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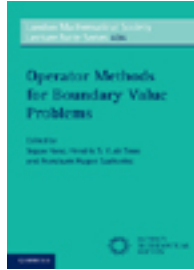
Operator Methods for Boundary Value Problems

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Chapter

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8
Extension theory for elliptic partial differential operators with pseudodifferential methods

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Abstract This is a short survey on the connection between general extension theories and the study of realizations of elliptic operators A on smooth domains in $\mathbb{R}^n$, $n \geq 2$. The theory of pseudodifferential boundary problems has turned out to be very useful here, not only as a formulational framework, but also for the solution of specific questions. We recall some elements of that theory, and show its application in several cases (including new results), namely to the lower boundedness question, and the question of spectral asymptotics for differences between resolvents.

8.1 Introduction

The general theory of extensions of a symmetric operator (or a dual pair of operators) in a Hilbert space, originating in the mid-1900's, has been applied in numerous works to ordinary differential equations (ODE), and also in a (smaller) number of works to partial differential equations (PDE).

There is a marked difference between the two cases: In ODE, the playground for boundary conditions is usually finite-dimensional vector spaces, where linear conditions can be expressed by the help of matrices. Moreover, the domains of differential operators defined by closure in L_2-based Hilbert spaces can usually all be expressed in terms of functions with the relevant number of absolutely continuous derivatives.

In contrast, boundary conditions for PDE (in space dimensions $n \geq 2$) are prescribed on infinite-dimensional vector spaces. Moreover, the domains of differential operators in L_2-based spaces will contain functions with distribution derivatives, not continuous and possibly highly irregular.

Whereas extensions of ODEs can usually be described in terms of

matrices, the tools to interpret extensions in PDE cases are therefore much more complicated. We shall give a survey of some tools developed through the years, and their applications, emphasizing the use of pseudodifferential operators.

Outline In Section 8.2 we recall the basic issues of elliptic boundary value problems. Pseudodifferential operators (ψdo's) are introduced in Section 8.3, and Section 8.4 introduces pseudodifferential boundary operators (ψdbo's). In Section 8.5 we recall the elements of a general abstract extension theory, and in Section 8.6 we show how this is implemented for realizations $\widetilde{A}$ of an elliptic operator A on a domain $\Omega \subset \mathbb{R}^n$. Section 8.7 focuses on the resolvent formulas that can be obtained via the general theory. In the last sections we go through several cases where pseudodifferential techniques have proved extremely useful (some of the results here are quite recent): In Section 8.8 it is the question of whether lower boundedness holds simultaneously for a realization $\widetilde{A}$ and the operator L over the boundary that enters in the corresponding boundary condition; the new results deal with unbounded domains. In Section 8.9 it is the question of showing Weyl-type spectral asymptotics formulas for differences between resolvents. The results there go back to the early theory, and Section 8.10 presents some additional new results.

8.2 Elliptic boundary value problems

In the following we use the customary multi-index notation for differential operators: $\partial = \partial_x = (\partial_1, \ldots, \partial_n)$, $\partial_j = \partial_{x_j} = \partial/\partial x_j$, and $D = D_x = (D_1, \ldots, D_n)$, $D_j = D_{x_j} = -i\partial/\partial x_j$; then $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, for $\alpha \in \mathbb{N}_0^n$; here $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

A differential operator of order $m > 0$,

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

is said to be *elliptic*, resp. *strongly elliptic*, on a set $U \subset \mathbb{R}^n$, when the principal symbol

$$a_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

satisfies

$$a_m(x, \xi) \neq 0, \text{ resp. } \operatorname{Re} a_m(x, \xi) > 0,$$

for $x \in U$, $\xi \in \mathbb{R}^n \setminus \{0\}$. As a basic example, the Laplacian $\Delta = \partial_1^2 + \cdots + \partial_n^2$ has principal symbol (and symbol) equal to $-|\xi|^2$, so it is elliptic, and $-\Delta$ is strongly elliptic. The Laplacian has been studied for several hundred years, and the problems around it solved by explicit solution formulas. It is the cases with variable (x-dependent) coefficients, and domains more general than simple geometric figures, that have been a challenge in more modern times.

The problem

$$Au = f \tag{8.1}$$

for a given function f on a subset Ω of $\mathbb{R}^n$ usually has infinitely many solutions. To get a problem with unique solvability, we must adjoin extra conditions such as suitable boundary conditions. We can consider A with a domain consisting of the functions satisfying the boundary condition, as an operator $\widetilde{A}$ acting between suitable spaces.

Then the question of *existence* of a solution corresponds to the question of whether $\widetilde{A}$ is *surjective*, and the question of *uniqueness* of a solution corresponds to the question of whether $\widetilde{A}$ is *injective*. In this way, the question of solvability of differential equations is turned into a question of properties of specific operators. The operator point of view became particularly fruitful when it was combined with appropriate scales of function spaces, such as the *Sobolev spaces*, [Sobolev, 1950], and with *Distribution theory*, [Schwartz, 1950].

When Ω is a smooth open subset of $\mathbb{R}^n$ with boundary $\partial\Omega = \Sigma$, we refer to the standard L_2-Sobolev spaces, with the following notation: $H^s(\mathbb{R}^n)$ ($s \in \mathbb{R}$) has the norm $\|v\|_s = \|\mathcal{F}^{-1}(\langle\xi\rangle^s \mathcal{F}v)\|_{L_2(\mathbb{R}^n)}$; here $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}: u(x) \mapsto (\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx,$$

and $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. Next, with r_Ω denoting restriction to Ω,

$$H^s(\Omega) = r_\Omega H^s(\mathbb{R}^n),$$

provided with the norm $\|u\|_s = \inf\{\|v\|_s \mid v \in H^s(\mathbb{R}^n),\, u = r_\Omega v\}$. Moreover,

$$H_0^s(\overline{\Omega}) = \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\},$$

closed subspace of $H^s(\mathbb{R}^n)$; it identifies with the antidual space of $H^{-s}(\Omega)$ (the space of antilinear, i.e., conjugate linear, functionals), with

a duality consistent with the L_2 scalar product. For s equal to a nonnegative integer k, $H_0^k(\overline{\Omega})$ is usually written $H_0^k(\Omega)$. Spaces over the boundary, $H^s(\Sigma)$, are defined by local coordinates from $H^s(\mathbb{R}^{n-1})$, $s \in \mathbb{R}$. (There are many equally justified equivalent choices of norms there; one can choose a particular norm when convenient.) When $s > 0$, there are dense continuous embeddings

$$H^s(\Sigma) \subset L_2(\Sigma) \subset H^{-s}(\Sigma),$$

and there is an identification of $H^{-s}(\Sigma)$ with the antidual space of $H^s(\Sigma)$, such that the duality $(\varphi, \psi)_{-s,s}$ coincides with the $L_2(\Sigma)$-scalar product when the elements lie there. Detailed explanations are found in many books, e.g. [Lions and Magenes, 1968; Hörmander, 1963; Grubb, 2009]. (There is a difference of notation: For $k \in \mathbb{N}$, the spaces denoted $H_0^{k+\frac{1}{2}}(\Omega)$ in [Lions and Magenes, 1968] are not the same as our $H_0^{k+\frac{1}{2}}(\overline{\Omega})$ that are consistent with [Hörmander, 1963]; the latter have the best duality and interpolation properties.)

Consider the case where A is defined on a smooth open subset Ω of $\mathbb{R}^n$ and has coefficients in $C^\infty(\overline{\Omega})$, and assume that A is elliptic on $\overline{\Omega}$. The results in this case are a model for results under weaker smoothness hypotheses. One defines the maximal realization $A_{\max}$ as the operator acting like A in the distribution sense with domain

$$D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\};$$

it is a closed, unbounded operator in $L_2(\Omega)$. The minimal realization $A_{\min}$ is defined as the closure of A acting on $C_0^\infty(\Omega)$ (the compactly supported C^∞-functions on Ω). When Ω is bounded, or is unbounded and there are suitable bounds on the coefficients of A,

$$D(A_{\min}) = H_0^m(\Omega).$$

The formal adjoint A' of A is the differential operator acting as follows:

$$A'u = \sum_{|\alpha| \leq m} D^\alpha(\bar{a}_\alpha(x)u).$$

By definition, $A'_{\min}$ and $A_{\max}$ are adjoints of one another (as unbounded operators in $L_2(\Omega)$).

The linear operators $\widetilde{A}$ satisfying

$$A_{\min} \subset \widetilde{A} \subset A_{\max}$$

are called *realizations* of A.

Generally, $A_{\max}$ is far from being injective, whereas $A_{\min}$ is far from

being surjective, but it may be possible to find realizations $\widetilde{A}$ that are bijective from $D(\widetilde{A})$ to $L_2(\Omega)$.

We see that the theory of distributions (which in this context was preceded by the definition of differential operators acting in the *weak sense*) allows defining operators representing the action of A in a generalized sense. Here invertibility can sometimes be achieved by methods of functional analysis. A fundamental example is the Dirichlet problem (where $\gamma_0 u = u|_\Sigma$)

$$Au = f \text{ in } \Omega, \quad \gamma_0 u = \varphi \text{ on } \Sigma, \tag{8.2}$$

for a strongly elliptic second-order operator having $\mathrm{Re}(Av, v) \geq c\|v\|^2_{L_2(\Omega)}$ with $c > 0$ for $v \in C_0^\infty(\Omega)$. By use of the so-called Lax-Milgram lemma one could define a realization A_γ of A with $D(A_\gamma) \subset H_0^1(\Omega)$, such that $A_\gamma \colon D(A_\gamma) \to L_2(\Omega)$ bijectively. (Details are found in many books, e.g. [Grubb, 2009], Ch. 12.)

But then the question was: How close is A_γ^{-1} to solving the problem in a more classical sense? Second-order derivatives have a meaning on $H^2(\Omega)$, by closure of the definition on $C^2(\overline{\Omega})$, so one can ask:

- If $f \in L_2(\Omega)$, is $u \in H^2(\Omega)$?
- More generally, if $f \in H^k(\Omega)$ for some $k \in \mathbb{N}_0$, is $u \in H^{k+2}(\Omega)$?

The answer was first found for the behavior of u in the interior of Ω: Indeed, when $f \in H^k(\Omega)$, u is in H^{k+2} over subsets of Ω with positive distance from the boundary. This is the so-called *interior regularity*.

There remained the question of *regularity at the boundary*. It was answered positively in [Nirenberg, 1955] and by Ladyzhenskaya (see the account in [Ladyzhenskaya, 1985]). This was followed up by research on higher-order operators A and more general boundary conditions $Tu = \varphi$ (possibly vector valued), where results on interior regularity and regularity at the boundary were established under suitable conditions. Besides ellipticity of the operator A one needs a condition on how the boundary condition fits together with A. Some authors called it the "covering condition" or the "complementing condition", but the name "the Shapiro-Lopatinskiĭ condition" (after [Shapiro, 1953; Lopatinskii, 1953]) has been more generally used. It is also customary to call the system $\{A, T\}$ *elliptic* when it holds (this was suggested in [Hörmander, 1963]; we return to a motivation in Section 8.4). A fundamental reference in this connection is in [Agmon, Douglas and Nirenberg, 1959] that collects and expands the knowledge on elliptic boundary value problems. An important point of view was to obtain so-called "à priori estimates"

(estimates of a Sobolev norm on u by norms on Au and Tu plus a lower order norm on u), shown for smooth functions at first, and extended to the considered solution.

Important monographs exposing the theories and the various authors' own contributions were written by [Agmon, 1965; Lions and Magenes, 1968]; the latter moreover contains valuable information on the surrounding literature. The early theory is exposed in [Courant and Hilbert, 1953, 1962].

In the works at that time, although the striving to show existence of a solution operator was always in the picture, the emphasis was more on showing qualitative properties of the unknown function u in terms of properties of the given data f and φ, regardless of whether u could be described by an operator acting on $\{f, \varphi\}$ or not.

Direct machinery to construct approximate solution operators in general came into the picture with the advent of pseudodifferential methods.

8.3 Pseudodifferential operators

One of the few cases where an elliptic differential operator has an explicit solution operator is the case of $I - \Delta$ on $\mathbb{R}^n$, whose action can be described by $(1 - \Delta)u = \mathcal{F}^{-1}\big((1 + |\xi|^2)\mathcal{F}u\big)$, where $\mathcal{F}$ is the Fourier transform, and whose solution operator is

$$\mathrm{Op}\Big(\frac{1}{1 + |\xi|^2}\Big)u = \mathcal{F}^{-1}\Big(\frac{1}{1 + |\xi|^2}\mathcal{F}u\Big).$$

A variable-coefficient elliptic differential operator on $\mathbb{R}^n$ can also be described by the help of the Fourier transform,

$$Au = \sum_{|\alpha| \le m} a_\alpha(x)D^\alpha u = \sum_{|\alpha| \le m} a_\alpha(x)\mathcal{F}^{-1}(\xi^\alpha \mathcal{F}u)$$
$$= \mathcal{F}^{-1}a(x, \xi)\mathcal{F}u = \mathrm{Op}(a(x, \xi))u,$$

where $a(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha$ is the symbol. But even when the symbol satisfies $a(x, \xi) \ne 0$ for all x, ξ, the operator

$$\mathrm{Op}(a(x, \xi)^{-1}) = \mathcal{F}^{-1}a(x, \xi)^{-1}\mathcal{F}$$

is not an exact inverse. Nevertheless, it is useful in the discussion of solutions, since one can show that

$$\mathrm{Op}(a(x, \xi))\,\mathrm{Op}(a(x, \xi)^{-1}) = I + \mathcal{R},$$

where the remainder $\mathcal{R}$ is of order -1 (lifts the exponent of a Sobolev space by 1).

A thorough treatment of $\mathrm{Op}(a(x,\xi)^{-1})$ and suitable generalizations that come closer to being an inverse of A (such "almost-inverses" are called parametrices) was obtained with the invention of *pseudo-differential operators*, ψdo's. Some of the initiators were [Mihlin, 1948; Calderón and Zygmund, 1957; Seeley, 1965; Kohn and Nirenberg, 1965; Hörmander, 1965, 1971].

General ψdo's are defined from general symbols $p(x,\xi)$ as

$$\mathrm{Op}(p(x,\xi))u = \mathcal{F}^{-1}p(x,\xi)\mathcal{F}u = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\,d\xi;$$

here $p(x,\xi)$ is required to belong to a suitable class of functions.

Not only do the operators make sense on $\mathbb{R}^n$ where the Fourier transform acts, they are also given a meaning on manifolds, by use of coordinate change formulas and cutoff functions. The theory is not altogether easy; it uses concepts from distribution theory in a refined way. Moreover, it is not exact but qualitative in many statements, so it can be something of a challenge to derive good results from its use. A fine achievement is that it leads to Fredholm operators, when applied to elliptic operators on compact manifolds. Here one is just a small step away from having invertible operators; this can sometimes be achieved by relying on additional knowledge of the situation.

A so-called "classical" ψdo is an operator defined from a symbol that has an asymptotic series expansion in homogeneous terms (a polyhomogeneous symbol):

$$p(x,\xi) \sim \sum_{j\in\mathbb{N}_0} p_{m-j}(x,\xi), \text{ where}$$
$$p_{m-j}(x,t\xi) = t^{m-j}p_{m-j}(x,\xi) \text{ for } |\xi| \geq 1,\ t \geq 1.$$

It is said to be of order m, and $\mathrm{Op}(p)$ maps H^s to H^{s-m} for all $s \in \mathbb{R}$. p_m is called the principal symbol, and p is said to be elliptic when $p_m(x,\xi) \neq 0$ for $|\xi| \geq 1$. Here one has that

$$\mathrm{Op}(p)\,\mathrm{Op}(p') = \mathrm{Op}(pp') + \mathcal{R}_1 = \mathrm{Op}(p_m p'_{m'}) + \mathcal{R}_2,$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are of order $m + m' - 1$. In this way, the principal part *dominates the behavior*. When p is elliptic, the principal part of a parametrix is found as p_m^{-1} (for $|\xi| \geq 1$, extended smoothly to $\xi \in \mathbb{R}^n$). Also the notation p^0 is used for the principal symbol.

There exist many different symbol classes with generalizations of the above properties, designed for particular purposes.

The very attractive feature of classical ψdo's is that they form a scale of operators of *all integer orders*, including differential operators among those of positive order, and including parametrices and inverses of elliptic differential operators among those of negative order. Moreover, it is an "algebra", in the sense that the elements by composition (and by addition) lead to other classical ψdo's.

The calculus is explained in the original papers and in several subsequent books such as [Treves, 1980; Hörmander, 1985]; a detailed introduction can be also found in [Grubb, 2009, Chapters 7–8].

The ψdo theory gives (after one has done the work to set it up) an easy proof of *interior regularity* of solutions to elliptic problems.

8.4 Pseudodifferential boundary operators

When an elliptic differential operator A is considered on a subset of $\mathbb{R}^n$ or on a manifold with boundary — let us here for simplicity just consider the case of a smooth bounded open subset Ω of $\mathbb{R}^n$ — we must impose boundary conditions to get uniquely solvable problems. Let us assume that we are in a case where the boundary condition $Tu = \varphi$ together with (8.1) gives a uniquely solvable problem; here T is a *trace operator* mapping functions on Ω into M-tuples of functions on $\Sigma = \partial\Omega$. We can formulate this in terms of matrices:

$$\binom{A}{T} : C^\infty(\overline{\Omega}) \to \begin{matrix} C^\infty(\overline{\Omega}) \\ \times \\ C^\infty(\Sigma)^M \end{matrix}$$

$$\text{has an inverse} \tag{8.3}$$

$$\begin{pmatrix} R & K \end{pmatrix} : \begin{matrix} C^\infty(\overline{\Omega}) \\ \times \\ C^\infty(\Sigma)^M \end{matrix} \to C^\infty(\overline{\Omega}).$$

Here K is called a *Poisson operator*; it solves the semi-homogeneous problem

$$Av = 0 \text{ in } \Omega, \quad Tv = \varphi \text{ on } \Sigma.$$

The operator R solves the other semi-homogeneous problem

$$Aw = f \text{ in } \Omega, \quad Tw = 0 \text{ on } \Sigma.$$

In a closer analysis of R, we can write it as a sum of two terms:

$$R = Q_+ + G, \tag{8.4}$$

where Q is the ψdo A^{-1} on $\mathbb{R}^n$, Q_+ is its *truncation* $r^+ Q e^+$ to Ω, and G is a supplementing operator adapted to the specific boundary condition, called a *singular Green operator* (s.g.o.). The operator e^+ stands for extension by 0 (to functions on $\mathbb{R}^n$), and the operator r^+ stands for restriction to Ω.

The calculus of pseudodifferential boundary operators (ψdbo's) was initated by Boutet de Monvel [Boutet de Monvel, 1971], who introduced operator systems encompassing both the systems $\binom{A}{T}$ and their solution operators $\begin{pmatrix} R & K \end{pmatrix}$. The original presentation is somewhat brief, and was followed up by extended expositions, in the detailed book [Rempel and Schulze, 1982], which elaborated the index theory, and in the paper [Grubb, 1984a] which completed some proofs of composition rules (with new points of view), and showed spectral asymptotic estimates for singuar Green operators. The book [Grubb, 1996, also 1986 edition] developed a calculus of parameter-dependent ψdbo's, leading to resolvent and heat operator constructions. The recent book [Grubb, 2009] gives a full introduction to the theory.

In the systematic calculus of Boutet de Monvel one considers systems (called Green operators):

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(\overline{\Omega})^N \\ \times \\ C^\infty(\Sigma)^M \end{matrix} \to \begin{matrix} C^\infty(\overline{\Omega})^{N'} \\ \times \\ C^\infty(\Sigma)^{M'} \end{matrix},$$

where

- P is a ψdo on $\mathbb{R}^n$, satisfying the so-called *transmission condition* at Σ (always true for operators stemming from elliptic differential operators);
- $P_+ = r^+ P e^+$ is the truncation to Ω (the transmission condition assures that P_+ maps $C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$);
- T is a *trace operator* from Ω to Σ, K is a *Poisson operator* from Σ to Ω, S is a ψdo on Σ;
- G is a *singular Green operator*, e.g. of the type KT.

The composition of two such systems is again a system belonging to the calculus.

The operators extend to act on Sobolev spaces. For T and G there is a condition expressing which differential trace operators $\gamma_j u = (\partial/\partial n)^j u|_\Sigma$

that enter: T or G is said to be of class r when only γ_j's with $j < r$ enter; and then they act on $H^s(\Omega)$ for $s > r - \frac{1}{2}$. The class 0 case is the case where they are purely integral operators, well-defined on $L_2(\Omega)$.

All entries can be matrix-formed. They are defined in local coordinates by formulas involving Fourier transformation and polyhomogeneous symbols. The idea is as follows: In local coordinates at the boundary, where Ω and Σ are replaced by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$ and $\mathbb{R}^{n-1}$ (with points $x' = (x_1, \ldots, x_{n-1})$), the system has for each (x', ξ') a *boundary symbol operator* acting in the x_n-variable:

$$
a(x', \xi', D_n) = \begin{pmatrix} p(x', 0, \xi', D_n) + g(x', \xi', D_n) & k(x', \xi', D_n) \\ t(x', \xi', D_n) & s(x', \xi') \end{pmatrix} :
$$

$$
\begin{matrix} H^m(\mathbb{R}_+)^N & & L_2(\mathbb{R}_+)^{N'} \\ \times & \to & \times \\ \mathbb{C}^M & & \mathbb{C}^{M'} \end{matrix} \tag{8.5}
$$

Here m is the order of the operator. Each entry in a acts in a specific way. E.g., when the matrix is $\binom{A}{T}$ in (8.3), the boundary symbol operator is the *model operator* obtained by freezing the coefficients at x' and replacing derivatives $D_{x'}^\alpha$ by their Fourier transforms $(\xi')^\alpha$ (with respect to $x' \in \mathbb{R}^{n-1}$). The principal boundary symbol operator $a^0(x', \xi', D_n)$ is formed of the top order terms. The principal boundary symbol operator for $\begin{pmatrix} R & K \end{pmatrix}$ is the inverse of the principal boundary symbol operator for $\binom{A}{T}$. (For (8.5), g and t must be of class $\leq m$.)

From the boundary symbol operator one defines a full operator by applying the ψdo definition in the x'-variable,

$$
\mathrm{Op}'(a(x', \xi', D_n))u = (2\pi)^{1-n} \int e^{ix' \cdot \xi'} a(x', \xi', D_n)(\mathcal{F}_{y' \to \xi'} u(y', x_n)) \, d\xi'.
$$

The symbols have asymptotic series of terms that are homogeneous in (ξ', ξ_n) (different rules apply to the different ingredients, and we must refer to the mentioned references for further details). One then defines $\mathcal{A}$ to be *elliptic*, when

(a) P is elliptic, i.e. its principal symbol $p^0(x, \xi)$ is invertible at each (x, ξ) with $|\xi| \geq 1$,

(b) the principal boundary symbol operator $a^0(x', \xi', D_n)$ is invertible at each (x', ξ') with $|\xi'| \geq 1$.

For a system $\binom{A}{T}$ formed of an elliptic differential operator A and a differential trace operator T, (b) is precisely the old covering-/complementing/Shapiro-Lopatinskiĭ condition for $\{A, T\}$.

In the elliptic case, one can construct a parametrix $\mathcal{B}^0$ from the inverses of the symbols in (a)–(b), such that $\mathcal{A}\mathcal{B}^0 - I$ and $\mathcal{B}^0\mathcal{A} - I$ have order ≤ -1, and the construction can be refined to give errors of arbitrarily low order. With supplementing information it can be possible to obtain an inverse.

For example, there holds a a solvability theorem for an elliptic differential operator problem as in (8.3), formulated in this framework as follows:

Theorem 8.1   *Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded open set, denote $\partial\Omega = \Sigma$, and let $A = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$ with $a_\alpha \in C^\infty(\overline{\Omega})$ be elliptic on $\overline{\Omega}$, i.e., $\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \neq 0$ for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n \setminus \{0\}$. Let $T = (T_j)_{j=1}^m$ be a column vector of trace operators $T_j = \gamma_0 B_j$, where the B_j are differential operators of order m_j with C^∞-coefficients, $0 \leq m_1 < \cdots < m_m \leq 2m - 1$. (Then T is of class $r = m_m + 1 \leq 2m$.) Assume that $\{A, T\}$ is elliptic.*

The operator $\mathcal{A} = \binom{A}{T}$ defines a continuous mapping

$$\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix} : H^{2m+s}(\Omega) \to \begin{matrix} H^s(\Omega) \\ \times \\ \prod_{j=1}^m H^{2m+s-m_j-\frac{1}{2}}(\Sigma) \end{matrix}, \ s > r - 2m - \tfrac{1}{2}, \ (8.6)$$

and there is a system $\mathcal{B} = \begin{pmatrix} R & K \end{pmatrix}$ (a parametrix) belonging to the calculus and continuous in the opposite direction, such that

$$\mathcal{A}\mathcal{B} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \mathcal{R}_1, \quad \mathcal{B}\mathcal{A} = I + \mathcal{R}_2,$$

$$\mathcal{R}_1 : \begin{matrix} H^s(\Omega) \\ \times \\ \prod_{j=1}^m H^{2m+s-m_j-\frac{1}{2}}(\Sigma) \end{matrix} \to \begin{matrix} H^{s'}(\Omega) \\ \times \\ \prod_{j=1}^m H^{2m+s'-m_j-\frac{1}{2}}(\Sigma) \end{matrix}$$

$$\mathcal{R}_2 : H^{2m+s}(\Omega) \to H^{2m+s'}(\Omega)$$

for all s as in (8.6), all $s' \geq s$. Here K is a row vector of Poisson operators $(K_j)_{j=1}^m$ of orders $-m_j$, and $R = Q_+ + G$, where Q is a parametrix of A on a neighborhood of $\overline{\Omega}$, and G is a singular Green operator.

The operator $\mathcal{A}$ in (8.6) is Fredholm for each s, with the same finite dimensional kernel and cokernel in C^∞ for all s.

If $\mathcal{A}$ is bijective, the inverse belongs to the calculus (and is of the same form as $\mathcal{B}$).

When $r = 2m$, the lower limit for s is $-\frac{1}{2}$; cases where it is $< -\frac{1}{2}$ occur for example for the Dirichlet problem, where $r = m$, and s can go down to $-m - \frac{1}{2}$. It is useful to know that the Poisson operator K in

fact has the mapping property

$$K \colon \prod_{j=1}^{m} H^{2m+s-m_j-\frac{1}{2}}(\Sigma) \to H^{2m+s}(\Omega)$$

for *all* $s \in \mathbb{R}$. The trace operator T is called *normal*, when $\gamma_0 B_j = b_j \gamma_{m_j} + \sum_{k<m_j} B_{jk}\gamma_k$ with an invertible coefficient b_j for each j. (More general normal boundary value problems are described below in Section 8.9.)

For example, for a second-order strongly elliptic operator with a Dirichlet condition, the operator in the theorem maps as follows:

$$\mathcal{A} = \begin{pmatrix} A \\ \gamma_0 \end{pmatrix} \colon H^{2+s}(\Omega) \to \begin{matrix} H^s(\Omega) \\ \times \\ H^{\frac{3}{2}+s}(\Sigma) \end{matrix} \quad \text{for } s > -\tfrac{3}{2},$$

with parametrices and solution operators continuous in the opposite direction.

Elliptic operators A of odd order occur mainly as square matrix-formed operators, and there is a similar theorem for such cases, where also the B_j can be matrix-formed. Operators of Dirac-type are a prominent first-order example. The matrix case is also interesting for even-order operators. The results can moreover be worked out for operators defined on manifolds, acting in vector bundles. (See e.g. [Grubb, 1974], on the even-order case, for notation and the appropriate definition of normal boundary conditions.)

8.5 Extension theories

We shall now recall some elements of the functional analysis theory of extensions of given operators. This has a long history, with prominent contributions from [von Neumann, 1929; Friedrichs, 1934; Krein, 1947; Vishik, 1952; Birman, 1956], and others. The present author made a number of contributions in [Grubb, 1968, 1970, 1971, 1973, 1974], completing the preceding theories and working out applications to elliptic boundary value problems; further developments are found e.g. in [Grubb, 1983, 1984a], and in recent works.

At the same time there was another, separate development of abstract extension theories, where the operator concept gradually began to be replaced by the concept of *relations*. This development has been aimed primarily towards applications to ODE, however including operator-valued

such equations and Schrödinger operators on $\mathbb{R}^n$; keywords in this connection are: boundary triples theory, Weyl-Titchmarsh m-functions and Kreĭn resolvent formulas. Cf. e.g. [Kochubei, 1975; Vainerman, 1980; Lyantze and Storozh, 1983; Gorbachuk and Gorbachuk, 1991; Derkach and Malamud, 1991; Arlinskii, 1999; Malamud and Mogilevski, 2002; Derkach, Hassi, Malamud and de Snoo, 2006; Brüning, Geyler and Pankrashkin, 2008], and their references. In later years there have also been applications to elliptic boundary value problems, cf. e.g. [Amrein and Pearson, 2004; Kopachevski and Krein, 2004; Behrndt and Langer, 2007; Ryzhov, 2007; Brown, Marletta, Naboko and Wood, 2008; Gesztesy and Mitrea, 2008], and references therein. See also Chapters 3, 6 and 7.

The connection between the two lines of extension theories has been clarified in a recent work [Brown, Grubb and Wood, 2009].

At this point we should also mention the recent efforts for problems on nonsmooth domains: [Posilicano and Raimondi, 2009; Grubb, 2008; Gesztesy and Mitrea, 2008, 2011; Abels, Grubb and Wood]; here [Grubb, 2008; Gesztesy and Mitrea, 2011; Abels, Grubb and Wood] use [Grubb, 1968].

In the following, we shall use the notation from [Grubb, 1968, 1970, 1971, 1973, 1974] and [Brown, Grubb and Wood, 2009].

Let there be given a pair $A_{\min}$, $A'_{\min}$ of closed, densely defined operators in a Hilbert space H, such that the following holds:

$$A_{\min} \subset (A'_{\min})^* =: A_{\max}, \quad A'_{\min} \subset (A_{\min})^* =: A'_{\max}.$$

Let $\mathcal{M} = \{\widetilde{A} \mid A_{\min} \subset \widetilde{A} \subset A_{\max}\}$. Write $\widetilde{A}u$ as Au, when $\widetilde{A} \in \mathcal{M}$.

We assume that there is given an operator $A_\gamma \in \mathcal{M}$, the reference operator, with $0 \in \varrho(A_\gamma)$ (the resolvent set); then

$$A_{\min} \subset A_\gamma \subset A_{\max}, \quad A'_{\min} \subset A_\gamma^* \subset A'_{\max}.$$

The case where $A_{\min} = A'_{\min}$ and A_γ is selfadjoint, is called the *symmetric* case.

Let $Z = \ker A_{\max}$, $Z' = \ker A'_{\max}$, and define the basic non-orthogonal decompositions

$$D(A_{\max}) = D(A_\gamma) \dotplus Z, \quad D(A'_{\max}) = D(A_\gamma^*) \dotplus Z',$$

denoted $u = u_\gamma + u_\zeta = \operatorname{pr}_\gamma u + \operatorname{pr}_\zeta u$, where $\operatorname{pr}_\gamma = A_\gamma^{-1} A_{\max}$, with a similar notation with primes.

By $\operatorname{pr}_X u = u_X$ we denote the *orthogonal projection* from H to X. The injection $X \hookrightarrow H$ is denoted i_X (it is the adjoint of $\operatorname{pr}_X \colon H \to X$).

There holds an "abstract Green's formula" for $u \in D(A_{\max})$, $v \in D(A'_{\max})$:

$$(Au, v) - (u, A'v) = ((Au)_{Z'}, v_{\zeta'}) - (u_\zeta, (A'v)_Z). \qquad (8.7)$$

It can be used to show that when $\widetilde{A} \in \mathcal{M}$, and we define

$$V = \overline{\operatorname{pr}_\zeta D(\widetilde{A})}, \quad W = \overline{\operatorname{pr}_{\zeta'} D(\widetilde{A}^*)},$$

then

$$\{\{u_\zeta, (Au)_W\} \mid u \in D(\widetilde{A})\} \text{ is a graph},$$

defining an operator T from $D(T) \subset V$ to W.

Theorem 8.2 [Grubb, 1968] *There is a 1–1 correspondence between the closed operators $\widetilde{A} \in \mathcal{M}$ and the closed densely defined operators $T \colon V \to W$, where $V \subset Z$, $W \subset Z'$ (arbitrary closed subspaces), such that $\widetilde{A}$ corresponds to $T \colon V \to W$ if and only if*

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \operatorname{pr}_\zeta u \in D(T), \ (Au)_W = T \operatorname{pr}_\zeta u\}. \qquad (8.8)$$

In this correspondence, $V = \overline{\operatorname{pr}_\zeta D(\widetilde{A})}$, $W = \overline{\operatorname{pr}_{\zeta'} D(\widetilde{A}^*)}$, and

- *$\widetilde{A}^*$ corresponds analogously to $T^* \colon W \to V$.*
- *$\ker \widetilde{A} = \ker T$; $\quad \operatorname{ran} \widetilde{A} = \operatorname{ran} T + (H \ominus W)$.*
- *$\widetilde{A}$ is bijective if and only if T is so, and then*

$$\widetilde{A}^{-1} = A_\gamma^{-1} + \mathrm{i}_V T^{-1} \operatorname{pr}_W.$$

One also has

$$D(\widetilde{A}) = \{u = v + A_\gamma^{-1}(Tz + f) + z \mid v \in D(A_{\min}), \ z \in D(T), \ f \in Z \ominus W\},$$

where v, z and f are uniquely determined from u.

The result builds on the works [Krein, 1947; Birman, 1956] (for selfadjoint operators) and [Vishik, 1952], and completes the latter: In Vishik's paper, the $\widetilde{A}$ were set in relation to operators over the nullspaces going in the opposite direction of our T's, and the results were focused on those $\widetilde{A}$'s that have closed range (the so-called normally solvable realizations). Our analysis covered all closed $\widetilde{A}$.

The condition in (8.8)

$$(Au)_W = T \operatorname{pr}_\zeta u \qquad (8.9)$$

can be viewed as an "abstract boundary condition".

When $\lambda \in \varrho(A_\gamma)$, one can do the same construction for the operators shifted by subtraction of λ. We denote

$$Z_\lambda = \ker(A_{\max} - \lambda), \quad Z'_{\bar\lambda} = \ker(A'_{\max} - \bar\lambda),$$

and have the decompositions (where $\mathrm{pr}^\lambda_\gamma = (A_\gamma - \lambda)^{-1}(A_{\max} - \lambda)$)

$$D(A_{\max}) = D(A_\gamma) \dotplus Z_\lambda, \quad u = u^\lambda_\gamma + u^\lambda_\zeta = \mathrm{pr}^\lambda_\gamma u + \mathrm{pr}^\lambda_\zeta u,$$

with a similar notation with primes.

Corollary 8.3   *Let $\lambda \in \varrho(A_\gamma)$. For the closed $\widetilde{A} \in \mathcal{M}$, there is a 1–1 correspondence*

$$\widetilde{A} - \lambda \longleftrightarrow \begin{cases} T^\lambda \colon V_\lambda \to W_{\bar\lambda}, \text{ closed, densely defined} \\ \text{with } V_\lambda \subset Z_\lambda, \ W_{\bar\lambda} \subset Z'_{\bar\lambda}, \text{ closed subspaces.} \end{cases}$$

Here $D(T^\lambda) = \mathrm{pr}^\lambda_\zeta D(\widetilde{A})$, $V_\lambda = \overline{D(T^\lambda)}$, $W_{\bar\lambda} = \overline{\mathrm{pr}^{\bar\lambda}_{\zeta'} D(\widetilde{A}^*)}$, and $D(\widetilde{A})$ consists of the functions $u \in D(A_{\max})$ such that $u^\lambda_\zeta \in D(T^\lambda)$ and

$$T^\lambda u^\lambda_\zeta = ((A - \lambda)u)_{W_{\bar\lambda}}.$$

Moreover,

- $\ker(\widetilde{A} - \lambda) = \ker T^\lambda; \quad \mathrm{ran}(\widetilde{A} - \lambda) = \mathrm{ran}\, T^\lambda + (H \ominus W_{\bar\lambda}).$
- $\widetilde{A} - \lambda$ *is bijective if and only if T^λ is so, and when $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$,*

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} + \mathrm{i}_{V_\lambda}(T^\lambda)^{-1}\,\mathrm{pr}_{W_{\bar\lambda}}.$$

This gives a Kreĭn-type resolvent formula for any closed $\widetilde{A} \in \mathcal{M}$ with $\varrho(\widetilde{A}) \cap \varrho(A_\gamma) \neq \emptyset$.

The relation between T and T^λ was determined in [Grubb, 1974] in the symmetric case, for real λ, and the proof given there extends immediately to the general situation (as shown in [Brown, Grubb and Wood, 2009]):

For $\lambda \in \varrho(A_\gamma)$, define

$$E^\lambda = I + \lambda(A_\gamma - \lambda)^{-1}, \text{ it has the inverse } F^\lambda = I - \lambda A^{-1}_\gamma,$$

and similarly $E'^{\bar\lambda} = I + \bar\lambda(A^*_\gamma - \bar\lambda)^{-1}$ has the inverse $F'^{\bar\lambda} = I - \bar\lambda(A^*_\gamma)^{-1}$ on H. Then $E^\lambda F^\lambda = F^\lambda E^\lambda = I$, $E'^{\bar\lambda} F'^{\bar\lambda} = F'^{\bar\lambda} E'^{\bar\lambda} = I$ on H.

Moreover, E^λ and $E'^{\bar\lambda}$ restrict to homeomorphisms

$$E^\lambda_V \colon V \xrightarrow{\sim} V_\lambda, \quad E'^{\bar\lambda}_W \colon W \xrightarrow{\sim} W_{\bar\lambda},$$

with inverses F^λ_V resp. $F'^{\bar\lambda}_W$. In particular, $D(T^\lambda) = E^\lambda_V D(T)$.

The operator families derived from E^λ are related to what was called

a gamma-field in other works from the 1970's and onwards, as a simple special case.

Theorem 8.4 *Let $G_{V,W}^\lambda = -\operatorname{pr}_W \lambda E^\lambda i_V$; then*

$$(E_W'^{\bar\lambda})^* T^\lambda E_V^\lambda = T + G_{V,W}^\lambda.$$

In other words, T and T^λ are related by the commutative diagram

$$
\begin{array}{ccc}
V_\lambda & \xleftarrow{\ \underset{E_V^\lambda}{\sim}\ } & V \\
{\scriptstyle T^\lambda}\Big\downarrow & & \Big\downarrow{\scriptstyle T+G_{V,W}^\lambda} \\
W_{\bar\lambda} & \xrightarrow[\ (E_W'^{\bar\lambda})^*\ ]{\sim} & W
\end{array}
\qquad\qquad D(T^\lambda) = E_V^\lambda D(T).
$$

In [Brown, Grubb and Wood, 2009] we moreover showed how the study relates to studies of boundary triples and M-functions by other researchers (as referred to in the start of this section; more references are given in [Brown, Grubb and Wood, 2009]):

Let $V = Z$, $W = Z'$, then, with $\mathcal{H} = Z'$, $\mathcal{K} = Z$, the mappings

$$\begin{pmatrix}\Gamma_1 u\\ \Gamma_0 u\end{pmatrix} = \begin{pmatrix}(Au)_{Z'}\\ u_\zeta\end{pmatrix} : D(A_{\max}) \to \mathcal{H} \times \mathcal{K},$$

$$\begin{pmatrix}\Gamma_1' v\\ \Gamma_0' v\end{pmatrix} = \begin{pmatrix}(A'v)_Z\\ v_{\zeta'}\end{pmatrix} : D(A_{\max}') \to \mathcal{K} \times \mathcal{H},$$

form a boundary triple: Both mappings $\binom{\Gamma_1}{\Gamma_0}$ and $\binom{\Gamma_1'}{\Gamma_0'}$ are surjective, their kernels are $D(A_{\min})$ resp. $D(A_{\min}')$, and they satisfy the Green's formula

$$(A_{\max}u, v) - (u, A_{\max}'v) = (\Gamma_1 u, \Gamma_0' v)_{\mathcal{H}} - (\Gamma_0 u, \Gamma_1' v)_{\mathcal{K}},$$

which is a rewriting of (8.7).

Here one can consider a boundary condition

$$\Gamma_1 u = T\Gamma_0 u, \qquad\qquad (8.10)$$

where we allow T to be unbounded (closed densely defined) from $\mathcal{K}$ to $\mathcal{H}$; it defines a restriction $\widetilde{A}$ of $A_{\max}$ by $D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \Gamma_0 u \in D(T), \Gamma_1 u = T\Gamma_0 u\}$. Then it is customary to define an M-function as follows:

Definition 8.5 For $\lambda \in \varrho(\widetilde{A})$, $M(\lambda)\colon \mathrm{ran}(\Gamma_1 - T\Gamma_0) \to \mathcal{K}$ is the operator satisfying

$$M(\lambda)(\Gamma_1 x - T\Gamma_0 x) = \Gamma_0 x, \text{ for all } x \in \ker(A_{\max} - \lambda) = Z_\lambda.$$

The analysis in [Brown, Grubb and Wood, 2009] showed that $M(\lambda)$ is a holomorphic family of operators in $\mathcal{L}(\mathcal{H}, \mathcal{K})$. On the other hand, when $\widetilde{A}$ and its boundary condition (8.10) are considered from the point of view of extensions [Grubb, 1968, 1970, 1971, 1973, 1974] recalled further above, $\widetilde{A}$ is the operator corresponding to $T\colon Z \to Z'$ by Theorem 8.2. Then we find moreover, in terms of the λ-dependent families introduced in that context:

$$M(\lambda) = -(T + G_{Z,Z'}^\lambda)^{-1} = -F_Z^\lambda (T^\lambda)^{-1} (F_{Z'}'^{\bar{\lambda}})^*, \text{ when } \lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma).$$

This gives the Kreǐn resolvent formula in the form

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} - \mathrm{i}_{Z_\lambda} E_Z^\lambda M(\lambda)(E_{Z'}'^{\bar{\lambda}})^* \, \mathrm{pr}_{Z'_{\bar{\lambda}}}.$$

For the case of general V and W, we could likewise construct an M-function from W to V for $\lambda \in \varrho(\widetilde{A})$, and establish a Kreǐn resolvent formula around it. The following result is shown in [Brown, Grubb and Wood, 2009]:

Theorem 8.6 *Let $\widetilde{A}$ correspond to $T\colon V \to W$. For $\lambda \in \varrho(\widetilde{A})$ there is a well-defined holomorphic family $M(\lambda) \in \mathcal{L}(W, V)$:*

$$M(\lambda) = \mathrm{pr}_\zeta \left(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda) \right) A_\gamma^{-1} \mathrm{i}_W.$$

When $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$, then

$$M(\lambda) = -(T + G_{V,W}^\lambda)^{-1} = -F_V^\lambda (T^\lambda)^{-1} (F_W'^{\bar{\lambda}})^*,$$

and

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} - \mathrm{i}_{V_\lambda} E_V^\lambda M(\lambda)(E_W'^{\bar{\lambda}})^* \, \mathrm{pr}_{W_{\bar{\lambda}}}.$$

To have both T^λ (for $\lambda \in \varrho(A_\gamma)$) and $M(\lambda)$ (for $\lambda \in \varrho(\widetilde{A})$) available is an advantage, since $\ker(\widetilde{A} - \lambda) = \ker T^\lambda$ and $\mathrm{ran}(\widetilde{A} - \lambda) = \mathrm{ran}\, T^\lambda + (H \ominus W_{\bar{\lambda}})$ give straightforward eigenvalue information at the poles of $M(\lambda)$ in $\varrho(A_\gamma)$.

Remark The name M-function is consistent with the notation in some papers that [Brown, Grubb and Wood, 2009] refers to, but possibly diverges from others (one could also use the longer name Weyl-Titchmarsh function). There is a recent publication [Malamud, 2010] that exposes related resolvent formulas on the basis of [Malamud and Mogilevski,

2002]. (Let us remark that [Malamud, 2010] gives the impression that only separate surjectiveness of Γ_0 and Γ_1 is assumed in [Brown, Grubb and Wood, 2009]; this is not so.)

8.6 Implementation of the abstract set-up for elliptic operators

We shall now recall how the abstract theory is applied to a concrete choice of elliptic operator A. Here $A_{\max}$ and $A_{\min}$ are defined as in Section 8.2; they are closed operators in $H = L_2(\Omega)$. In [Grubb, 1968] general even-order operators were considered, and the reference operator (called A_γ in Section 8.5) was taken to represent a general normal boundary condition. To simplify our explanation, we shall here just consider a second-order strongly elliptic operator A and let A_γ stand for the Dirichlet realization, mentioned after (8.2). We have by elliptic regularity that $D(A_\gamma) = H^2(\Omega) \cap H_0^1(\Omega)$, and we can assume that (a constant has been added to A such that) the lower bound $m(A_\gamma)$ is positive. The lower bound $m(P)$ of an operator P is defined by

$$m(P) = \inf\{\operatorname{Re}(Pu, u) \mid u \in D(P), \|u\| = 1\} \geq -\infty; \qquad (8.11)$$

when it is finite, P is said to be lower bounded.

The trace operator γ_0 defines a continuous mapping $H^s(\Omega) \to H^{s-\frac{1}{2}}(\Sigma)$ for $s > -\frac{1}{2}$. We shall also need a more advanced fact, namely that, as shown by Lions and Magenes (see e.g. [Lions and Magenes, 1968]), γ_0 extends to a mapping $D(A_{\max}) \to H^{-\frac{1}{2}}(\Sigma)$, and defines *homeomorphisms*

$$\gamma_Z \colon Z \xrightarrow{\sim} H^{-\frac{1}{2}}(\Sigma), \quad \gamma_{Z'} \colon Z' \xrightarrow{\sim} H^{-\frac{1}{2}}(\Sigma),$$

where Z and Z' are the nullspaces of $A_{\max}$ and $A'_{\max}$ (not contained in $H^s(\Omega)$ for $s > 0$). The inverse of γ_Z is consistent with the Poisson operator K_γ solving the semi-homogeneous Dirichlet problems (8.2) with $f = 0$, in the sense that

$$K_\gamma = \mathrm{i}_Z \gamma_Z^{-1}.$$

Similarly, the inverse $\gamma_{Z'}^{-1}$ is consistent with the Poisson solution operator K'_γ solving the Dirichlet problem for A' with $f = 0$, $K'_\gamma = \mathrm{i}_{Z'} \gamma_{Z'}^{-1}$. Moreover, with λ-dependence,

$$K_\gamma^\lambda = \mathrm{i}_{Z_\lambda} \gamma_{Z_\lambda}^{-1}, \quad K_\gamma'^{\bar\lambda} = \mathrm{i}_{Z'_{\bar\lambda}} \gamma_{Z'_{\bar\lambda}}^{-1},$$

solve the semi-homogeneous Dirichlet problems for $A - \lambda$, $A' - \bar{\lambda}$, when $\lambda \in \varrho(A_\gamma)$.

For a closed subspace $V \subset Z$, let $X = \gamma_0 V \subset H^{-\frac{1}{2}}(\Sigma)$. Here we denote the restriction of γ_0 by γ_V;

$$\gamma_V \colon V \xrightarrow{\sim} X, \tag{8.12}$$

with a similar notation for $Y = \gamma_0 W$ and λ-dependent cases. The map $\gamma_V \colon V \xrightarrow{\sim} X$ has the adjoint $\gamma_V^* \colon X^* \xrightarrow{\sim} V$. Here X^* denotes the antidual space of X, with a duality coinciding with the scalar product in $L_2(\Sigma)$ when applied to elements that come from $L_2(\Sigma)$. The duality is written $(\psi, \varphi)_{X^*, X}$.
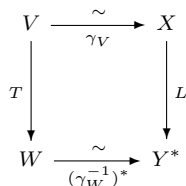
We denote

$$K_{\gamma, X} = i_V \gamma_V^{-1} \colon X \to V \subset H; \tag{8.13}$$

it is a Poisson operator when $X = H^{-\frac{1}{2}}(\Sigma)$.

Now a given $T \colon V \to W$ is carried over to a closed, densely defined operator $L \colon X \to Y^*$ by the definition

$$L = (\gamma_W^{-1})^* T \gamma_V^{-1}, \quad D(L) = \gamma_V D(T);$$

it is expressed in the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\underset{\gamma_V}{\sim}} & X \\
{\scriptstyle T}\Big\downarrow & & \Big\downarrow{\scriptstyle L} \\
W & \xrightarrow[(\gamma_W^{-1})^*]{\sim} & Y^*
\end{array}
$$

There is a similar definition in the λ-dependent case.

Before formulating the results in a theorem, we shall interpret the abstract boundary condition (8.9) defining the realization $\widetilde{A}$, as a concrete condition in terms of L.

A has a Green's formula (for sufficiently smooth u, v)

$$(Au, v)_\Omega - (u, A'v)_\Omega = (\nu_1 u, \gamma_0 v)_\Sigma - (\gamma_0 u, \nu_1' v)_\Sigma, \tag{8.14}$$

where

$$\nu_1 = s\gamma_1, \quad \nu_1' = \bar{s}\gamma_1 + \mathcal{A}'\gamma_0,$$

with a nonvanishing smooth function s and a suitable first-order differential operator $\mathcal{A}'$ on Σ.

Let $\lambda \in \varrho(A_\gamma)$. In addition to the Poisson operators K_γ^λ resp. $K_\gamma'^{\bar{\lambda}}$

solving the Dirichlet problems for $A - \lambda$ resp. $A' - \bar{\lambda}$, we shall need the *Dirichlet-to-Neumann* operators

$$P^\lambda_{\gamma_0, \nu_1} = \nu_1 K^\lambda_\gamma, \quad P'^{\bar{\lambda}}_{\gamma_0, \nu'_1} = \nu'_1 K^{\bar{\lambda}}_\gamma,$$

that map the Dirichlet boundary value into the Neumann boundary value for null-solutions. By the composition rules for ψdbo's, they are pseudodifferential operators of order 1; moreover, it is known that $P^\lambda_{\gamma_0, \nu_1}$ is elliptic.

Theorem 8.7 *Define the* **reduced Neumann trace operator** Γ *by*

$$\Gamma = \nu_1 - P^0_{\gamma_0, \nu_1} \gamma_0 = \nu_1 A^{-1}_\gamma A_{\max} \colon D(A_{\max}) \to H^{\frac{1}{2}}(\Sigma).$$

It is continuous and surjective, and vanishes on Z. *With the analogous definition for* A' *one has the* **reduced Green's formula**:

$$(Au, v) - (u, A'v) = (\Gamma u, \gamma_0 v)_{\frac{1}{2}, -\frac{1}{2}} - (\gamma_0 u, \Gamma' v)_{-\frac{1}{2}, \frac{1}{2}},$$

valid for all $u \in D(A_{\max})$, $v \in D(A'_{\max})$. *In particular*,

$$(Au, w) = (\Gamma u, \gamma_0 w)_{\frac{1}{2}, -\frac{1}{2}}, \text{ when } u \in D(A_{\max}), w \in Z'. \quad (8.15)$$

For $\lambda \in \varrho(A_\gamma)$ *we similarly define*

$$\Gamma^\lambda = \nu_1 - P^\lambda_{\gamma_0, \nu_1} \gamma_0 = \nu_1 (A_\gamma - \lambda)^{-1}(A_{\max} - \lambda),$$
$$\Gamma'^{\bar{\lambda}} = \nu_1 - P'^{\bar{\lambda}}_{\gamma_0, \nu'_1} \gamma_0 = \nu'_1 (A^*_\gamma - \bar{\lambda})^{-1}(A'_{\max} - \bar{\lambda}),$$

continuous and surjective from $D(A_{\max})$ *resp.* $D(A'_{\max})$ *to* $H^{\frac{1}{2}}(\Sigma)$; *then there holds a reduced Green's formula*

$$(Au, v) - (u, A'v) = (\Gamma^\lambda u, \gamma_0 v)_{\frac{1}{2}, -\frac{1}{2}} - (\gamma_0 u, \Gamma'^{\bar{\lambda}} v)_{-\frac{1}{2}, \frac{1}{2}},$$

for $u \in D(A_{\max})$, $v \in D(A^*_{\max})$.

Here $D(A_{\max})$ is provided with the graph-norm.

Now let $\widetilde{A}$ be a closed operator lying between $A_{\min}$ and $A_{\max}$, so $\widetilde{A} \in \mathcal{M}$. The abstract boundary condition (8.9) for $\widetilde{A}$ may be written:

$$(Au, w) = (Tu_\zeta, w), \text{ all } w \in W. \quad (8.16)$$

The left-hand side equals $(\Gamma u, \gamma_0 w)_{\frac{1}{2}, -\frac{1}{2}}$ by (8.15). The right-hand side equals

$$(Tu_\zeta, w) = (T\gamma_V^{-1}\gamma_0 u, \gamma_W^{-1}\gamma_0 w) = (L\gamma_0 u, \gamma_0 w)_{Y^*, Y},$$

by definition of L (it is used that when $u_\zeta \in V$, $u_\zeta = \gamma_V^{-1}\gamma_V u_\zeta = \gamma_V^{-1}\gamma_0 u$).

Hence (8.16) may be rewritten as

$$(\Gamma u, \gamma_0 w)_{\frac{1}{2}, -\frac{1}{2}} = (L\gamma_0 u, \gamma_0 w)_{Y^*, Y}, \text{ all } w \in W. \qquad (8.17)$$

The injection $i_Y \colon Y \to H^{-\frac{1}{2}}(\Sigma)$ has the adjoint $i_Y^* \colon H^{\frac{1}{2}}(\Sigma) \to Y^*$ that sends a functional ψ on $H^{-\frac{1}{2}}(\Sigma)$ over into a functional $i_Y^*\psi$ on Y by:

$$(i_Y^*\psi, \varphi)_{Y^*, Y} = (\psi, \varphi)_{\frac{1}{2}, -\frac{1}{2}} \text{ for all } \varphi \in Y.$$

Then (8.17) may be rewritten as

$$i_Y^* \Gamma u = L\gamma_0 u,$$

or, when we use that $\Gamma = \nu_1 - P^0_{\gamma_0, \nu_1}\gamma_0$,

$$i_Y^* \nu_1 u = (L + i_Y^* P^0_{\gamma_0, \nu_1})\gamma_0 u. \qquad (8.18)$$

This is the boundary condition derived from (8.9), when $\widetilde{A}$ corresponds to $T \colon V \to W$ by Theorem 8.2, carried over to $L \colon X \to Y^*$ by (8.12).

Then Theorem 8.2 implies:

Theorem 8.8 *There is a 1–1 correspondence between the closed operators $\widetilde{A} \in \mathcal{M}$ and the closed densely defined operators $L \colon X \to Y^*$, where X and Y are closed subspaces of $H^{-\frac{1}{2}}(\Sigma)$, such that $\widetilde{A}$ corresponds to $L \colon X \to Y^*$ if and only if $D(\widetilde{A})$ consists of the functions in $D(A_{\max})$ for which*

$$\gamma_0 u \in D(L), \quad i_Y^* \nu_1 u = (L + i_Y^* P^0_{\gamma_0, \nu_1})\gamma_0 u. \qquad (8.19)$$

In this correspondence, $X = \overline{\gamma_0 D(\widetilde{A})}$, $Y = \overline{\gamma_0 D(\widetilde{A}^*)}$, and

- *$\widetilde{A}^*$ corresponds analogously to $L^* \colon Y \to X^*$.*
- *$\ker \widetilde{A} = i_V \gamma_V^{-1} \ker L; \quad \operatorname{ran} \widetilde{A} = \gamma_W^* \operatorname{ran} L \dotplus (H \ominus W)$, cf. (8.12), (8.13).*
- *$\widetilde{A}$ is bijective if and only if L is so, and then*

$$\widetilde{A}^{-1} = A_\gamma^{-1} + i_V \gamma_V^{-1} L^{-1} \gamma_W^* \operatorname{pr}_W = A_\gamma^{-1} + K_{\gamma, X} L^{-1}(K'_{\gamma, Y})^*. \qquad (8.20)$$

Theorems 8.7 and 8.8 are from [Grubb, 1968], except that we have modified the notation a little.

In [Brown, Grubb and Wood, 2009], the subspace cases are treated with insertion of an *isometry* $\Lambda_{\frac{1}{2}} \colon L_2(\Sigma) \xrightarrow{\sim} H^{-\frac{1}{2}}(\Sigma)$, that allows replacing X and Y by closed subspaces X_1 and Y_1 of $L_2(\Sigma)$, identified with their dual spaces; then i_Y^* is replaced by an orthogonal projection pr_{Y_1}.

In the case where $Y = H^{-\frac{1}{2}}(\Sigma)$, i.e., $W = Z'$, the map i_Y^* is superfluous, and the second condition in (8.19) takes the form

$$\nu_1 u = (L + P^0_{\gamma_0,\nu_1})\gamma_0 u.$$

When also $X = H^{-\frac{1}{2}}(\Sigma)$, we say that $\widetilde{A}$ represents a *Neumann-type condition*

$$\nu_1 u = C\gamma_0 u; \text{ here } C = L + P^0_{\gamma_0,\nu_1} \text{ on } D(L).$$

In this case, L can act like a pseudodifferential operator, namely when C (in the condition $\nu_1 u = C\gamma_0 u$) is a differential or pseudodifferential operator.

Let us consider a slightly different set-up where C is a *given* first-order differential or pseudodifferential operator on Σ, and we *define* $\widetilde{A}$ by

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \nu_1 u = C\gamma_0 u\}, \tag{8.21}$$

where γ_0 and ν_1 are considered as mappings from $D(A_{\max})$ to $H^{-\frac{1}{2}}(\Sigma)$ resp. $H^{-\frac{3}{2}}(\Sigma)$. We shall discuss the corresponding operator $L\colon X \to Y^*$. Since $\{\gamma_0 u, \nu_1 u\}$ maps $H^2(\Omega)$ onto $H^{\frac{3}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$, $D(L) = \gamma_0 D(\widetilde{A}) \supset H^{\frac{3}{2}}(\Sigma)$. Then since $H^{\frac{3}{2}}(\Sigma)$ is dense in $H^{-\frac{1}{2}}(\Sigma)$, $X = H^{-\frac{1}{2}}(\Sigma)$. By use of Green's formula (8.14) it is checked that the adjoint $\widetilde{A}^*$ extends the realization of A' with domain consisting of the functions $v \in H^2(\Omega)$ satisfying

$$\nu_1' v = C^*\gamma_0 v,$$

so also $Y = H^{-\frac{1}{2}}(\Sigma)$. Thus we are in the case of Neumann-type boundary conditions, so by comparison with (8.6), it is seen that L acts like $C - P^0_{\gamma_0,\nu_1}$.

The domain $D(L)$ equals $\{\varphi \in H^{-\frac{1}{2}}(\Sigma) \mid (C - P^0_{\gamma_0,\nu_1})\varphi \in H^{\frac{1}{2}}(\Sigma)\}$; it may not be easy to determine more exactly. Note that L is used as a map from its domain in $H^{-\frac{1}{2}}(\Sigma)$ to $H^{+\frac{1}{2}}(\Sigma)$, although it acts like a ψdo of order 1.

One case is clear, though: If $C - P^0_{\gamma_0,\nu_1}$ is *elliptic* of order 1, then $L\varphi \in H^{\frac{1}{2}}(\Sigma)$ implies $\varphi \in H^{\frac{3}{2}}(\Sigma)$; in this case $D(L) = H^{\frac{3}{2}}(\Sigma)$. Moreover, $D(\widetilde{A}) \subset H^2(\Omega)$. A check of the boundary symbol rules shows that this is precisely the case where the system $\{A, \nu_1 - C\gamma_0\}$ is *elliptic*. Here we have:

Theorem 8.9 *Let C be a first-order differential or pseudodifferential operator on Σ and define the realization $\widetilde{A}$ of A by (8.21). Then if*

$C - P^0_{\gamma_0,\nu_1}$ *is elliptic, the operator* $L\colon X \to Y^*$ *corresponding to* $\widetilde{A}$ *by Theorem 8.8 acts like* $C - P^0_{\gamma_0,\nu_1}$ *and has*

$$X = Y = H^{-\frac{1}{2}}(\Sigma), \quad D(L) = H^{\frac{3}{2}}(\Sigma).$$

Moreover, $D(\widetilde{A}) \subset H^2(\Omega)$. *Related statements hold for the adjoint* $\widetilde{A}^*$.

A Robin condition $\nu_1 u = b\gamma_0 u$, with a smooth function $b(x)$ on Σ, is elliptic, since L acts like $b - P^0_{\gamma_0,\nu_1}$, where $P^0_{\gamma_0,\nu_1}$ is elliptic of order 1 and b is of order 0.

In the case of Theorem 8.9, when L is bijective, the formula (8.20) has the form

$$\widetilde{A}^{-1} = A_\gamma^{-1} + K_\gamma L^{-1}(K'_\gamma)^*, \qquad (8.22)$$

where all ingredients belong to the ψdbo calculus: K_γ is a Poisson operator, L^{-1} is a ψdo on Σ, $(K'_\gamma)^*$ is a trace operator of class 0, and the composition $K_\gamma L^{-1}(K'_\gamma)^*$ is a singular Green operator (of class 0). A_γ^{-1} is the sum $Q_+ + G$ of a truncated ψdo Q on $\mathbb{R}^n$ and a singular Green operator, as in (8.4).

Realizations defined by boundary conditions of the type $\nu_1 u = C\gamma_0 u$ are studied by means of quasi-boundary triples techniques for selfadjoint A in Theorem 6.24 ff.; here C is a selfadjoint bounded operator in $L_2(\Sigma)$ and $D(\widetilde{A}) \subset H^{\frac{3}{2}}(\Omega)$.

When A is of order $2m$, there is a Green's formula generalizing (8.14), where γ_0 is replaced by an m-vector $\gamma = \{\gamma_0, \dots, \gamma_{m-1}\}$, and ν_1 and ν'_1 are replaced by m-vectors of trace operators of orders m, $\dots$, $2m-1$, mapping into products of Sobolev spaces of different orders over Σ. One then gets vector versions of the reduced Neumann trace operators Γ and Γ', with matrix-formed versions of the Dirichlet-to-Neumann pseudodifferential operators, but the basic mechanisms in the interpretation are the same. There are interesting cases of subspaces X, Y of the products of Sobolev spaces over Σ, where ellipticity considerations are relevant. Details are given in [Grubb, 1968]–[Grubb, 1974] and [Brown, Grubb and Wood, 2009].

8.7 Resolvent formulas

When $\lambda \in \varrho(A_\gamma)$, there is a similar representation of $\widetilde{A} - \lambda$ in terms of a boundary condition defined from an operator L^λ acting over the boundary. Here it is of particular interest to find the connection between

L and L^λ, just as we found the connection between T and T^λ. It turns out that the relation between L and L^λ is simpler: They both go from X to Y^*, whereas T resp. T^λ map between different spaces due to the shift from Z to Z_λ. This holds, since

$$D(L) = \gamma_0 D(\widetilde{A}) = \gamma_0 D(\widetilde{A} - \lambda) = D(L^\lambda), \quad X = \overline{D(L)} = \overline{D(L^\lambda)},$$

with similar statements for $D(L^*)$, $D((L^\lambda)^*)$ and Y. Then we have:

The horizontal maps compose as $\gamma_{V_\lambda} E_V^\lambda = \gamma_V$, $(\gamma_{W_{\bar\lambda}}^*)^{-1}(F_W'^{\bar\lambda})^* = (\gamma_W^*)^{-1}$, so

$$L^\lambda = \gamma_V^{-1}(T + G_{V,W}^\lambda)(\gamma_W^*)^{-1}.$$

In terms of L^λ, the boundary condition reads (analogously to (8.18)):

$$i_Y^* \nu_1 u = (L^\lambda + i_Y^* P_{\gamma_0,\nu_1}^\lambda)\gamma_0 u, \ \gamma_0 u \in D(L). \tag{8.23}$$

Since $D(\widetilde{A} - \lambda) = D(\widetilde{A})$ is at the same time defined by the boundary condition $i_Y^* \nu_1 u = (L + i_Y^* P_{\gamma_0,\nu_1}^0)\gamma_0 u$ for $\gamma_0 u \in D(L)$, we have that $L^\lambda + i_Y^* P_{\gamma_0,\nu_1}^\lambda = L + i_Y^* P_{\gamma_0,\nu_1}^0$ on $D(L)$, so

$$L^\lambda = L + i_Y^*(P_{\gamma_0,\nu_1}^0 - P_{\gamma_0,\nu_1}^\lambda) \text{ on } D(L).$$

The last formula is convenient, since $P_{\gamma_0,\nu_1}^0 - P_{\gamma_0,\nu_1}^\lambda$ can be shown to be *bounded* from $H^{-\frac{1}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$; hence L^λ is a perturbation of L by a bounded operator.

Also the general M-function defined in Section 8.5 carries over to an M-function on the boundary, a holomorphic family of operators $M_L(\lambda) \in \mathcal{L}(Y^*, X)$ defined for $\lambda \in \varrho(\widetilde{A})$.

The results are collected in the following theorem:

Theorem 8.10 *Let $\widetilde{A}$ correspond to $T\colon V \to W$ as in Theorem 8.2, carried over to $L\colon X \to Y^*$ as in Theorem 8.8. For $\lambda \in \varrho(A_\gamma)$ it is also described by the boundary condition (8.23), and there holds:*
(i) For $\lambda \in \varrho(A_\gamma)$, $P_{\gamma_0,\nu_1}^0 - P_{\gamma_0,\nu_1}^\lambda \in \mathcal{L}(H^{-\frac{1}{2}}(\Sigma), H^{\frac{1}{2}}(\Sigma))$ and

$$L^\lambda = L + i_Y^*(P_{\gamma_0,\nu_1}^0 - P_{\gamma_0,\nu_1}^\lambda) \text{ on } D(L).$$

(ii) *For $\lambda \in \varrho(\widetilde{A})$, there is a related M-function $\in \mathcal{L}(Y^*, X)$,*

$$M_L(\lambda) = \gamma_0 \big(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda) \big) A_\gamma^{-1} \mathrm{i}_W \gamma_W^*.$$

(iii) *For $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$,*

$$M_L(\lambda) = -(L^\lambda)^{-1} = -\big(L + \mathrm{i}_Y^*(P_{\gamma_0,\nu_1}^0 - P_{\gamma_0,\nu_1}^\lambda)\mathrm{i}_X \big)^{-1},$$

and we have the Kreĭn-type resolvent formulas:

$$(\widetilde{A} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = K_{\gamma,X}^\lambda (L^\lambda)^{-1} (K_{\gamma,Y}'^{\bar{\lambda}})^*$$
$$= -K_{\gamma,X}^\lambda M_L(\lambda)(K_{\gamma,Y}'^{\bar{\lambda}})^*.$$

In the case of a Neumann-type boundary condition as in (8.21), $L^\lambda = C - P_{\gamma_0,\nu_1}^\lambda$ on $D(L^\lambda) = D(L)$.

8.8 Applications of pseudodifferential methods I: Conditions for lower boundedness

The formulas we have shown so far use the terminology of ψdbo's mainly as a way to indicate what the ingredients in certain operator compositions are. The next question to consider is how properties of $\widetilde{A}$ are reflected in properties of L. Part of the analysis can be carried out with methods of functional analysis, but there also exist problems that are solved most efficiently by involving deeper pseudodifferential principles.

An example of how functional analytic and pseudodifferential methods are useful together, is the question of lower boundedness inequalities.

We here restrict the attention to the symmetric set-up where A is formally selfadjoint (so $A_{\max} = A_{\min}^*$) and A_γ is selfadjoint; methods for extending the results to nonsymmetric set-ups are found in [Grubb, 1974]. We assume that A_γ has positive lower bound $m(A_\gamma)$ (cf. (8.11)).

In the symmetric case, the general extensions $\widetilde{A}$ can of course be nonsymmetric (since $A_{\max}$ is so). Let us speak of the "selfadjoint case" when only selfadjoint $\widetilde{A}$'s are considered.

In the following, we assume throughout that $\widetilde{A}$ corresponds to $T \colon V \to W$ as in Theorem 8.2, and to $L \colon X \to Y^*$ as in Theorem 8.8. The lower boundedness problem is the problem of how information on lower bounds on $\widetilde{A}$ is related to similar information on T or L. The following was shown in [Grubb, 1970] (together with studies of coerciveness estimates):

Theorem 8.11 *In the symmetric set-up with A_γ selfadjoint positive, let $\widetilde{A}$ correspond to $T \colon V \to W$ as in Theorem 8.2.*

1° *If $V \subset W$ and T has a lower bound $m(T)$ satisfying $m(T) > -m(A_\gamma)$, then $m(\widetilde{A}) \geq m(T)m(A_\gamma)/(m(T) + m(A_\gamma))$.*

2° *Assume that A_γ is the Friedrichs extension of $A_{\min}$. If $m(\widetilde{A}) > -\infty$, then $V \subset W$ and $m(T) \geq m(\widetilde{A})$.*

In the selfadjoint case these rules go back to [Birman, 1956], preceded by sesquilinear form results of [Krein, 1947]. For studies where $m(A_\gamma)$ is only assumed ≥ 0, see Chapter 3.

The properties of T are easily translated to similar properties of L using the homeomorphism (8.6); here when $X \subset Y$, we set

$$m_{-\frac{1}{2}}(L) = \inf\{\operatorname{Re}(L\varphi, \varphi)_{Y^*, Y} \mid \varphi \in D(L), \|\varphi\|_{-\frac{1}{2}} = 1\}, \qquad (8.24)$$

for some choice of norm $\|\varphi\|_{-\frac{1}{2}}$ on $H^{-\frac{1}{2}}(\Sigma)$. (One can let γ_Z be an isometry, to carry numerical information over between T and L. Sometimes qualitative objects such as the sign of $m_{-\frac{1}{2}}(L)$ are sufficiently interesting.) To take A_γ as the Friedrichs extension of $A_{\min}$ means that it is taken as the Dirichlet realization.

In 1° we see that the statement "$m(T) > -\infty \implies m(\widetilde{A}) > -\infty$" holds under the additional assumption that $m(T) > -m(A_\gamma)$; there is a nontrivial question of when that assumption can be removed. In [Grubb, 1974] it was shown that when A_γ is the Friedrichs extension and A_γ^{-1} is a *compact* operator, then $m(T) > -\infty$ does imply $m(\widetilde{A}) > -\infty$. This same result was also announced in [Gorbachuk and Mikhailets, 1976] for the selfadjoint case.

In the application to boundary value problems, we therefore have from this early result that lower boundedness of $\widetilde{A}$ and L hold simultaneously when we consider problems on *bounded domains* Ω, for then A_γ^{-1} is indeed compact.

For unbounded domains, the question has, to our knowledge, remained unsolved up until recently. The question is closely connected with the comparison of T with T^λ as in Theorem 8.4. Indeed, as shown in [Grubb, 1974]:

Proposition 8.12   *Let $G^\lambda_{V,W}$ be as defined in Theorem 8.4. The property*

$$m(G^\mu_{Z,Z}) \to \infty \text{ for } \mu \to -\infty, \ \mu \in \mathbb{R}, \qquad (8.25)$$

is necessary and sufficient for the validity of

$$m(T) > -\infty \implies m(\widetilde{A}) > -\infty \qquad (8.26)$$

for general closed $\widetilde{A} \in \mathcal{M}$.

The question was also studied later in [Derkach and Malamud, 1991] who worked out an analysis that generalizes Proposition 8.12 and gives further conditions for the validity of the conclusion from $m(T)$ to $m(\widetilde{A})$. However this did not capture elliptic problems on unbounded domains ($n \geq 2$).

Because of the recent interest in the analysis of extensions, we have considered the problem again, and found a solution in [Grubb, 2012] for *exterior domains* (complements in $\mathbb{R}^n$ of bounded domains).

Theorem 8.13 *Let Ω be the complement of a smooth bounded set $\overline{\Omega}_-$ in $\mathbb{R}^n$, and let A be symmetric and uniformly strongly elliptic on Ω with coefficients in $C_b^\infty(\overline{\Omega})$, and with a positive lower bound for A_γ. In the application of the extension theory to this situation, (8.25) holds, and hence also (8.26).*

Here $C_b^\infty(\overline{\Omega})$ stands for the C^∞-functions that are bounded with bounded derivatives.

The proof relies on the "translation" of abstract operators $\widetilde{A}$ to concrete operators defined by boundary conditions. Indeed, it turns out that the lower bound of $G_{Z,Z}^\mu$ behaves like the lower bound $m_{-\frac{1}{2}}(Q^\mu)$ (cf. (8.24)) of $Q^\mu = P_{\gamma_0,\nu_1}^0 - P_{\gamma_0,\nu_1}^\mu$. Then the deep part of the proof lies in setting the operator Q^μ in relation to the analogous operator for the interior domain Ω_-, Q_-^μ, which does have the desired property, in view of our knowledge of problems on bounded domains. The point is to show that $|((Q^\mu - Q_-^\mu)\varphi, \varphi)|$ is bounded by $c\|\varphi\|_{-\frac{1}{2}}^2$ uniformly for $\mu \to -\infty$, so that addition of $Q^\mu - Q_-^\mu$ to Q_-^μ does not violate the growth of the lower bound. This goes by a delicate application of the ψdbo calculus. Details are in [Grubb, 2012].

Let us mention that there is a considerably easier result that holds regardless of boundedness of $\partial\Omega$ and only requires some uniformity in the estimates of coefficients of the operators, namely preservation of coerciveness inequalities (Gårding inequalities). We here assume that Ω is a subset of $\mathbb{R}^n$ with smooth boundary, admissible as defined in [Grubb, 1996] (besides bounded domains, this allows exterior domains, perturbed halfspaces and other cases that can be covered with a finite system of local coordinates of a relatively simple kind). Moreover, we assume that A is uniformly strongly elliptic on $\overline{\Omega}$ with coefficients in $C_b^\infty(\overline{\Omega})$. The result is that then $\widetilde{A}$ satisfies a Gårding inequality (with $c > 0$, $k \in \mathbb{R}$)

$$\mathrm{Re}(\widetilde{A}u, u) \geq c\|u\|_1^2 - k\|u\|_0^2, \quad u \in D(\widetilde{A}),$$

if and only if $X \subset Y$ and L satisfies an inequality

$$\mathrm{Re}(L\varphi, \varphi) \geq c' \|\varphi\|^2_{H^{\frac{1}{2}}(\Sigma)} - k' \|\varphi\|^2_{H^{-\frac{1}{2}}(\Sigma)}, \quad \varphi \in D(L). \qquad (8.27)$$

In the case of differential or pseudodifferential Neumann-type boundary conditions, the inequality (8.27) for L holds precisely when the pseudodifferential operator it acts like, is strongly elliptic. Details for bounded sets are in [Grubb, 1970] for realizations of $2m$-order operators; the extension to unbounded sets is shown in [Grubb, 2012] — the argumentation just involves standard trace theorems and interpolation inequalities.

8.9 Applications of pseudodifferential methods II: Spectral asymptotics

In the symmetric set-up, when Ω is bounded, the eigenvalues of the selfadjoint operator A_γ form a sequence λ_j going to ∞ on $\mathbb{R}$. In [Weyl, 1912] appeared the famous estimate for $A = -\Delta$, for $n = 2, 3$, $m = 2$:

$$\lambda_j(A_\gamma) - c_0 j^{m/n} \text{ is } o(j^{m/n}) \text{ for } j \to \infty,$$

where c_0 is a constant defined from the volume of Ω; the eigenvalues are repeated according to multiplicities. Equivalently, the counting function $N(t; A_\gamma)$ (counting the number of eigenvalues in $[0, t]$), and the eigenvalues μ_j of the inverse A_γ^{-1}, satisfy

$$N(t; A_\gamma) - c_A t^{n/m} \text{ is } o(t^{n/m}) \text{ for } t \to \infty,$$
$$\mu_j(A_\gamma^{-1}) - c_A^{m/n} j^{-m/n} \text{ is } o(j^{-m/n}) \text{ for } j \to \infty,$$

where

$$c_A = (2\pi)^{-n} \int_{x \in \Omega, \, a^0(x,\xi) < 1} dx d\xi$$

(and $c_0 = c_A^{-m/n}$). The estimates have been shown for general n and sharpened since then, with more precision on the remainder, and the validity has been extended to general elliptic operators A and boundary conditions, and to elliptic pseudodifferential operators P on compact manifolds. These improvements have a long history that we shall not try to account for here in detail; they are interesting not only because of the results but even more because of the refined theories that were invented in connection with the proofs (for example: Fourier integral operators).

See e.g. [Hörmander, 1968, 1971; Brüning, 1974; Seeley, 1978; Ivrii, 1982, 1984, 1991; Safarov and Vassiliev, 1997].

We shall here be concerned with a slightly different question, namely the spectral behavior of the difference between the resolvents of two realizations of A.

It is shown in [Birman, 1962] for second-order symmetric uniformly strongly elliptic operators A that the singular numbers $s_j(B) = \mu_j(B^*B)^{\frac{1}{2}}$ of the compact operator $B = \widetilde{A}^{-1} - A_\gamma^{-1}$ satisfy an upper estimate:

$$s_j(\widetilde{A}^{-1} - A_\gamma^{-1}) \leq Cj^{-2/(n-1)}, \text{ for all } j, \tag{8.28}$$

when $\widetilde{A}$ is a selfadjoint realization of A defined by a Neumann or Robin condition. In other words, $\widetilde{A}^{-1} - A_\gamma^{-1}$ belongs $\mathfrak{S}_{(n-1)/2,\infty}$, where $\mathfrak{S}_{p,\infty}$ is the space of compact operators B for which $s_j(B)$ is $O(j^{-\frac{1}{p}})$ (often called a weak Schatten class). It is particularly interesting that Birman showed this not just for interior, but also for exterior domains, and under low smoothness assumptions.

We note in passing that the estimate (8.28) implies that

$$\widetilde{A}^{-1} - A_\gamma^{-1} \in \mathcal{C}_p \text{ for } p > (n-1)/2, \tag{8.29}$$

where $\mathcal{C}_p$ is the p-th Schatten class (consisting of the operators B such that $(s_j(B))_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$); but this is less informative than (8.28).

One of the fundamental ingredients in these studies is embedding properties, more precisely the knowledge that an operator B that is continuous from $L^2(\Omega)$ to $H^s(\Omega)$ for some $s > 0$ (Ω bounded smooth $\subset \mathbb{R}^n$) is compact in $L_2(\Omega)$ and belongs to $\mathfrak{S}_{n/s,\infty}$. However, this alone only gives upper bounds on the behavior of singular numbers. To get Weyl-type limit properties one must know more about the differential or pseudodifferential structure of the operators.

The estimate (8.28) was sharpened to a Weyl-type asymptotic estimate in [Birman and Solomyak, 1977, 1980]; the latter showed a general principle for the spectrum of a ratio of two quadratic forms, implying that

$$s_j(\widetilde{A}^{-1} - A_\gamma^{-1})j^{2/(n-1)} \to c, \text{ for } j \to \infty, \tag{8.30}$$

for interior and exterior smooth domains.

Prior to this, a far-reaching result had been shown in [Grubb, 1974], Section 8: We consider a symmetric, strongly elliptic $2m$-order operator acting in an N-dimensional vector bundle E over a smooth compact Riemanninan manifold $\overline{\Omega}$ with boundary Σ, assuming that the Dirichlet

realization A_γ is invertible. Let A_B be a selfadjoint invertible realization defined by a normal boundary condition

$$\sum_{k \leq j} B_{jk}\gamma_k u = 0, \ j = 0, 1, \ldots, 2m - 1,$$

where the B_{jk} are differential operators of order $j - k$ from $E|_\Sigma$ to given vector bundles F_j over Σ (with $\dim F_j \geq 0$); *normality* of the boundary condition means that the B_{jj} are *surjective morphisms*. (That A acts in the vectorbundle E means that it is locally $(N \times N)$-matrix-formed. In the scalar case, $N = 1$ and the F_j are 0- or 1-dimensional, with B_{jj} an invertible function when $\dim F_j = 1$. Ellipticity of the boundary condition requires in particular that $\sum_j \dim F_j = mN$.) Denote $\oplus_{j>m}F_j = F^1$.

Theorem 8.14 *Let $T \colon V \to V$ be the operator corresponding to A_B by Theorem 8.2. There exists an* **isometry** *$J \colon L_2(\Sigma, F^1) \overset{\sim}{\to} V$ with inverse $J^{-1} = J^*$ (in the ψdbo calculus), such that*

$$\mathcal{T}_1 = J^*TJ$$

acts like an elliptic invertible ψdo $\mathcal{T}$ in F^1 of order $2m$, and $D(\mathcal{T}_1) = \{\varphi \in L_2(\Sigma, F^1) \mid \mathcal{T}\varphi \in L_2(\Sigma, F^1)\} = H^{2m}(\Sigma, F^1)$. Here $\mathcal{T}_1$ has the same spectrum as T, and its eigenvectors are mapped to the corresponding eigenvectors of $\mathcal{T}_1$ by the isometry J. Moreover,

$$A_B^{-1} - A_\gamma^{-1} = i_V J \mathcal{T}_1^{-1} J^* \operatorname{pr}_V, \tag{8.31}$$

whereby the positive resp. negative eigenvalues satisfy

$$\mu_j^\pm(A_B^{-1} - A_\gamma^{-1}) = \mu_j^\pm(\mathcal{T}_1^{-1}) \text{ for all } j.$$

It follows that with constants determined from the principal symbols,

$$N'^\pm(t; A_B^{-1} - A_\gamma^{-1}) = C^\pm t^{(n-1)/(2m)} + O(t^{(n-1)/(2m)}) \text{ for } t \to \infty,$$
$$\mu_j^\pm(A_B^{-1} - A_\gamma^{-1}) = (C^\pm)^{2m/(n-1)} j^{-2m/(n-1)} + O(j^{-(2m+1)/(n-1)}); \tag{8.32}$$

here $N'^\pm(t; S)$ indicates the number of positive, resp. negative eigenvalues of S outside the interval $\,]-1/t, 1/t[\,$.

The two statements in (8.32) are equivalent, cf. e.g. [Grubb, 1996], Lemma A.5. They follow from [Hörmander, 1968] for elliptic ψdo's, when the principal symbol eigenvalues of $\mathcal{T}$ are simple; this restriction is removed by results of [Ivrii, 1982]. See also Theorem 8.17 below.

We have recently checked that the proof extends to exterior domains, for uniformly strongly elliptic systems.

The fine estimates with remainders depend on the ellipticity of the ψdo. For simple Weyl-type estimates, the ellipticity hypothesis is removed in [Birman and Solomyak, 1977], showing that

$$s_j(P)j^{k/n} \to c(p^0) \text{ for } j \to \infty$$

holds for any classical ψdo P of order $-k < 0$ on a compact manifold of dimension n. They even allowed a certain nonsmoothness of the homogeneous principal symbol, both in x and ξ, needing only a little more than continuity. In the elliptic case, there are recent works of Ivrii dealing with remainder estimates under weak smoothness hypotheses.

In [Grubb, 1984a] an effort was made to increase the accessibility of the ψdbo calculus by publishing an introduction to it with several improvements, and showing as a main result that any singular Green operator of negative order and class 0 has a Weyl-type spectral asymptotics formula:

Theorem 8.15 *Let G be a classical singular Green operator of order $-k < 0$ and class 0 on an n-dimensional compact manifold with boundary. It has a spectral asymptotics behavior*

$$s_j(G)j^{k/(n-1)} \to c(g^0) \text{ for } j \to \infty, \tag{8.33}$$

where $c(g^0)$ is a constant defined from on the principal symbol of G.

This was moreover used to show asymptotic formulas generalizing (8.30), both for interior and exterior domains, followed up in another study [Grubb, 1984b] including also the dependence on a spectral parameter λ. Indeed, we have as an immediate corollary of Theorem 8.15, also for nonselfadjoint cases:

Corollary 8.16 *Let A be elliptic of order $2m$ with invertible Dirichlet realization and let A_B be an invertible realization defined by a normal elliptic boundary condition. For any positive integer N, $A_B^{-N} - A_\gamma^{-N}$ is a singular Green operator of order $-2mN$ and class 0, and hence satisfies*

$$s_j(A_B^{-N} - A_\gamma^{-N})j^{2mN/(n-1)} \to c_N \text{ for } j \to \infty, \tag{8.34}$$

for a constant c_N defined from the principal symbols.

Also exterior domains are considered in [Grubb, 1984a,b], where (8.34) is shown for realizations of second-order operators and their iterates. The results apply of course to resolvents by replacement of A by $A - \lambda$; the λ-dependence is studied in [Grubb, 1984b]. It is seen that the ψdbo theory provides a forceful tool for such questions, and we strongly recommend its use.

8.10 New spectral results

Spectral estimates of resolvent differences have been taken up in recent papers [Alpay and Behrndt, 2009; Behrndt, Langer, Lobanov, Lotore-ichik and Popov, 2010; Behrndt, Langer and Lotoreichik] for second-order operators and [Malamud, 2010] for $2m$-order operators with normal boundary conditions, based on boundary triples methods. Here Schatten class and weak Schatten class estimates are shown, relying on such estimates for Sobolev space embeddings.

We have returned to the subject in [Grubb, 2011a] where we, besides showing new results on perturbations of essential spectra, have reformulated and extended results in [Grubb, 1984a] on estimates like (8.34), including general differences and exterior domains. The central ingredient is the estimate (8.33) for singular Green operators, plus the fact that s.g.o.s give their essential contribution in a small neighborhood of the boundary, also for exterior domains, allowing cutoffs eliminating infinity.

An inspection of the results of [Grubb, 1974] shows that the spectral estimates in Theorem 8.14 can be further sharpened by use of results of [Ivrii, 1982]:

Theorem 8.17 *In the setting of Theorem* 8.14*, assume in addition that the principal symbol of $\mathcal{T}$ satifies Ivrii's conditions* $(\mathrm{H}_{\pm})$ *from* [Ivrii, 1982] *(the bicharateristics through points of $T^*(\Sigma) \setminus 0$ are nonperiodic except for a set of measure zero). Then there are constants $C_1^{\pm}$ such that*

$$N'^{\pm}(t; A_B^{-1} - A_\gamma^{-1}) = C^{\pm} t^{(n-1)/(2m)} + C_1^{\pm} t^{(n-2)/(2m)} + o(t^{(n-2)/(2m)}).$$

The proof is a direct application of [Ivrii, 1982] Th. 0.2 to $\mathcal{T}$.

The formula (8.31) is a special type of Kreĭn resolvent formula with isometries, valid for selfadjoint realizations, but the analysis in [Grubb, 1974] also implies Kreĭn formulas in the nonselfadjoint cases. Namely, Th. 6.4 there shows how $T \colon V \to W$ is represented by a realization $\mathcal{L}_1$ of a ψdo $\mathcal{L}$ acting between vector bundles over Σ, and here A_B is elliptic if and only if $\mathcal{L}$ is elliptic (Cor. 6.10). In the invertible elliptic case, formula (8.20) then takes the form

$$A_B^{-1} - A_\gamma^{-1} = \mathrm{i}_V \gamma_V^{-1} \Phi \mathcal{L}_1^{-1} \Psi^* (\gamma_W^*)^{-1} \mathrm{pr}_W,$$

with $\mathcal{L}_1$ acting like $-(B^{10} + B^{11} P_{\gamma,\chi})\Phi$ (notation explained in [Grubb, 1974]); the right-hand side is a composition of a Poisson operator, an elliptic ψdo and the adjoint of a Poisson operator, all of mixed order. Its s-numbers can be studied by reduction to an elliptic ψdo over Σ, where Ivrii's sharp results can be used.

Let us just demonstrate this for second-order operators, for the formula (8.22), with L^{-1} elliptic of order -1: Denote $\widetilde{A}^{-1} - A_\gamma^{-1} = S$. Then

$$
\begin{aligned}
s_j(S)^2 &= s_j(K_\gamma L^{-1}(K_\gamma')^*)^2 = \mu_j(K_\gamma L^{-1}(K_\gamma')^* K_\gamma'(L^{-1})^* K_\gamma^*) \\
&= \mu_j(L^{-1}(K_\gamma')^* K_\gamma'(L^{-1})^* K_\gamma^* K_\gamma),
\end{aligned}
$$

where we used the general rule $\mu_j(B_1 B_2) = \mu_j(B_2 B_1)$. Both operators $P_1 = K_\gamma^* K_\gamma$ and $P_1' = (K_\gamma')^* K_\gamma'$ are selfadjoint positive elliptic ψdo's of order -1 (cf. e.g. [Grubb, 2011b, proof of Theorem 4.4]). Let $P_2 = P_1^{\frac{1}{2}}$, then we continue the calculations as follows:

$$
s_j(S)^2 = \mu_j(L^{-1} P_1'(L^{-1})^* P_1) = \mu_j(P_2 L^{-1} P_1'(L^{-1})^* P_2) = \mu_j(P_3),
$$

where $P_3 = P_2 L^{-1} P_1'(L^{-1})^* P_2$ is a selfadjoint positive elliptic ψdo on Σ of order -4. Applying Ivrii's theorem to P_3^{-1}, we conclude:

Theorem 8.18 *For the operator considered in Theorem 8.9, the s-numbers satisfy*

$$
N'(t; \widetilde{A}^{-1} - A_\gamma^{-1}) = C t^{(n-1)/2} + O(t^{(n-2)/2}).
$$

Moreover, if the principal symbol of P_3^{-1} satifies Ivrii's condition from [Ivrii, 1982] *(the bicharateristics through points of $T^*(\Sigma) \setminus 0$ are non-periodic except for a set of measure zero), there is a constant C_1 such that*

$$
N'(t; \widetilde{A}^{-1} - A_\gamma^{-1}) = C t^{(n-1)/2} + C_1 t^{(n-2)/2} + o(t^{(n-2)/2}).
$$

Sharpened asymptotic formulas can also be obtained for differences between resolvents of two realizations that both differ from the Dirichlet realization, by use of the analysis in [Grubb, 1968] with a general invertible realization A_β as reference operator.

To give another example of applications of the ψdbo theory, the following result is found straightforwardly as a consequence of [Grubb, 1984a]:

Theorem 8.19 *Let A_B and $A_{B'}$ be elliptic invertible realizations of A such that B and B' map into the same bundles and have the* **same principal part**. *Then $A_B^{-1} - A_{B'}^{-1}$ is a singular Green operator of order $-2m - 1$ (since its principal part is zero), and hence, by (8.33),*

$$
s_j(A_B^{-1} - A_{B'}^{-1}) j^{(2m+1)/(n-1)} \to c \text{ for } j \to \infty.
$$

The singular Green operator will be of a still lower order $-2m - r$ if the first $r > 1$ terms in the symbols of B and B' coincide.

Example 8.20 As a special case, we can compare two Robin conditions for a second-order operator A:

$$\widetilde{A}_1 \text{ defined by } \nu_1 u = b_1 \gamma_0 u,$$
$$\widetilde{A}_2 \text{ defined by } \nu_1 u = b_2 \gamma_0 u;$$

$b_1, b_2 \in C^\infty(\Sigma)$. When regarded from the point of view of Theorem 8.19, these are normal boundary conditions $\nu_1 u - B_i \gamma_0 u = 0$, where $B_1 = b_1$ and $B_2 = b_2$ considered as first-order operators have principal part 0, so the boundary operators have the same principal part. Then the s.g.o. $\widetilde{A}_1^{-1} - \widetilde{A}_2^{-1}$ is of order -3, and by (8.33),

$$s_j(\widetilde{A}_1^{-1} - \widetilde{A}_2^{-1})j^{3/(n-1)} \to c \text{ for } j \to \infty. \tag{8.35}$$

In [Behrndt, Langer, Lobanov, Lotoreichik and Popov, 2010] upper estimates are shown for this difference in the case $A = -\Delta - \lambda$, namely Schatten class estimates of $s_j(\widetilde{A}_1^{-1} - \widetilde{A}_2^{-1})$ as in (8.29) with $(n-1)/2$ replaced by $(n-1)/3$.

In case the b_i are C^∞, the result is covered by (8.35) as explained above. However, the b_i in [Behrndt, Langer, Lobanov, Lotoreichik and Popov, 2010] are allowed to be nonsmooth, namely to be in $L_\infty(\Sigma)$, which goes outside the range covered by the smooth ψdbo theory.

This led us to investigate how far we could push the proof of asymptotic estimates (8.35) to make them valid for nonsmooth choices of b_i. The outcome is published in [Grubb, 2011b], where it is shown that (8.35) holds for symmetric second-order strongly elliptic operators on smooth domains, when b_1 and b_2 are piecewise C^ε on Σ, having jumps at a smooth hypersurface.

The Schatten class estimates have been followed up by [Behrndt, Langer and Lotoreichik] in a study of selfadjoint realizations; see also Chapter 6 in this volume.

Unsolved questions of asymptotic estimates lie primarily in the range of situations with limited smoothness. Resolvent formulas have been studied in such general cases, [Posilicano and Raimondi, 2009; Grubb, 2008] for $C^{1,1}$-domains, [Gesztesy and Mitrea, 2008, 2011] for Lipschitz and quasi-convex domains, [Abels, Grubb and Wood] for a class of domains containing $C^{3/2+\varepsilon}$, with a nonsmooth generalization of ψdbo's. To our knowledge, spectral asymptotic estimates have not yet been worked

out for such resolvent differences. Some upper estimates are in selfadjoint cases known from [Birman, 1962].

A problem with a different flavor is the case of a *mixed boundary condition*, such as prescribing for $-\Delta$ the Dirichlet condition on a part Σ_- of the boundary and a Neumann-type condition on the other part Σ_+. Here there is a jump in the *order* of the boundary condition. The domain of the realization is contained in $H^{\frac{3}{2}-\varepsilon}(\Omega)$ only for $\varepsilon > 0$, so the mixed problem is not covered by those boundary triples methods that require the domain to be in $H^{\frac{3}{2}}(\Omega)$. Spectral upper estimates are known from [Birman, 1962]. A spectral asymptotic estimate was obtained recently in [Grubb, 2011c], based on somewhat intricate applications of results on nonstandard pseudodifferential operators.

There are also other questions that can benefit from pseudodifferential methods, for example the study of spectral asymptotics of the nonelliptic Kreĭn-like extensions, cf. [Grubb, 2012].

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