorem 3.15. *When $\varphi \in C_0^\infty(\mathbb{R}^n)$, the convolution map $u \mapsto \varphi * u$ defined by* (3.40) *is continuous in $\mathcal{D}'(\mathbb{R}^n)$; it satisfies* (3.41) *and* (3.42) *there.*

One can define the convolution in higher generality, with more general objects in the place of φ, for example a distribution $v \in \mathcal{E}'(\mathbb{R}^n)$. The procedure does not extend to completely general $v \in \mathcal{D}'(\mathbb{R}^n)$ without any support- or growth-conditions. But if for example u and v are distributions with support in $[0, \infty[^n$, then $v * u$ can be given a sense. (More about convolutions in [S 1950] and [H 1983].)

When $u \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, then we have as in Lemma 2.9 that $\varphi * u$ is a C^∞ function. Note moreover that for each $x \in \mathbb{R}^n$,

$$\varphi * u(x) = \int \varphi(x-y)u(y)\,dy = \langle u, \varphi(x - \cdot)\rangle. \tag{3.43}$$

We shall show that this formula extends to general distributions and defines a C^∞ function even then:

Theorem 3.16. *When $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $\varphi * u$ equals the function of $x \in \mathbb{R}^n$ defined by $\langle u, \varphi(x - \cdot)\rangle$, it is in $C^\infty(\mathbb{R}^n)$.*

Proof. Note first that $x \mapsto \varphi(x - \cdot)$ is continuous from $\mathbb{R}^n$ to $\mathcal{D}(\mathbb{R}^n)$. Then the map $x \mapsto \langle u, \varphi(x - \cdot)\rangle$ is continuous from $\mathbb{R}^n$ to $\mathbb{C}$ (you are asked to think about such situations in Exercise 3.14); let us denote this continuous function $v(x) = \langle u, \varphi(x - \cdot)\rangle$. To see that v is differentiable, one can use the mean value theorem to verify that $\frac{1}{h}[\varphi(x + he_i - \cdot) - \varphi(x - \cdot)]$ converges to $\partial_i\varphi(x - \cdot)$ in $\mathcal{D}(\mathbb{R}^n)$ for $h \to 0$; then (b) in Exercise 3.14 applies. Higher derivatives are included by iteration of the argument.

We have to show that

$$\langle v, \psi\rangle = \langle \varphi * u, \psi\rangle \text{ for all } \psi \in \mathcal{D}.$$

To do this, denote $\operatorname{supp}\psi = K$ and write $\langle v, \psi\rangle$ as a limit of Riemann sums:

$$\begin{aligned}\langle v, \psi\rangle = \int v(x)\psi(x)\,dx &= \lim_{h\to 0+} \sum_{z\in\mathbb{Z}^n, hz\in K} h^n v(hz)\psi(hz)\\ &= \lim_{h\to 0+} \sum_{z\in\mathbb{Z}^n, hz\in K} h^n \langle u, \varphi(hz - \cdot)\rangle\psi(hz)\\ &= \lim_{h\to 0+} \langle u, \sum_{z\in\mathbb{Z}^n, hz\in K} h^n \varphi(hz - \cdot)\psi(hz)\rangle.\end{aligned}$$

Here we observe that $\sum_{z\in\mathbb{Z}^n, hz\in K} h^n\varphi(hz - y)\psi(hz)$ is a Riemann sum for $(\check\varphi * \psi)(y)$, so it converges to $(\check\varphi * \psi)(y)$ for $h \to 0+$, each y. The reader can check that this holds not only pointwise, but uniformly in y; uniform convergence can also be shown for the y-derivatives, and the support (with respect to y) is contained in the compact set $\operatorname{supp}\check\varphi + \operatorname{supp}\psi$ for all h. Thus

$$\sum_{z\in\mathbb{Z}^n, hz\in K} h^n\varphi(hz - \cdot)\psi(hz) \to \check\varphi * \psi \text{ in } \mathcal{D}(\mathbb{R}^n), \text{ for } h \to 0+. \tag{3.44}$$

Applying this to the preceding calculation, we find that

$$\langle v, \psi\rangle = \langle u, \check\varphi * \psi\rangle = \langle \varphi * u, \psi\rangle,$$

as was to be shown. ☐

Here follows a useful special application of convolutions:

Lemma 3.17. *Let $(h_j)_{j\in\mathbb{N}}$ be a sequence as in (2.32). Then (for $j \to \infty$) $h_j * \varphi \to \varphi$ in $C_0^\infty(\mathbb{R}^n)$ when φ in $C_0^\infty(\mathbb{R}^n)$, and $h_j * u \to u$ in $\mathcal{D}'(\mathbb{R}^n)$ when u in $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. For any α, $\partial^\alpha(h_j * \varphi) = h_j * \partial^\alpha \varphi \to \partial^\alpha \varphi$ uniformly (cf. (2.41)). Then $h_j * \varphi \to \varphi$ in $C_0^\infty(\mathbb{R}^n)$. Moreover, $(\check{h}_j)_{j\in\mathbb{N}}$ has the properties (2.32), so

$$\langle h_j * u, \varphi\rangle = \langle u, \check{h}_j * \varphi\rangle \to \langle u, \varphi\rangle, \text{ when } u \in \mathcal{D}'(\mathbb{R}^n),\ \varphi \in C_0^\infty(\mathbb{R}^n).$$

This shows that $h_j * u \to u$ in $\mathcal{D}'(\mathbb{R}^n)$. $\square$

Because of these convergence properties we call a sequence $\{h_j\}$ as in (2.36) an *approximative unit in* $C_0^\infty(\mathbb{R}^n)$ (the name was mentioned already in Chapter 2). Note that the approximating sequence $h_j * u$ consists of C^∞ functions, by Theorem 3.16.

The idea can be modified to show that any distribution in $\mathcal{D}'(\Omega)$ is a limit of functions in $C_0^\infty(\Omega)$:

Theorem 3.18. *Let Ω be open $\subset \mathbb{R}^n$. For any $u \in \mathcal{D}'(\Omega)$ there exists a sequence of functions $u_j \in C_0^\infty(\Omega)$ so that $u_j \to u$ in $\mathcal{D}'(\Omega)$ for $j \to \infty$.*

Proof. Choose K_j and η_j as in Corollary 2.14 2°; then $\eta_j u \to u$ for $j \to \infty$, and each $\eta_j u$ identifies with a distribution in $\mathcal{D}'(\mathbb{R}^n)$. For each j, choose $k_j \geq j$ so large that $\operatorname{supp} \eta_j + \underline{B}(0, \frac{1}{k_j}) \subset K_{j+1}^\circ$; then $u_j = h_{k_j} * (\eta_j u)$ is well-defined and belongs to $C_0^\infty(\Omega)$ (by Theorem 3.16). When $\varphi \in C_0^\infty(\Omega)$, write

$$\langle u_j, \varphi\rangle = \langle h_{k_j} * (\eta_j u), \varphi\rangle = \langle \eta_j u, \check{h}_{k_j} * \varphi\rangle = I_j.$$

Since φ has compact support, there is a j_0 such that for $j \geq j_0$, $\check{h}_{k_j} * \varphi$ is supported in K_{j_0}, hence in K_j for $j \geq j_0$. For such j, we can continue the calculation as follows:

$$I_j = \langle u, \eta_j \cdot (\check{h}_{k_j} * \varphi)\rangle = \langle u, \check{h}_{k_j} * \varphi\rangle \to \langle u, \varphi\rangle, \text{ for } j \to \infty.$$

In the last step we used that $\check{h}_k$ has similar properties as h_k, so that Lemma 3.17 applies to $\check{h}_k * \varphi$. $\square$

Hence $C_0^\infty(\Omega)$ is dense in $\mathcal{D}'(\Omega)$. Thanks to this theorem, we can carry many rules of calculus over from C_0^∞ to $\mathcal{D}'$ by approximation instead of via adjoints. For example, the Leibniz formula (Lemma 3.7) can be deduced from the C_0^∞ case as follows: We know that (3.26) holds if $f \in C^\infty(\Omega)$ and $u \in C_0^\infty(\Omega)$. If $u \in \mathcal{D}'(\Omega)$, let $u_j \to u$ in $\mathcal{D}'(\Omega)$, $u_j \in C_0^\infty(\Omega)$. We have that

$$\partial^\alpha(f\, u_j) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} u_j$$

holds for each j. By Theorem 3.8, each side converges to the corresponding expression with u_j replaced by u, so the rule for u follows.

Finally, we consider coordinate changes. A C^∞-coordinate change (a diffeomorphism) carries C^∞ functions resp. $L_{1,\mathrm{loc}}$ functions into C^∞ functions resp. $L_{1,\mathrm{loc}}$ functions. We sometimes need a similar concept for distributions. As usual, we base the concept on analogy with functions.

Let Ω and Ξ be open sets in $\mathbb{R}^n$, and let κ be a diffeomorphism of Ω onto Ξ. More precisely, κ is a bijective map

$$\kappa\colon x = (x_1, \dots, x_n) \mapsto (\kappa_1(x_1, \dots, x_n), \dots, \kappa_n(x_1, \dots, x_n)), \tag{3.45}$$

where each κ_j is a C^∞ function from Ω to $\mathbb{R}$, and the modulus of the functional determinant

$$J(x) = \left| \det \begin{pmatrix} \frac{\partial \kappa_1}{\partial x_1} & \cdots & \frac{\partial \kappa_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \kappa_n}{\partial x_1} & \cdots & \frac{\partial \kappa_n}{\partial x_n} \end{pmatrix} \right| \tag{3.46}$$

is > 0 for all $x \in \Omega$ (so that $J(x)$ and $1/J(x)$ are C^∞ functions). A function $f(x)$ on Ω is carried over to a function $(Tf)(y)$ on Ξ by the definition

$$(Tf)(y) = f(\kappa^{-1}(y)). \tag{3.47}$$

The usual rules for coordinate changes show that T is a linear operator from $C_0^\infty(\Omega)$ to $C_0^\infty(\Xi)$, from $C^\infty(\Omega)$ to $C^\infty(\Xi)$, and from $L_{1,\mathrm{loc}}(\Omega)$ to $L_{1,\mathrm{loc}}(\Xi)$. Concerning integration, we have when $f \in L_{1,\mathrm{loc}}(\Omega)$ and $\psi \in C_0^\infty(\Xi)$,

$$\begin{aligned} \langle Tf, \psi(y) \rangle_\Xi &= \int_\Xi f(\kappa^{-1}(y))\psi(y)dy = \int_\Omega f(x)\psi(\kappa(x))J(x)dx \\ &= \langle f, J(x)\psi(\kappa(x)) \rangle_\Omega = \langle f, JT^{-1}\psi \rangle_\Omega. \end{aligned} \tag{3.48}$$

We carry this over to distributions by analogy:

Definition 3.19. *When $\kappa = (\kappa_1, \dots, \kappa_n)$ is a diffeomorphism of Ω onto Ξ and $J(x) = |\det\big(\frac{\partial \kappa_i}{\partial x_j}(x)\big)_{i,j=1,\dots,n}|$, we define the coordinate change map $T\colon \mathcal{D}'(\Omega) \to \mathcal{D}'(\Xi)$ by*

$$\langle Tu, \psi(y) \rangle_\Xi = \langle u, J(x)\psi(\kappa(x)) \rangle_\Omega, \tag{3.49}$$

for $\psi \in C_0^\infty(\Xi)$.

Clearly, Tu is a linear functional on $C_0^\infty(\Xi)$. The continuity of this functional follows from the fact that one has for $\psi \in C_0^\infty(\Xi)$ supported in K:

$$|D_x^\alpha(J(x)\psi(\kappa(x)))| \le c_K \sup \left\{ |D_y^\beta \psi(y)| \mid y \in K,\, \beta \le \alpha \right\} \tag{3.50}$$

by the Leibniz formula and the chain rule for differentiation of composed functions. In this way, the map T has been defined such that it is consistent with (3.47) when u is a locally integrable function. (There is a peculiar asymmetry in the transformation rule for f and for ψ in (3.48). In some texts this is removed by introduction of a definition where one views the distributions as a generalization of measures with the functional determinant built in, in a suitable way; so-called *densities*. See e.g. [H 1983, Sect. 6.3].)

Since $T\colon \mathcal{D}'(\Omega) \to \mathcal{D}'(\Xi)$ is defined as the adjoint of the map $J \circ T^{-1}$ from $\mathcal{D}(\Xi)$ to $\mathcal{D}(\Omega)$, T is *continuous* from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Xi)$ by Theorem 3.8 (generalized to the case of a map from $\mathcal{D}(\Omega)$ to $\mathcal{D}(\Xi)$ with two different open sets Ω and Ξ).

Definition 3.19 is useful for example when we consider smooth open subsets of $\mathbb{R}^n$, where we use a coordinate change to "straighten out" the boundary; cf. Appendix C. It can also be used to extend Lemma 3.6 to functions with discontinuities along curved surfaces:

Theorem 3.20. *Let Ω be a smooth open bounded subset of $\mathbb{R}^n$. Let $k \in \mathbb{N}$. If $u \in C^{k-1}(\mathbb{R}^n)$ is such that its k'th derivatives in Ω and in $\mathbb{R}^n \setminus \overline{\Omega}$ exist and can be extended to continuous functions on $\overline{\Omega}$ resp. $\mathbb{R}^n \setminus \Omega$, then the distribution derivatives of u of order k are in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ and coincide with the usual derivatives in Ω and in $\mathbb{R}^n \setminus \overline{\Omega}$ (this determines the derivatives).*

Proof. For each boundary point x we have an open set U_x and a diffeomorphism $\kappa_x\colon U_x \to B(0,1)$ according to Definition C.1; let $U'_x = \kappa_x^{-1}(B(0,\frac{1}{2}))$. Since $\overline{\Omega}$ is compact, the covering of $\partial\Omega$ with the sets U'_x can be reduced to a finite covering system $(\Omega_i)_{i=1,\dots,N}$; the associated diffeomorphisms from Ω_i onto $B(0,\frac{1}{2})$ will be denoted $\kappa_{(i)}$. By the diffeomorphism $\kappa_{(i)}$, $u|_{\Omega_i}$ is carried over to a function v on $B(0,\frac{1}{2})$ satisfying the hypotheses of Lemma 3.6. Thus the k'th derivatives of v in the distribution sense are functions, defined by the usual rules for differentiation inside the two parts of $B(0,\frac{1}{2})$. Since the effect of the diffeomorphism on distributions is consistent with the effect on functions, we see that $u|_{\Omega_i}$ has k'th derivatives that are functions, coinciding with the functions defined by the usual rules of differentiation in $\Omega_i \cap \Omega$ resp. $\Omega_i \cap (\mathbb{R}^n \setminus \overline{\Omega})$. Finally, since u is C^k in the open sets Ω and $\mathbb{R}^n \setminus \overline{\Omega}$, we get the final result by use of the fact that u equals the distribution (function) obtained by gluing the distributions $u|_{\Omega_i} (i = 1, \dots, N)$, $u|_\Omega$ and $u|_{\mathbb{R}^n \setminus \overline{\Omega}}$ together (cf. Theorem 3.14). □

We shall also use the coordinate changes in Chapter 4 (where for example translation plays a role in some proofs) and in Chapter 5, where the Fourier transformed of some particular functions are determined by use of their invariance properties under certain coordinate changes. Moreover, one needs to know what happens under coordinate changes when one wants to consider differential operators on *manifolds*; this will be taken up in Chapter 8.

Example 3.21. Some simple coordinate changes in $\mathbb{R}^n$ that are often used, are *translation*

$$\tau_a(x) = x - a \quad (\text{where } \ a \in \mathbb{R}^n), \tag{3.51}$$

and *dilation*

$$\mu_\lambda(x) = \lambda x \quad (\text{where } \ \lambda \in \mathbb{R} \setminus \{0\}). \tag{3.52}$$

They lead to the coordinate change maps $T(\tau_a)$ and $T(\mu_\lambda)$, which look as follows for functions on $\mathbb{R}^n$:

$$(T(\tau_a)u)(y) = u(\tau_a^{-1}y) = u(y+a) = u(x)\ , \quad \text{where } \ y = x - a, \tag{3.53}$$

$$(T(\mu_\lambda)u)(y) = u(\mu_\lambda^{-1}y) = u(y/\lambda) = u(x)\ , \quad \text{where } \ y = \lambda x, \tag{3.54}$$

and therefore look as follows for distributions:

$$\langle T(\tau_a)u, \psi(y)\rangle_{\mathbb{R}^n_y} = \langle u, \psi(x-a)\rangle_{\mathbb{R}^n_x} = \langle u, T(\tau_{-a})\psi\rangle, \tag{3.55}$$

$$\langle T(\mu_\lambda)u, \psi(y)\rangle_{\mathbb{R}^n_y} = \langle u, |\lambda^n|\psi(\lambda x)\rangle_{\mathbb{R}^n_x} = \langle u, |\lambda^n| T(\mu_{1/\lambda})\psi\rangle, \tag{3.56}$$

since the functional determinants are 1 resp. λ^n.

Another example is an *orthogonal transformation* O (a unitary operator in the real Hilbert space $\mathbb{R}^n$), where the coordinate change for functions on $\mathbb{R}^n$ is described by the formula

$$(T(O)u)(y) = u(O^{-1}y) = u(x)\ , \quad \text{where } \ y = Ox, \tag{3.57}$$

and for distributions then must take the form

$$\langle T(O)u, \psi(y)\rangle_{\mathbb{R}^n_y} = \langle u, \psi(Ox)\rangle_{\mathbb{R}^n_x} = \langle u, T(O^{-1})\psi\rangle, \tag{3.58}$$

since the modulus of the functional determinant is 1.

We shall *write* the coordinate changes as in (3.53), (3.54), (3.57) also when they are applied to distributions; the precise interpretation is then (3.55), (3.56), resp. (3.58).

The chain rule for coordinate changes is easily carried over to distributions by use of Theorem 3.18: When $u \in C_0^\infty(\Omega)$, differentiation of $Tu = u \circ \kappa^{-1} \in C_0^\infty(\Xi)$ is governed by the rule

$$\partial_i(u \circ \kappa^{-1})(y) = \sum_{l=1}^n \frac{\partial u}{\partial x_l}(\kappa^{-1}(y)) \frac{\partial {\kappa^{-1}}_l}{\partial y_i}(y), \tag{3.59}$$

that may also be written

$$\partial_i(Tu) = \sum_{l=1}^{n} \frac{\partial \kappa^{-1}{}_l}{\partial y_i} T(\partial_l u)\,, \tag{3.60}$$

by definition of T. For a general distribution u, choose a sequence u_j in $C_0^\infty(\Omega)$ that converges to u in $\mathcal{D}'(\Omega)$. Since (3.60) holds with u replaced by u_j, and T is continuous from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Xi)$, the validity for u follows by convergence from the validity for the u_j, in view of Theorem 3.8.

3.5 The calculation rules and the weak* topology on $\mathcal{D}'$.

For completeness, we also include a more formal and fast deduction of the rules given above, obtained by a direct appeal to general results for topological vector spaces. (Thanks are due to Esben Kehlet for providing this supplement to an earlier version of the text.)

Let E be a locally convex Hausdorff topological vector space over $\mathbb{C}$. Let E' denote the dual space consisting of the continuous linear maps of E into $\mathbb{C}$. The topology $\sigma(E, E')$ on E defined by the family $(e \mapsto |\eta(e)|)_{\eta\in E'}$ of seminorms is called the weak topology on E, and E provided with the weak topology is a locally convex Hausdorff topological vector space with the dual space E'.

The topology $\sigma(E', E)$ on E' defined by the family $(\eta \mapsto |\eta(e)|)_{e\in E}$ of seminorms is called the weak* topology on E', and E' provided with the weak* topology is a locally convex Hausdorff topological vector space with dual space E.

Let also F denote a locally convex Hausdorff topological vector space, and let T be a linear map of E into F.

If T is continuous, then $\varphi \circ T$ is in E' for φ in F', and $\varphi \mapsto \varphi \circ T$ defines a linear map $T^\times$ of F' into E'. This adjoint map $T^\times$ is weak*-weak* continuous. Indeed, for each e in E, $\varphi \mapsto (T^\times\varphi)(e) = \varphi(Te)$, $\varphi \in F'$, is weak* continuous on F'. The situation is symmetrical: If S is a weak*-weak* continuous linear map of F' into E', then $S^\times$, defined by $\varphi(S^\times e) = (S\varphi)(e)$ for e in E and φ in F', is a weak-weak continuous linear map of E into F.

If T is continuous, it is also weak-weak continuous, since $e \mapsto \varphi(Te) = (T^\times\varphi)(e)$, $e \in E$, is weakly continuous for each φ in F'. (When E and F are Fréchet spaces, the converse also holds, since a weak-weak continuous linear map has a closed graph.)

Lemma 3.22. *Let M be a subspace of E'. If*

$$\{\, e \in E \mid \forall \eta \in M\colon \eta(e) = 0 \,\} = \{0\}, \tag{3.61}$$

then M is weak* dense in E'.

Proof. Assume that there is an η_0 in E' which does not lie in the weak* closure of M. Let U be an open convex neighborhood of η_0 disjoint from M. According to a Hahn–Banach theorem there exists e_0 in E and t in $\mathbb{R}$ such that $\operatorname{Re}\psi(e_0) \le t$ for ψ in M and $\operatorname{Re}\eta_0(e_0) > t$. Since $0 \in M$, $0 \le t$. For ψ in M and an arbitrary scalar λ in $\mathbb{C}$, one has that $\operatorname{Re}[\lambda\psi(e_0)] \le t$; thus $\psi(e_0) = 0$ for ψ in M. By hypothesis, e_0 must be 0, but this contradicts the fact that $\operatorname{Re}\eta_0(e_0) > t \ge 0$. $\square$

Let Ω be a given open set in a Euclidean space $\mathbb{R}^a$, $a \in \mathbb{N}$. We consider the space $C_0^\infty(\Omega)$ of test functions on Ω provided with the (locally convex) topology as an inductive limit of the Fréchet spaces $C_K^\infty(\Omega)$, K compact $\subset \Omega$, and the dual space $\mathcal{D}'(\Omega)$ of distributions on Ω provided with the weak* topology.

For each f in $L_{1,\mathrm{loc}}(\Omega)$, the map $\varphi \mapsto \int_\Omega f\varphi\,dx$, $\varphi \in C_0^\infty(\Omega)$ is a distribution Λ_f on Ω. The map $f \mapsto \Lambda_f$ is a continuous injective linear map of $L_{1,\mathrm{loc}}(\Omega)$ into $\mathcal{D}'(\Omega)$ (in view of the Du Bois-Reymond lemma, Lemma 3.2).

Theorem 3.23. *The subspace $\{\Lambda_\varphi \mid \varphi \in C_0^\infty(\Omega)\}$ is weak* dense in $\mathcal{D}'(\Omega)$.*

Proof. It suffices to show that 0 is the only function ψ in $C_0^\infty(\Omega)$ for which $0 = \Lambda_\varphi(\psi) = \int_\Omega \varphi\psi\,dx$ for every function φ in $C_0^\infty(\Omega)$; this follows from the Du Bois-Reymond lemma. $\square$

Theorem 3.24. *Let there be given open sets Ω in $\mathbb{R}^a$ and Ξ in $\mathbb{R}^b$, $a, b \in \mathbb{N}$ together with a weak-weak continuous linear map A of $C_0^\infty(\Omega)$ into $C_0^\infty(\Xi)$. There is at most one weak*-weak* continuous linear map $\tilde{A}$ of $\mathcal{D}'(\Omega)$ into $\mathcal{D}'(\Xi)$ with $\tilde{A}\,\Lambda_\varphi = \Lambda_{A\varphi}$ for all φ in $C_0^\infty(\Omega)$. Such a map $\tilde{A}$ exists if and only if there is a weak-weak continuous linear map B of $C_0^\infty(\Xi)$ into $C_0^\infty(\Omega)$ so that*

$$\int_\Xi (A\varphi)\psi\,dy = \int_\Omega \varphi(B\psi)dx \quad \text{for} \quad \varphi \in C_0^\infty(\Omega)\,,\ \psi \in C_0^\infty(\Xi)\,. \tag{3.62}$$

In the affirmative case, $B = \tilde{A}^\times$, $\tilde{A} = B^\times$, and $\Lambda_{B\psi} = A^\times(\Lambda_\psi)$, $\psi \in C_0^\infty(\Xi)$.

Remark 3.25. The symbol Λ is here used both for the map of $C_0^\infty(\Omega)$ into $\mathcal{D}'(\Omega)$ and for the corresponding map of $C_0^\infty(\Xi)$ into $\mathcal{D}'(\Xi)$.

Proof. The uniqueness is an immediate consequence of Theorem 3.23.

In the rest of the proof, we set $E = C_0^\infty(\Omega)$, $F = C_0^\infty(\Xi)$.

Assume that $\tilde{A}$ exists as desired, then $\tilde{A}^\times$ is a weak-weak continuous map of F into E with

$$\begin{aligned}\int_\Omega \varphi(\tilde{A}^\times\psi)dx &= \Lambda_\varphi(\tilde{A}^\times\psi) = \tilde{A}(\Lambda_\varphi)(\psi) = \Lambda_{A\varphi}(\psi)\\ &= \int_\Xi (A\varphi)\psi\,dy\,,\ \text{for } \varphi \in E\,,\ \psi \in F\,;\end{aligned}$$

so we can use $\tilde{A}^\times$ as B.

Assume instead that B exists; then $B^\times$ is a weak*-weak* continuous linear map of $\mathcal{D}'(\Omega)$ into $\mathcal{D}'(\Xi)$, and

$$\begin{aligned}(B^\times\Lambda_\varphi)(\psi) = \Lambda_\varphi(B\psi) = \int_\Omega \varphi(B\psi)dx = \int_\Xi (A\varphi)\psi\,dy \\ = \Lambda_{A\varphi}(\psi)\,, \text{ for } \varphi \in E\,,\ \psi \in F\,,\end{aligned}$$

so that $B^\times\Lambda_\varphi = \Lambda_{A\varphi}$, $\varphi \in E$; hence we can use $B^\times$ as $\tilde{A}$. Moreover we observe that

$$\begin{aligned}\Lambda_{B\psi}(\varphi) = \int_\Omega \varphi(B\psi)dx = \int_\Xi (A\varphi)\psi\,dy = \Lambda_\psi(A\varphi) \\ = A^\times(\Lambda_\psi)(\varphi), \text{ for } \varphi \in E,\ \psi \in F\,,\end{aligned}$$

so that $\Lambda_{B\psi} = A^\times(\Lambda_\psi)$, $\psi \in F$. $\square$

Remark 3.26. If a weak-weak continuous linear map

$$A\colon C_0^\infty(\Omega) \to C_0^\infty(\Xi)$$

has the property that there for each compact subset K of Ω exists a compact subset L of Ξ so that $A(C_K^\infty(\Omega)) \subseteq C_L^\infty(\Xi)$, then A is continuous, since $A|_{C_K^\infty(\Omega)}$ is closed for each K. Actually, all the operators we shall consider are continuous.

PROGRAM: When you meet a continuous linear map $A\colon C_0^\infty(\Omega) \to C_0^\infty(\Xi)$, you should look for a corresponding map B. When B has been found, drop the tildas ("lägg bort tildarna"[1]) and define $(Au)(\psi) = u(B\psi)\,,\ u \in \mathcal{D}'(\Omega)\,,\ \psi \in C_0^\infty(\Xi)$. It often happens so that A is the restriction to $C_0^\infty(\Omega)$ of an operator defined on a larger space of functions. One should therefore think about which functions f in $L_{1,\mathrm{loc}}(\Omega)$ that have the property $A\Lambda_f = \Lambda_{Af}$.

The program looks as follows for the operators discussed above in Sections 3.2 and 3.4.

The example: Multiplication. Let f be a function in $C^\infty(\Omega)$. The multiplication by f defines a continuous operator $M_f\colon \varphi \mapsto f\varphi$ on $C_0^\infty(\Omega)$. Since

$$\int_\Omega (f\varphi)\psi\,dx = \int_\Omega \varphi(f\psi)dx\,,\ \varphi, \psi \in C_0^\infty(\Omega)\,,$$

[1] In Swedish, "lägg bort titlarna" means "put away titles" — go over to using first names.

we define $M_f u = fu$ by

$$(fu)(\varphi) = u(f\varphi)\,,\; u \in \mathcal{D}'(\Omega)\,,\; \varphi \in C_0^\infty(\Omega)\,;$$

M_f is a continuous operator on $\mathcal{D}'(\Omega)$.

For g in $L_{1,\mathrm{loc}}(\Omega)$,

$$(f\Lambda_g)(\varphi) = \int_\Omega gf\varphi\, dx = \Lambda_{fg}(\varphi)\,,\; \varphi \in C_0^\infty(\Omega)\,.$$

The example: Differentiation. For $\alpha \in \mathbb{N}_0^a$, ∂^α is a continuous operator on $C_0^\infty(\Omega)$. For φ and ψ in $C_0^\infty(\Omega)$,

$$\int_\Omega (\partial^\alpha \varphi)\psi\, dx = (-1)^{|\alpha|} \int_\Omega \varphi(\partial^\alpha \psi) dx\,.$$

We therefore define a continuous operator ∂^α on $\mathcal{D}'(\Omega)$ by

$$(\partial^\alpha u)(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi)\,,\; u \in \mathcal{D}'(\Omega)\,,\; \varphi \in C_0^\infty(\Omega)\,.$$

If we identify f with Λ_f for f in $L_{1,\mathrm{loc}}(\Omega)$, we have given $\partial^\alpha f$ a sense for any function f in $L_{1,\mathrm{loc}}(\Omega)$.

When f is so smooth that we can use the formula for integration by parts, e.g. for f in $C^{|\alpha|}(\Omega)$,

$$\begin{aligned}(\partial^\alpha \Lambda_f)(\varphi) = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \varphi\, dx &= \int_\Omega (\partial^\alpha f)\varphi\, dx \\ &= \Lambda_{\partial^\alpha f}(\varphi),\ \text{for } \varphi \in C_0^\infty(\Omega)\,.\end{aligned}$$

The Leibniz formula now follows directly from the smooth case by extension by continuity in view of Theorem 3.23.

The example: Convolution. When φ and ψ are in $C_0^\infty(\mathbb{R}^n)$, then, as noted earlier, $\varphi * \psi$ is in $C_0^\infty(\mathbb{R}^n)$ and satisfies $\partial^\alpha(\varphi * \psi) = \varphi * \partial^\alpha \psi$ for each α, and the map $\psi \mapsto \varphi * \psi$ is continuous.

For φ, ψ and χ in $C_0^\infty(\mathbb{R}^n)$ we have, denoting $\varphi(-x)$ by $\check{\varphi}(x)$, that

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi * \psi(y)\chi(y) dy &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(x)\varphi(y-x)\chi(y) dx dy \\ &= \int_{\mathbb{R}^n} \psi(x) \chi * \check{\varphi}(x) dx\,;\end{aligned}$$

therefore we define

$$(\varphi * u)(\chi) = u(\check{\varphi} * \chi)\,,\; u \in \mathcal{D}'(\mathbb{R}^n)\,,\; \varphi, \chi \in C_0^\infty(\mathbb{R}^n)\,;$$

this makes $u \mapsto \varphi * u$ a continuous operator on $\mathcal{D}'(\mathbb{R}^n)$.

For f in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$,

$$(\varphi * \Lambda_f)(\psi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\varphi(x-y)\psi(x)dxdy = \int_{\mathbb{R}^n} \varphi * f(x)\psi(x)dx$$

by the Fubini theorem, so $\varphi * \Lambda_f = \Lambda_{\varphi * f}$ for $\varphi \in C_0^\infty(\mathbb{R}^n)$, $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$. For $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$, the property

$$\partial^\alpha(\varphi * u) = (\partial^\alpha \varphi) * u = \varphi * (\partial^\alpha u),$$

now follows simply by extension by continuity.

The example: Change of coordinates. Coordinate changes can also be handled in this way. Let κ be a C^∞ diffeomorphism of Ω onto Ξ with the modulus of the functional determinant equal to J. Define $T(\kappa)\colon C_0^\infty(\Omega) \to C_0^\infty(\Xi)$ by

$$(T(\kappa)\varphi)(y) = \varphi(\kappa^{-1}(y))\,,\; \varphi \in C_0^\infty(\Omega)\,,\; y \in \Xi\,.$$

The map $T(\kappa)$ is continuous according to the chain rule and the Leibniz formula and we have that

$$\int_\Xi T(\kappa)\varphi \cdot \psi\, dy = \int_\Omega \varphi \cdot \psi \circ \kappa \cdot J\, dx\,,$$
$$\text{for } \varphi \in C_0^\infty(\Omega),\; \psi \in C_0^\infty(\Xi).$$

Then

$$(T(\kappa)u)(\psi) = u(\psi \circ \kappa \cdot J)\,,\; \psi \in C_0^\infty(\Xi),\; u \in \mathcal{D}'(\Omega),$$

defines a continuous linear map $T(\kappa)$ of $\mathcal{D}'(\Omega)$ into $\mathcal{D}'(\Xi)$.

It is easily seen that $T(\kappa)\Lambda_f = \Lambda_{f\circ\kappa^{-1}}$ for $f \in L_{1,\mathrm{loc}}(\Omega)$.

Exercises for Chapter 3.

3.1. Show that convergence of a sequence in $C_0^\infty(\Omega)$, $C^\infty(\Omega)$, $L_p(\Omega)$ or $L_{p,\mathrm{loc}}(\Omega)$ ($p \in [1,\infty]$) implies convergence in $\mathcal{D}'(\Omega)$.

3.2. (a) With $f_n(x)$ defined by

$$f_n(x) = \begin{cases} n & \text{for } x \in \left[-\frac{1}{2n}, \frac{1}{2n}\right], \\ 0 & \text{for } x \in \mathbb{R} \setminus \left[-\frac{1}{2n}, \frac{1}{2n}\right], \end{cases}$$

show that $f_n \to \delta$ in $\mathcal{D}'(\mathbb{R})$ for $n \to \infty$.
(b) With

$$g_n(x) = \frac{1}{\pi} \frac{\sin nx}{x},$$

show that $g_n \to \delta$ in $\mathcal{D}'(\mathbb{R})$, for $n \to \infty$.
(One can use the Riemann-Lebesgue lemma from Fourier theory.)

3.3. Let $f(x)$ be a function on $\mathbb{R}$ such that f is C^∞ on each of the intervals $]-\infty, x_0[$ and $]x_0, +\infty[$, and such that the limits $\lim_{x\to x_0+} f^{(k)}(x)$ and $\lim_{x\to x_0-} f^{(k)}(x)$ exist for all $k \in \mathbb{N}_0$. Denote by $f_k(x)$ the *function* that equals $f^{(k)}(x)$ for $x \neq x_0$. Show that the *distribution* $f \in \mathcal{D}'(\mathbb{R})$ is such that its derivative ∂f identifies with the sum of the function f_1 (considered as a distribution) and the distribution $c\delta_{x_0}$, where $c = \lim_{x\to x_0+} f(x) - \lim_{x\to x_0-} f(x)$; briefly expressed:

$$\partial f = f_1 + c\delta_{x_0} \text{ in } \mathcal{D}'(\mathbb{R}).$$

Find similar expressions for $\partial^k f$, for all $k \in \mathbb{N}$.

3.4. Consider the series $\sum_{k\in\mathbb{Z}} e^{ikx}$ for $x \in I =]-\pi, \pi[$ (this series is in the usual sense divergent at all points $x \in I$).
(a) Show that the sequences $\sum_{0\le k\le M} e^{ikx}$ and $\sum_{-M\le k<0} e^{ikx}$ converge to distributions Λ_+ resp. Λ_- in $\mathcal{D}'(I)$ for $M \to \infty$, and find $\Lambda = \Lambda_+ + \Lambda_-$. (We say that the series $\sum_{k\in\mathbb{Z}} e^{ikx}$ converges to Λ in $\mathcal{D}'(I)$.)
(b) Show that for any $N \in \mathbb{N}$, the series $\sum_{k\in\mathbb{Z}} k^N e^{ikx}$ converges to a distribution Λ_N in $\mathcal{D}'(I)$, and show that $\Lambda_N = D^N \Lambda$.

3.5. For $a \in \mathbb{R}_+$, let

$$f_a(x) = \frac{a}{\pi} \frac{1}{x^2 + a^2} \text{ for } x \in \mathbb{R}.$$

Show that $f_a \to \delta$ in $\mathcal{D}'(\mathbb{R})$ for $a \to 0+$.

3.6. (Distributions supported in a point.) Let u be a distribution on $\mathbb{R}^n$ with support $= \{0\}$. Then there exists an N so that u has order N. Denote $\chi(x/r) = \zeta_r(x)$ for $r \in]0,1]$.

(a) *The case* $N = 0$. Show that there is a constant c_1 so that

$$|\langle u, \varphi\rangle| \le c_1 |\varphi(0)| \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^n). \tag{1}$$

(Apply the distribution to $\varphi = \zeta_r\varphi + (1-\zeta_r)\varphi$ and let $r \to 0$.) Show that there is a constant a so that

$$u = a\delta. \tag{2}$$

(*Hint.* One can show that $\langle u, \varphi\rangle = \langle u, \zeta_1\varphi\rangle + 0 = \langle u, \zeta_1\rangle\varphi(0)$.)

(b) *The case* $N > 0$. Show that the function ζ_r satisfies

$$|\partial^\alpha \zeta_r(x)| \le c_\alpha r^{-|\alpha|} \text{ for each } \alpha \in \mathbb{N}_0^n, \tag{3}$$

when $r \in]0,1]$.

Let $V = \{\psi \in C_0^\infty(\mathbb{R}^n) \mid \partial^\alpha\psi(0) = 0 \text{ for all } |\alpha| \le N\}$, and show that there are inequalities for each $\psi \in V$:

$$|\psi(x)| \le c|x|^{N+1} \text{ for } x \in \mathbb{R}^n; \tag{4}$$

$$|\partial^\alpha(\zeta_r(x)\psi(x))| \le c' r^{N+1-|\alpha|} \text{ for } x \in \mathbb{R}^n, r \in]0,1] \text{ and } |\alpha| \le N; \tag{5}$$

$$|\langle u, \zeta_r\psi\rangle| \le c' r \text{ for all } r \in]0,1]; \tag{6}$$

and hence

$$\langle u, \psi\rangle = 0 \text{ when } \psi \in V. \tag{7}$$

Show that there are constants a_α so that

$$u = \sum_{|\alpha|\le N} a_\alpha \partial^\alpha \delta. \tag{8}$$

(*Hint.* One may use that $\langle u, \varphi\rangle = \langle u, \zeta_1\varphi\rangle = \langle u, \zeta_1 \sum_{|\alpha|\le N} \frac{\partial^\alpha\varphi(0)}{\alpha!} x^\alpha + \psi(x)\rangle = \sum_{|\alpha|\le N} \langle u, \zeta_1 \frac{(-x)^\alpha}{\alpha!}\rangle (-\partial)^\alpha\varphi(0)$.)

3.7. We consider $\mathcal{D}'(\mathbb{R}^n)$ for $n \ge 2$.

(a) Show that the function $f(x) = \dfrac{x_1}{|x|}$ is bounded and belongs to $L_{1,\mathrm{loc}}(\mathbb{R}^n)$.

(b) Show that the first-order classical derivatives of f, defined for $x \ne 0$, are functions in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$.

(c) Show that the first-order derivatives of f defined in the distribution sense on $\mathbb{R}^n$ equal the functions defined under (b).
(*Hint.* It is sufficient to consider f and $\partial_{x_j} f$ on $B(0,1)$. One can here calculate $\langle \partial_{x_j} f, \varphi\rangle = -\langle f, \partial_{x_j}\varphi\rangle$ for $\varphi \in C_0^\infty(B(0,1))$ as an integral over $B(0,1) = [B(0,1)\setminus B(0,\varepsilon)] \cup B(0,\varepsilon)$, using formula (A.20) and letting $\varepsilon \to 0$.)

3.8. (a) Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Show that $\langle \delta, \varphi\rangle = 0 \implies \varphi\delta = 0$.
(b) Consider $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. Find out whether one of the following implications holds for arbitrary u and φ:

$$\langle u, \varphi\rangle = 0 \implies \varphi u = 0; \tag{i}$$

or

$$\varphi u = 0 \implies \langle u, \varphi\rangle = 0. \tag{ii}$$

3.9. (a) Let $\Omega = \mathbb{R}^n$. Show that *the order* of the distribution $D^\alpha\delta$ equals $|\alpha|$. Show that when M is an interval $[a,b]$ of $\mathbb{R}$ ($a < b$), then the order of $D^j 1_M$ equals $j-1$.
(b) Let $\Omega = \mathbb{R}$. Show that the functional Λ_1 defined by

$$\langle \Lambda_1, \varphi\rangle = \sum_{N=1}^{\infty} \varphi^{(N)}(N) \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}),$$

is a distribution on $\mathbb{R}$ whose order equals ∞. Show that the functional Λ defined by (3.18) is a distribution whose order equals ∞.

3.10. Let Ω be a smooth open subset of $\mathbb{R}^n$, or let $\Omega = \mathbb{R}^n_+$.
(a) Show that $\operatorname{supp} \partial_j 1_\Omega \subset \partial\Omega$.
(b) Show that the distribution $(-\Delta)1_\Omega$ on $\mathbb{R}^n$ satisfies

$$\langle (-\Delta)1_\Omega, \varphi\rangle = \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\, d\sigma \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^n)$$

(cf. (A.20)), and determine the order and support of the distribution in the case $\Omega = \mathbb{R}^n_+$.

3.11. Show that the space $C^\infty(\Omega)'$ of continuous linear functionals on $C^\infty(\Omega)$ can be identified with the subspace $\mathcal{E}'(\Omega)$ of $\mathcal{D}'(\Omega)$, in such a way that when $\Lambda \in C^\infty(\Omega)'$ is identified with $\Lambda_1 \in \mathcal{D}'(\Omega)$, then

$$\Lambda(\varphi) = \langle \Lambda_1, \varphi\rangle \quad \text{for } \varphi \in C_0^\infty(\Omega).$$

3.12. On often meets the notation $\delta(x)$ for the distribution δ_0. Moreover it is customary (e.g. in physics texts) to write $\delta(x-a)$ for the distribution δ_a, $a \in \mathbb{R}$; this is motivated by the heuristic calculation

$$\int_{-\infty}^{\infty} \delta(x-a)\varphi(x)\,dx = \int_{-\infty}^{\infty} \delta(y)\varphi(y+a)\,dy = \varphi(a), \text{ for } \varphi \in C_0^\infty(\mathbb{R}).$$

(a) Motivate by a similar calculation the formula

$$\delta(ax) = \frac{1}{|a|}\delta(x), \quad \text{for } a \in \mathbb{R} \setminus \{0\}.$$

(b) Motivate the following formula:

$$\delta(x^2 - a^2) = \frac{1}{2a}\big(\delta(x-a) + \delta(x+a)\big), \quad \text{for } a > 0.$$

(*Hint.* One can calculate the integral $\int_{-\infty}^{\infty} \delta(x^2-a^2)\varphi(x)\,dx$ heuristically by decomposing it into integrals over $]-\infty, 0[$ and $]0, \infty[$ and use of the change of variables $x = \pm\sqrt{y}$. A precise account of how to compose distributions and functions — in the present case δ composed with $f(x) = x^2 - a^2$ — can be found in [H 1983, Ch. 3.1].)

3.13. Denote by e_j the j'th coordinate vector in $\mathbb{R}^n$ and define the difference quotient $\Delta_{j,h}u$ of an arbitrary distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ by

$$\Delta_{j,h}u = \frac{1}{h}(T(\tau_{he_j})u - u)\,, \quad \text{for } \ h \in \mathbb{R} \setminus \{0\},$$

cf. Example 3.21. Show that $\Delta_{j,h}u \to \partial_j u$ in $\mathcal{D}'(\mathbb{R}^n)$ for $h \to 0$.

3.14. For an open subset Ω of $\mathbb{R}^n$ and an open interval I of $\mathbb{R}$ we consider a parametrized family of functions $\varphi(x,t)$ belonging to $C_0^\infty(\Omega)$ as functions of x for each value of the parameter t.

(a) Show that when the map $t \mapsto \varphi(x,t)$ is continuous from I to $C_0^\infty(\Omega)$, then the function $t \mapsto f(t) = \langle u, \varphi(x,t)\rangle$ is continuous from I to $\mathbb{C}$, for any distribution $u \in \mathcal{D}'(\Omega)$.

(b) We say that the map $t \mapsto \varphi(x,t)$ is differentiable from I til $C_0^\infty(\Omega)$, when we have for each $t \in I$ that $[\varphi(x,t+h) - \varphi(x,t)]/h$ (defined for h so small that $t + h \in I$) converges in $C_0^\infty(\Omega)$ to a function $\psi(x,t)$ for $h \to 0$; observe that ψ in that case is the usual partial derivative $\partial_t\varphi$. If this function $\partial_t\varphi(x,t)$ moreover is continuous from I to $C_0^\infty(\Omega)$, we say that $\varphi(x,t)$ is C^1 from I to $C_0^\infty(\Omega)$. C^k-maps are similarly defined.

Show that when $t \mapsto \varphi(x,t)$ is differentiable (resp. C^k for some $k \geq 1$) from I to $C_0^\infty(\Omega)$, then the function $t \mapsto f(t) = \langle u, \varphi(x,t)\rangle$ is differentiable (resp. C^k) for any distribution $u \in \mathcal{D}'(\Omega)$, and one has at each $t \in I$:

$$\partial_t f(t) = \langle u, \partial_t \varphi(x,t)\rangle; \quad \text{resp.} \quad \partial_t^k f(t) = \langle u, \partial_t^k \varphi(x,t)\rangle.$$

3.15. Let $\Omega = \Omega' \times \mathbb{R}$, where Ω' is an open subset of $\mathbb{R}^{n-1}$ (the points in Ω, Ω' resp. $\mathbb{R}$ are denoted x, x' resp. x_n).

(a) Show that if $u \in \mathcal{D}'(\Omega)$ satisfies $\partial_{x_n} u = 0$, then u is *invariant under x_n-translation*, i.e., $T_h u = u$ for all $h \in \mathbb{R}$, where T_h is the translation coordinate change (denoted $T(\tau_{he_n})$ in Example 3.21) defined by

$$\langle T_h u, \varphi(x', x_n)\rangle = \langle u, \varphi(x', x_n + h)\rangle \quad \text{for } \varphi \in C_0^\infty(\Omega).$$

(*Hint.* Introduce the function $f(h) = \langle T_h u, \varphi\rangle$ and apply the Taylor formula (A.8) and Exercise 3.14 to this function.)

(b) Show that if u is a continuous function on Ω satisfying $\partial_{x_n} u = 0$ in the distribution sense, then $u(x', x_n) = u(x', 0)$ for all $x' \in \Omega'$.

3.16. (a) Let $\Omega = \Omega' \times \mathbb{R}$, as in Exercise 3.15. Show that if u and $u_1 = \partial_{x_n} u$ are continuous functions on Ω (where $\partial_{x_n} u$ is defined in the distribution sense), then u is differentiable in the original sense with respect to x_n at every point $x \in \Omega$, with the derivative $u_1(x)$.

(*Hint.* Let v be the function defined by

$$v(x', x_n) = \int_0^{x_n} u_1(x', t)\, dt,$$

show that $\partial_{x_n}(u - v) = 0$ in $\mathcal{D}'(\Omega)$, and apply Exercise 3.15.)

(b) Show that the conclusion in (a) also holds when Ω is replaced by an arbitrary open set in $\mathbb{R}^n$.

3.17. The distribution $\frac{d^k}{dx^k}\delta$ is often denoted $\delta^{(k)}$; for $k = 1, 2, 3$, the notation δ', δ'', δ''' (respectively) is also used. Let $f \in C^\infty(\mathbb{R})$.

(a) Show that there are constants c_0 and c_1 such that one has the identity:

$$f\delta' = c_0\delta + c_1\delta';$$

find these.

(b) For general $k \in \mathbb{N}_0$, show that there are constants c_{kj} such that one has the identity:

$$f\delta^{(k)} = \sum_{j=0}^{k} c_{kj}\delta^{(j)};$$

find these.