I. DISTRIBUTION SPACES

§1. Motivation and overview

1.1. Introduction.

In the study of ordinary differential equations one can get very far by using just the classical concept of differentiability, working with spaces of continuously differentiable functions on an interval $I \subset \mathbb{R}$:

$$C^m(I) = \{\, u\colon I \to \mathbb{C} \mid \tfrac{d^j}{dx^j} u \text{ exists and is continuous on } I \text{ for } 0 \le j \le m \,\}. \tag{1.1}$$

The need for more general concepts comes up for example in the study of eigenvalue problems for second order operators on an interval $[a,b]$ with boundary conditions at the endpoints a, b, by Hilbert space methods. But here it usually suffices to extend the notions to *absolutely continuous functions*, i.e., functions $u(x)$ of the form

$$u(x) = \int_{x_0}^{x} v(y)\, dy + c, \quad v \text{ locally integrable on } I \tag{1.2}$$

(integrable on compact subsets of I); c denotes a constant. Here v is regarded as the derivative $\frac{d}{dx}u$ of u, and the fundamental formula

$$u(x) = u(x_0) + \int_{x_0}^{x} \tfrac{d}{dy} u(y)\, dy \tag{1.3}$$

still holds.

But for partial differential equations one finds when using methods from functional analysis that the spaces C^m are inadequate, and there is no good concept of absolute continuity in the case of functions of several real variables. One can get some ways by using the concept of *weak derivatives*: When u and v are locally integrable on an open subset Ω of $\mathbb{R}^n$, we say that $v = \frac{\partial}{\partial x_j} u$ in the weak sense, when

$$-\int_\Omega u \tfrac{\partial}{\partial x_j} \varphi\, dx = \int_\Omega v\varphi\, dx, \text{ for all } \varphi \in C_0^\infty(\Omega); \tag{1.4}$$

here $C_0^\infty(\Omega)$ denotes the space of C^∞ functions on Ω with compact support in Ω. (The support $\operatorname{supp} f$ of a function f is the complement of the largest open set where the function is zero.) This criterion is modeled after the fact that the formula (1.4) holds when $u \in C^1(\Omega)$, with $v = \frac{\partial}{\partial x_j} u$.

Sometimes even the concept of weak derivatives is not sufficient, and the need arises to define derivatives that are not functions, but are more general objects. Some measures and derivatives of measures will enter. For example, there is the Dirac measure δ_0 that assigns 1 to every Lebesgue measurable set in $\mathbb{R}^n$ containing $\{0\}$, and 0 to any Lebesgue measurable set not containing $\{0\}$. For $n = 1$, δ_0 is the derivative of the Heaviside function defined in (1.8) below. In the book of Laurent Schwartz [S 1961] there is also a description of the *derivative of* δ_0 (on $\mathbb{R}$) — which is not even a measure — as a "dipole", with some kind of physical explanation.

For the purpose of setting up the rules for a general theory of differention where classical differentiability fails, Schwartz brought forward around 1950 the concept of *distributions:* a class of objects containing the locally integrable functions and allowing differentiations of any order.

This book gives an introduction to distribution theory, based on the work of Schwartz and of many other people. Our aim is also to show how the theory is combined with the study of operators in Hilbert space by methods of functional analysis, with applications to ordinary and partial differential equations. In some chapters of a more advanced character, we show how the distribution theory is used to define pseudodifferential operators and how they are applied in the discussion of solvability of PDE, with or without boundary conditions. A bibliography of relevant books and papers is collected at the end.

Plan.

Part I gives an introduction to distributions.

In the rest of Chapter 1 we begin the discussion of taking derivatives in the distribution sense, motivating the study of function spaces in the following chapter.

Notation and prerequisites are collected in Appendix A.

Chapter 2 studies the spaces of C^∞ functions (and C^k functions) needed in the theory, and their relations to L_p spaces.

The relevant topological considerations are collected in Appendix B.

In Chapter 3 we introduce distributions in full generality and show the most prominent rules of calculus for them.

Part II connects the distribution concept with differential equations and Fourier transformation.

Chapter 4 is aimed at linking distribution theory to the treatment of partial differential equations (PDE) by Hilbert space methods. Here we introduce Sobolev spaces and realizations of differential operators, both in the (relatively simple) one-dimensional case and in n-space, and study some applications.

Here we use some of the basic results on unbounded operators in Hilbert space that are collected in Chapter 12.

In Chapter 5, we study the Fourier transformation in the framework of temperate distributions.

Chapter 6 gives a further development of Sobolev spaces as well as applications to PDE by use of Fourier theory, and shows a fundamental result on the structure of distributions.

Part III contains more advanced material, primarily on pseudodifferential operators (ψdo's), a generalization of partial differential operators containing also the solution operators for elliptic problems.

Chapter 7 gives the basic ingredients of the local calculus of pseudodifferential operators.

Chapter 8 shows how to define ψdo's on manifolds, and how they in the elliptic case define Fredholm operators, with solvability properties modulo finite-dimensional spaces. (An introduction to Fredholm operators is included.)

Chapter 9 takes up the study of boundary value problems by use of Fourier transformation. The main effort is spent on an important constant-coefficient case which, as an example, shows how Sobolev spaces of noninteger and negative order can enter. Also, a connection is made to the abstract theory of Chapter 13. This chapter can be read directly after Parts I and II.

In Chapter 10 we present the basic ingredients in a pseudodifferential theory of boundary value problems introduced originally by L. Boutet de Monvel; this builds on the methods of Chapters 7 and 8 and the example in Chapter 9, introducing new operator types.

Chapter 11 shows how the theory of Chapter 10 can be used to discuss solvability of elliptic boundary value problems, by use of the Calderón projector, that we construct in detail.

Part IV gives the supplementing topics needed from Hilbert space theory.

Chapter 12, departing from the knowledge of bounded linear operators in Hilbert spaces, shows some basic results for unbounded operators, and develops the theory of variational operators.

Chapter 13 gives a thorough presentation of certain families of extensions of closed operators, of interest for the study of boundary value problems for elliptic PDE and their positivity properties.

Chapter 14 establishes some basic results on semigroups of operators, relevant for parabolic PDE (problems with a time-parameter), and appealing to positivity and variationality properties discussed in earlier chapters.

Finally, there are three appendics. In Appendix A, we recall some basic rules of calculus and set up the notation.

Appendix B gives some elements of the theory of topological vector spaces, that can be invoked when one wants the correct topological formulation of the properties of distributions.

Appendix C introduces some function spaces, as a continuation of Chapter 2, but needed only later in the text.

1.2 On the definition of distributions.

The definition of a weak derivative $\partial_j u$ was mentioned in (1.4) above. Here both u and its weak derivative v are locally integrable functions on (Ω). Observe that the right hand side is a linear functional on $C_0^\infty(\Omega)$, i.e., a linear mapping Λ_v of $C_0^\infty(\Omega)$ into $\mathbb{C}$, here defined by

$$\Lambda_v \colon \varphi \mapsto \Lambda_v(\varphi) = \int_\Omega v\varphi \, dx. \tag{1.5}$$

The idea of Distribution Theory is to allow much more general functionals than this one. In fact, when Λ is any linear functional on $C_0^\infty(\Omega)$ such that

$$-\int_\Omega u\partial_j\varphi \, dx = \Lambda(\varphi) \text{ for all } \varphi \in C_0^\infty(\Omega); \tag{1.6}$$

we shall say that

$$\partial_j u = \Lambda \text{ in the distribution sense,} \tag{1.7}$$

even if there is no function v (locally integrable) such that Λ can be defined from it as in (1.5).

Example 1.1. Here is the most famous example in the theory: Let $\Omega = \mathbb{R}$ and consider the Heaviside function $H(x)$; it is defined by

$$H(x) = \begin{cases} 1 & \text{for } \ x > 0 \\ 0 & \text{for } \ x \le 0\,. \end{cases} \tag{1.8}$$

It is locally integrable on $\mathbb{R}$. But there is no locally integrable function v such that (1.4) holds with $u = H$:

$$-\int_\mathbb{R} H \tfrac{d}{dx}\varphi \, dx = \int v\varphi \, dx, \text{ for all } \varphi \in C_0^\infty(\mathbb{R}). \tag{1.9}$$

For, assume that v were such a function, and let $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(0) = 1$ and set $\varphi_N(x) = \varphi(Nx)$. Note that $\max|\varphi(x)| = \max|\varphi_N(x)|$ for all N, and that when φ is supported in $[-R, R]$, φ_N is supported in $[-R/N, R/N]$. Thus by the theorem of Lebesgue,

$$\int_{\mathbb{R}} v\varphi_N \, dx \to 0 \text{ for } N \to \infty, \tag{1.10}$$

but on the other hand,

$$-\int_{\mathbb{R}} H \tfrac{d}{dx}\varphi_N \, dx = -\int_0^\infty N\varphi'(Nx)\, dx = -\int_0^\infty \varphi'(y)\, dy = \varphi(0) = 1. \tag{1.11}$$

So (1.9) cannot hold for this sequence of functions φ_N, and we conclude that a locally integrable function v for which (1.9) holds for all $\varphi \in C_0^\infty(\mathbb{R})$ cannot exist.

A linear functional that does match H in a formula (1.6) is the following one:

$$\Lambda \colon \varphi \to \varphi(0) \tag{1.12}$$

(as seen by a calculation as in (1.11)). This is the famous delta-distribution, usually denoted δ_0. (It identifies with the delta-measure mentioned earlier.)

There are some technical things that have to be cleared up before we can define distributions in a proper way.

For one thing, we have to look more carefully at the elements of $C_0^\infty(\Omega)$. We must demonstrate that such functions really do exist, and we need to show that there are elements with convenient properties (such as having the support in a prescribed set and being 1 on a smaller prescribed set).

Moreover, we have to describe what is meant by convergence in $C_0^\infty(\Omega)$, and provide it with a topology. There are also some other spaces of C^∞ or C^k functions with suitable support or integrability properties that we need to introduce.

These preparatory steps will take some time, before we begin to introduce distributions in full generality. (The theories that go into giving $C_0^\infty(\Omega)$ a good topology are quite advanced, and will partly be relegated to Appendix B. In fact, the urge to do this in all details has been something of an obstacle to making the tool of distributions available to everybody working with PDE — so we shall here take the point of view of giving full details of how one *operates with* distributions, but tone down the topological discussion to some statements one can use without necessarily checking all proofs.)

The reader is urged to consult Appendix A (with notation and prerequisites) before starting to read the next chapters.