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Regularity in L_p Sobolev spaces of solutions to fractional heat equations

Gerd Grubb

Department of Mathematical Sciences, Copenhagen University, Universitetsparken 5, DK-2100 Copenhagen, Denmark

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ABSTRACT

This work contributes in two areas, with sharp results, to the current investigation of regularity of solutions of heat equations with a nonlocal operator P:

$$Pu + \partial_t u = f(x,t), \text{ for } x \in \Omega \subset \mathbb{R}^n,\ t \in I \subset \mathbb{R}. \qquad (*)$$

1) For strongly elliptic pseudodifferential operators (ψdo's) P on $\mathbb{R}^n$ of order $d \in \mathbb{R}_+$, a symbol calculus on $\mathbb{R}^{n+1}$ is introduced that allows showing optimal regularity results, globally over $\mathbb{R}^{n+1}$ and locally over $\Omega \times I$:

$$f \in H^{(s,s/d)}_{p,\mathrm{loc}}(\Omega \times I) \implies u \in H^{(s+d,s/d+1)}_{p,\mathrm{loc}}(\Omega \times I),$$

for $s \in \mathbb{R}$, $1 < p < \infty$. The $H^{(s,s/d)}_p$ are anisotropic Sobolev spaces of Bessel-potential type, and there is a similar result for Besov spaces.
2) Let Ω be smooth bounded, and let P equal $(-\Delta)^a$ ($0 < a < 1$), or its generalizations to singular integral operators with regular kernels, generating stable Lévy processes. With the Dirichlet condition $\operatorname{supp} u \subset \overline{\Omega}$, the initial condition $u|_{t=0} = 0$, and $f \in L_p(\Omega \times I)$, (*) has a unique solution $u \in L_p(I; H^{a(2a)}_p(\overline{\Omega}))$ with $\partial_t u \in L_p(\Omega \times I)$. Here $H^{a(2a)}_p(\overline{\Omega}) = \dot H^{2a}_p(\overline{\Omega})$ if $a < 1/p$, and is contained in $\dot H^{2a-\varepsilon}_p(\overline{\Omega})$ if $a = 1/p$, but contains nontrivial elements from $d^a \overline{H}^a_p(\Omega)$ if $a > 1/p$

E-mail address: grubb@math.ku.dk.

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(where $d(x) = \operatorname{dist}(x, \partial\Omega)$). The interior regularity of u is lifted when f is more smooth.

0. Introduction

There is currently a great interest for evolution problems (heat equations)

$$Pu(x,t) + \partial_t u(x,t) = f(x,t) \text{ on } \Omega \times I,\ \Omega \text{ open } \subset \mathbb{R}^n,\ I =]0,T[\,, \tag{0.1}$$

where P is a nonlocal operator, as for example the fractional Laplacian $(-\Delta)^a$ $(0 < a < 1)$ or other pseudodifferential operators (ψdo's) or singular integral operators. For differential operators P, there are classical treatises such as Ladyzhenskaya, Solonnikov and Uraltseva [30] with L_p-methods, Lions and Magenes [33] with L_2-methods, Friedman [11] with L_2 semigroup methods, and numerous more recent studies. Motivated by the linearized Navier–Stokes problem, which can be reduced to the form (0.1) with nonlocal ingredients, the author jointly with Solonnikov treated such problems in [24] (for L_2-spaces) and [17] (for L_p-spaces). In those papers, the operator P fits into the Boutet de Monvel calculus [5,15,16,19], and is necessarily of integer order.

This does not cover fractional order operators, and the present paper aims to find techniques to handle (0.1) in fractional cases. Firstly, we treat ψdo's without boundary conditions in Sections 2 and 3, where we introduce a systematic calculus that allows showing regularity results globally in $\mathbb{R}^{n+1}$, and locally in arbitrary open subset $\Sigma \subset \mathbb{R}^{n+1}$, in terms of anisotropic function spaces described in detail in Appendix A:

Theorem 0.1. *Let P be a classical strongly elliptic ψdo $P = \mathrm{OP}(p(x,\xi))$ on $\mathbb{R}^n$ of order $d \in \mathbb{R}_+$. Let $s \in \mathbb{R}$, $1 < p < \infty$. Then $P + \partial_t$ maps $H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R}) \to H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$.*

1° *Let $u \in H_p^{(r,r/d)}(\mathbb{R}^n \times \mathbb{R})$ for some large negative r (this holds in particular if $u \in \mathcal{E}'(\mathbb{R}^{n+1})$ or e.g. $L_p(\mathbb{R}; \mathcal{E}'(\mathbb{R}^n))$). Then*

$$(P + \partial_t)u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) \implies u \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R}). \tag{0.2}$$

2° *Let Σ be an open subset of $\mathbb{R}^{n+1}$, and let $u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$. Then*

$$(P + \partial_t)u|_\Sigma \in H_{p,\mathrm{loc}}^{(s,s/d)}(\Sigma) \implies u \in H_{p,\mathrm{loc}}^{(s+d,s/d+1)}(\Sigma). \tag{0.3}$$

The analogous result holds in Besov-spaces $B_p^{(s,s/d)}$, and there is also a result in anisotropic Hölder spaces that can be derived from (0.3) by letting $p \to \infty$.

A celebrated example to which the above theorem applies is the fractional Laplacian

$$(-\Delta)^a u = \operatorname{Op}(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)), \text{ also defined by}$$
$$(-\Delta)^a u(x) = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2a}}\, dy, \tag{0.4}$$

which has interesting applications in probability, finance, mathematical physics and differential geometry. Along with (0.4), one considers more general translation-invariant singular integral operators

$$Pu(x) = PV \int_{\mathbb{R}^n} \frac{(u(x)-u(x+y))k(y/|y|)}{|y|^{n+2a}}\, dy, \tag{0.5}$$

with $k(y)$ even and positive on S^{n-1} (cf. e.g. the survey of Ros-Oton [37]); they are infinitesimal generators of stable Lévy processes. Further generalizations with nonhomogeneous or nonsmooth kernels are also studied. When $k \in C^\infty$, the operator (0.5) is a ψdo of order $2a$ with positive homogeneous symbol $p(\xi)$ even in ξ, and Theorem 0.1 applies with $d = 2a$.

We underline that the above regularity results apply not only to such operators, but also to x-dependent ψdo's, and to ψdo's with complex symbol, without special symmetries and with a different behavior at boundaries (no "transmission property"). (An example is the square-root Laplacian with drift $(-\Delta)^{\frac12} + b(x)\cdot\nabla$.)

For (0.5), regularity questions for solutions of (0.1) have been treated recently by Leonori, Peral, Primo and Soria [32] in $L_r(I; L_q(\Omega))$-spaces, by Fernandez-Real and Ros-Oton [10] in anisotropic Hölder spaces, and by Biccari, Warma and Zuazua [3] for $(-\Delta)^a$ in L_p-spaces valued in local Sobolev spaces over Ω. Earlier results are shown e.g. in Felsinger and Kassmann [9] and Chang-Lara and Davila [7] (Hölder properties), and Jin and Xiong [28] (Schauder estimates); the references in the mentioned works give further information, also on related heat kernel estimates.

The second aim of our paper is to obtain a global result in L_p Sobolev spaces for the heat equation (0.1) for $(-\Delta)^a$ or (0.5), with Dirichlet boundary condition on a bounded smooth domain Ω, giving a detailed description of the solution. By combining the characterization of the Dirichlet domain obtained in [21] with a semigroup theorem of Lamberton [31] put forward in [3], we show in Section 4:

Theorem 0.2. *Let $1 < p < \infty$. When $P = (-\Delta)^a$, or is an operator as in (0.5) with $k \in C^\infty$, and $\Omega \subset \mathbb{R}^n$ is bounded smooth, then the evolution problem*

$$\begin{aligned} Pu + \partial_t u &= f \text{ on } \Omega \times I, \\ u &= 0 \text{ on } (\mathbb{R}^n \setminus \Omega) \times I, \\ u &= 0 \text{ for } t = 0, \end{aligned} \tag{0.6}$$

has for any $f \in L_p(\Omega \times I)$ a unique solution $u(x,t) \in C^0(\overline{I}; L_p(\Omega))$, that satisfies:

$$u \in L_p(I; H_p^{a(2a)}(\overline{\Omega})) \cap H_p^1(I; L_p(\Omega)). \tag{0.7}$$

Here $H_p^{a(2a)}(\overline{\Omega})$ is the domain of the Dirichlet L_p-realization $P_{\mathrm{Dir},p}$ of P and equals $\dot{H}_p^{2a}(\overline{\Omega}) + V$, where $V = 0$ if $a < 1/p$, $V \subset \dot{H}_p^{2a-\varepsilon}(\overline{\Omega})$ if $a = 1/p$, $V \subset d^a \overline{H}_p^a(\Omega)$ if $a > 1/p$ (here $d(x) = \mathrm{dist}(x, \partial\Omega)$).

For the time-dependent problem, this precision is new.

An application of Theorem 0.1 2° gives moreover:

Corollary 0.3. *Let u be as in Theorem 0.2, and let $r = 2a$ if $a < 1/p$, $r = a + 1/p - \varepsilon$ if $a \geq 1/p$ (for some small $\varepsilon > 0$). Then for $0 < s \leq r$,*

$$f \in H_{p,\mathrm{loc}}^{(s,s/(2a))}(\Omega \times I) \implies u \in H_{p,\mathrm{loc}}^{(s+2a,s/(2a)+1)}(\Omega \times I). \tag{0.8}$$

For larger s, the local regularity (0.8) can be obtained via Theorem 0.1 2° if one knows on beforehand that $u \in H_p^{(s,s/(2a))}(\mathbb{R}^n \times I)$.

Plan of the paper Section 1 gives some definitions and prerequisites. The definitions and properties of anisotropic Sobolev spaces (of Bessel-potential and Besov type) are collected in Appendix A. In Section 2 we show how an anisotropic symbol calculus can be introduced, that covers the operators $P + \partial_t$ with a pseudodifferential P of order $d \in \mathbb{R}_+$. Section 3 gives the proofs of the global and local regularity stated in Theorem 0.1. In Section 4 we start by introducing some further prerequisites needed for the global results on a bounded smooth subset, and then give the proof of Theorem 0.2.

1. Preliminaries

We shall use the notation set up in [21], also used in [20,22], and will just list some important points here.

The function $\langle\xi\rangle$ equals $(|\xi|^2 + 1)^{\frac{1}{2}}$. The Fourier transform $\mathcal{F}$ is defined by $\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx$; it maps the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing C^∞-functions into itself, and extends by duality to the temperate distributions $\mathcal{S}'(\mathbb{R}^n)$.

A *pseudodifferential operator* (ψdo) P on $\mathbb{R}^n$ is defined from a symbol $p(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$Pu = \mathrm{OP}(p(x,\xi))u = (2\pi)^{-n} \int e^{ix\cdot\xi} p(x,\xi) \hat{u}\, d\xi = \mathcal{F}^{-1}_{\xi\to x}(p(x,\xi)\mathcal{F}u(\xi)), \tag{1.1}$$

using the Fourier transform $\mathcal{F}$. We refer to textbooks such as Hörmander [26], Taylor [45], Grubb [19] for the rules of calculus (in particular the definition by oscillatory integrals in [26]). The symbols p of order $m \in \mathbb{R}$ we shall use are generally taken to lie in the symbol space $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, consisting of C^∞-functions $p(x,\xi)$ such that $\partial_x^\beta \partial_\xi^\alpha p(x,\xi)$ is $O(\langle\xi\rangle^{m-|\alpha|})$ for all α, β, for some $m \in \mathbb{R}$, with global estimates for $x \in \mathbb{R}^n$ (as in [26] start of Sect. 18.1, and [18]). P (of order m) then maps $H^s_p(\mathbb{R}^n)$ continuously into $H^{s-m}_p(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ (cf. (1.4)). P is said to be *classical* when p moreover has an asymptotic expansion $p(x,\xi) \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi)$ with p_j homogeneous in ξ of degree $m-j$ for $|\xi| \geq 1$, all j, and

$$p(x,\xi) - \sum\nolimits_{j<J} p_j(x,\xi) \in S^{m-J}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n), \text{ for all } J. \tag{1.2}$$

P is then said to be (uniformly) *elliptic* when $|p_0(x,\xi)| \geq c|\xi|^m$ for $|\xi| \geq 1$, with $c > 0$. To these operators one can add the smoothing operators (mapping any $H^s_p(\mathbb{R}^n)$ into $\bigcap_t H^t_p(\mathbb{R}^n)$), regarded as operators of order $-\infty$. $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ will also be written $S^m_{1,0}(\mathbb{R}^{2n})$ for short.

Formula (1.1) will also be used in some cases of more general functions p for which the definition can be given a sense, for example in case of the symbol $(\langle\xi'\rangle + i\xi_n)^t$ in (4.2), the anisotropic symbols in Definition 2.1, the symbol $|\xi|^a$ in Example 2.10.

Recall the composition rule: When $PQ = R$, then R has a symbol $r(x,\xi)$ with the following asymptotic expansion, called the Leibniz product:

$$r(x,\xi) \sim p(x,\xi)\#q(x,\xi) = \sum_{\alpha\in\mathbb{N}_0^n} D_\xi^\alpha p(x,\xi)\partial_x^\alpha q(x,\xi)/\alpha!. \tag{1.3}$$

We shall also define ψdo's on $\mathbb{R}^{n+1}$ with variables denoted (x,t), the dual variables denoted (ξ,τ). The symbols $h(x,t,\xi,\tau)$ may satisfy other types of estimates with respect to (ξ,τ) than the $S^m_{1,0}$ estimates mentioned above. To distinguish between operators on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, we may write $\mathrm{OP}_x(p(x,\xi))$ resp. $\mathrm{OP}_{x,t}(h(x,t,\xi,\tau))$.

The standard Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$ and $s \geq 0$, have a different character according to whether s is integer or not. Namely, for s integer, they consist of L_p-functions with derivatives in L_p up to order s, hence coincide with the Bessel-potential spaces $H^s_p(\mathbb{R}^n)$, defined for $s \in \mathbb{R}$ by

$$H^s_p(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_p(\mathbb{R}^n)\}. \tag{1.4}$$

For noninteger s, the $W^{s,p}$-spaces coincide with the Besov spaces, defined e.g. as follows: For $0 < s < 2$,

$$f \in B^s_p(\mathbb{R}^n) \iff \|f\|^p_{L_p} + \int_{\mathbb{R}^{2n}} \frac{|f(x) + f(y) - 2f((x+y)/2)|^p}{|x+y|^{n+ps}}\, dxdy < \infty; \tag{1.5}$$

and $B^{s+t}_p(\mathbb{R}^n) = (1-\Delta)^{-t/2} B^s_p(\mathbb{R}^n)$ for all $t \in \mathbb{R}$. The Bessel-potential spaces are important because they are most directly related to $L_p(\mathbb{R}^n)$; the Besov spaces have other convenient properties, and are needed for boundary value problems in an H^s_p-context, because they are the correct range spaces for trace maps (both from H_p^s and B_p^s-spaces); see e.g. the overview in the introduction to [16]. For $p = 2$, the two scales are identical, but for $p \neq 2$ they are related by strict inclusions:

$$H_p^s \subset B_p^s \text{ when } p > 2, \quad H_p^s \supset B_p^s \text{ when } p < 2. \tag{1.6}$$

When working with operators of possibly noninteger order, it is much preferable to use the different notations for the two scales, rather than formulating results in the $W^{s,p}$-scale, where the definition changes when s changes between integer and noninteger values, so that mapping properties risk not being optimal.

For any open subset Ω of $\mathbb{R}^n$, one can define the local variants:

$$\begin{aligned} H^s_{p,\mathrm{loc}}(\Omega) &= \{u \in \mathcal{D}'(\Omega) \mid \psi u \in H_p^s(\mathbb{R}^n) \text{ for all } \psi \in C_0^\infty(\Omega)\}, \\ H^s_{p,\mathrm{comp}}(\Omega) &= \{u \in H_p^s(\mathbb{R}^n) \mid \operatorname{supp} u \text{ compact } \subset \Omega\}, \end{aligned} \tag{1.7}$$

and similar spaces with B.

In Appendix A we list anisotropic variants of the Bessel-potential and Besov scales with weights $(d, 1)$, and explain their relation to Sobolev scales; this can also be read as a supplementing information on the isotropic case where $d = 1$.

Further notation from [21] is recalled in the start of Section 4 below where it is needed.

2. Anisotropic symbols

When P is a pseudodifferential operator on $\mathbb{R}^n$ of order $d \in \mathbb{R}$, it is a well-known fact that if P is elliptic, the solutions u to the equation $Pu(x) = g(x)$ on an open subset Ω are d values more regular than g, e.g., $g \in H^s_{p,\mathrm{loc}}(\Omega)$ implies $u \in H^{s+d}_{p,\mathrm{loc}}(\Omega)$ for $s \in \mathbb{R}$. (Cf. e.g. Seeley [42], Kohn and Nirenberg [29], Hörmander [25,26] and the exposition in Taylor [45]). It was one of the purposes of setting up the rules of symbol calculus to have easy access to such regularity results.

For the parabolic (heat operator) problem $Pu(x,t) + \partial_t u(x,t) = f(x,t)$ on $\mathbb{R}^{n+1}$ it is not quite as well-known what there holds of regularity. In the differential operator case, when P is strongly elliptic, then $P + \partial_t$ and its solution operator belong to a natural class of anisotropic ψdo's on $\mathbb{R}^{n+1}$ where there are straightforward results. But if P is truly pseudodifferential, the symbol of $P + \partial_t$ does not satisfy all the estimates required in a standard ψdo calculus on $\mathbb{R}^{n+1}$, but something weaker (see the discussion in [15,18] in Remark 1.5.1ff and at the end of Section 4.1, with references). The operators were analyzed briefly in an L_2-framework in [15,18], Sect. 4.2 (which focused on kernel estimates). The mapping properties were extended to L_p-based spaces $H_p^{(s,s/d)}$ and $B_p^{(s,s/d)}$ in [17] (cf. Th. 3.1(1) there), for operators P of integer order d, in connection with a study of boundary value problems in the Boutet de Monvel framework. This depended on a symbol calculus carried over from the calculus developed in the book [15,18], and relied on L_p-boundedness theorems of Lizorkin [35] and Yamazaki [47].

In the present paper we are interested in heat problems with operators P of primarily noninteger order (such as $(-\Delta)^a$), not covered by [17]. There is the question of mapping properties of $P+\partial_t$, and of the existence of (approximate) solution operators under suitable parabolicity hypotheses, that allow showing regularity of solutions. In addition there is the question of *local regularity*, often shown by use of commutations with cut-off functions. All this can be handled by setting up a systematic calculus, including composition rules. Let us now present the appropriate symbols and estimates.

In the following, $d \in \mathbb{R}_+$ is fixed. The basic anisotropic invertible symbol in the calculus is $\{\xi,\tau\}$, with definition

$$\{\xi,\tau\} \equiv (\langle\xi\rangle^{2d} + \tau^2)^{1/(2d)}, \tag{2.1}$$

leading to the "order-reducing" operators

$$\Theta^s u = \mathrm{OP}(\{\xi,\tau\}^s)u \equiv \mathcal{F}^{-1}_{(\xi,\tau)\to(x,t)}(\{\xi,\tau\}^s \mathcal{F}_{(x,t)\to(\xi,\tau)}u), \tag{2.2}$$

for all $s \in \mathbb{R}$. Then the anisotropic Bessel-potential spaces can be defined by

$$H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) = \Theta^{-s} L_p(\mathbb{R}^{n+1}); \tag{2.3}$$

see more about such spaces and the related Besov family $B_p^{(s,s/d)}$ below in Appendix A.

Definition 2.1. Let m and $\nu \in \mathbb{R}$. The space $S_{1,0}^{m,\nu}(\mathbb{R}^{2n+1})$ of d-anisotropic (or just anisotropic) symbols of order m and with **regularity number** ν consists of the C^∞-functions $h(x,\xi,\tau)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ satisfying the estimates

$$|\partial_x^\beta \partial_\xi^\alpha \partial_\tau^j h(x,\xi,\tau)| \le C_{\alpha,\beta,j}(\langle\xi\rangle^{\nu-|\alpha|} + \{\xi,\tau\}^{\nu-|\alpha|})\{\xi,\tau\}^{m-\nu-dj}, \tag{2.4}$$

for all indices $\alpha, \beta \in \mathbb{N}_0^n$, $j \in \mathbb{N}_0$.

In particular, in case $m=0$ and $\nu \ge 0$, the symbols satisfy

$$|\partial_x^\beta \partial_\xi^\alpha \partial_\tau^j h(x,\xi,\tau)| \le C_{\alpha,\beta,j}\langle\xi\rangle^{-|\alpha|}\{\xi,\tau\}^{-dj} \le C_{\alpha,\beta,j}\langle\xi\rangle^{-|\alpha|}\langle\tau\rangle^{-j}. \tag{2.5}$$

These symbol classes are very similar to those introduced in [15,18], Section 2.1. The difference is that $\langle\xi,\mu\rangle$ there has been replaced by the anisotropic $\{\xi,\tau\}$ here, and that the rule for differentiation in τ is that it lowers the order by dj instead of j (still without changing the regularity number). Therefore the symbols allow a very similar calculus.

Let us first show that some symbols of interest lie in these classes.

Lemma 2.2. 1° *For $s \in \mathbb{R}$, the estimates (2.4) are satisfied by $\{\xi,\tau\}^s$ with $\nu = 2d$, $m = s$.*

2° *Let $p(x,\xi)$ be a ψdo symbol in $S_{1,0}^m(\mathbb{R}^{2n})$ for some $m \in \mathbb{R}$. Then, considered as a symbol on $\mathbb{R}^{2n+1}$, constant in τ, it belongs to $S_{1,0}^{m,m}(\mathbb{R}^{2n+1})$.*

3° Let $p(x,\xi)$ be a ψdo symbol in $S^d_{1,0}(\mathbb{R}^{2n})$. Then $h(x,\xi,\tau)=p(x,\xi)+i\tau$ satisfies the estimates (2.4) with $\nu=d$, $m=d$, i.e., belongs to $S^{d,d}_{1,0}(\mathbb{R}^{2n+1})$.

Moreover, if $|p(x,\xi)+i\tau|\ge c(\langle\xi\rangle^d+|\tau|)$ with $c>0$, then $(p(x,\xi)+i\tau)^{-1}$ satisfies the estimates with $\nu=d$, $m=-d$.

Proof. 1°. Denote for short

$$\langle\xi\rangle=\sigma,\quad \{\xi,\tau\}=\kappa, \tag{2.6}$$

and observe that

$$(\sigma^{\nu-|\alpha|}+\kappa^{\nu-|\alpha|})\kappa^{m-\nu}\simeq\begin{cases}\kappa^{m-|\alpha|} & \text{if } \nu\ge 0,\\ \sigma^{\nu-|\alpha|}\kappa^{m-\nu} & \text{if } \nu\le 0.\end{cases} \tag{2.7}$$

For $\{\xi,\tau\}=(\langle\xi\rangle^{2d}+\tau^2)^{1/2d}$ itself, we have that

$$\partial_{\xi_j}(\langle\xi\rangle^{2d}+\tau^2)^{1/2d}=\tfrac{1}{2d}(\langle\xi\rangle^{2d}+\tau^2)^{(1/2d)-1}2d\langle\xi\rangle^{2d-1}\partial_{\xi_j}\langle\xi\rangle=\kappa^{1-2d}\sigma^{2d-1}\partial_{\xi_j}\langle\xi\rangle,$$
$$\partial_\tau(\langle\xi\rangle^{2d}+\tau^2)^{1/2d}=\tfrac{1}{2d}(\langle\xi\rangle^{2d}+\tau^2)^{(1/2d)-1}2\tau=c\kappa^{1-2d}\tau,$$

which satisfy the estimates with $m=1$ and $\nu=2d$ (note that $\partial_{\xi_j}\langle\xi\rangle$ is bounded and $\langle\tau\rangle\le\kappa$). Further differentiations give linear combinations of such expressions, where ∂_ξ^α produces a factor $\sigma^{2d-|\alpha|}$ in at least one of the terms, whereas ∂_τ^j only gives factors comparable with powers of κ. The weakest term in each expression is the one with the lowest power of σ; for $\partial_\xi^\alpha\partial_\tau^j\kappa$ the power is $2d-|\alpha|$, so (2.4) holds with $m=1$, $\nu=2d$.

For κ^s we use that its derivatives are linear combinations of products of κ^{s-k} ($k\in\mathbb{N}_0$) with expressions as above.

2°. We have for all α:

$$|\partial_\xi^\alpha p|\le C\langle\xi\rangle^{m-|\alpha|}\le C(\sigma^{m-|\alpha|}+\kappa^{m-|\alpha|})\kappa^{m-m}, \tag{2.8}$$

showing the asserted estimates.

3°. For $p(x,\xi)+i\tau$ we have that clearly $|p+i\tau|\le C\kappa^d$. Moreover, $\partial_\xi^\alpha(p+i\tau)$ satisfies the estimates (2.8) with $m=d$ for $|\alpha|>0$, and $\partial_\tau(p+i\tau)=i$, the higher derivatives being 0. This shows the first assertion.

For the second assertion, the given hypothesis shows that $|(p(x,\xi)+i\tau)^{-1}|\le C\kappa^{-d}$; then since the derivatives produce negative integer powers $(p(x,\xi)+i\tau)^{-k}$ times derivatives of $p(x,\xi)+i\tau$, the assertion is seen using the first estimates. □

We have as in [15,18], Prop. 2.1.5:

Lemma 2.3. *The product of two symbols of order and regularity m, ν resp. m',ν' is of order $m''=m+m'$ and regularity $\nu''=\min\{\nu,\nu',\nu+\nu'\}$.*

Likewise, when $h(x,\xi,\tau) \in S^{m,\nu}_{1,0}(\mathbb{R}^{2n+1})$ and $h'(x,\xi,\tau) \in S^{m',\nu'}_{1,0}(\mathbb{R}^{2n+1})$, then the Leibniz product

$$h''(x,\xi,\tau) \sim \sum\nolimits_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!} D^\alpha_\xi h(x,\xi,\tau)\partial^\alpha_x h'(x,\xi,\tau) \tag{2.9}$$

belongs to $S^{m'',\nu''}_{1,0}(\mathbb{R}^{2n+1})$.

Proof. The first assertion is seen by use of the elementary fact that

$$((\sigma/\kappa)^\nu + 1)((\sigma/\kappa)^{\nu'} + 1) \le 3(\sigma/\kappa)^{\nu''} + 1. \tag{2.10}$$

(Details in the proof of [15,18], Prop. 2.1.6.) The second assertion now follows by application of the first assertion to each term in the asymptotic series. □

Here the Leibniz product represents the symbol of the composition of $\mathrm{OP}(h)$ and $\mathrm{OP}(h')$, as in [15,18], (2.1.56). Note that since the symbols are constant in t, there are only terms with x,ξ-derivatives.

Remark 2.4. The proofs in [18] are formulated in the framework of globally estimated ψdo's of Hörmander [26], Sect. 18.1 (estimates in $x \in \mathbb{R}^n$), whereas the proofs in [15] are based on a more pedestrian local ψdo calculus. The global calculus has the advantage that remainders of order $-\infty$ are treated in a simpler way, and the Leibniz product has a precise meaning. We will take advantage of this fact in the following, and refer to [26] and [23,18] for more detailed explanations.

Remark 2.5. A classical ψdo P of order $d > 0$ is said to be *strongly elliptic*, when the principal symbol $p_0(x,\xi)$ takes values in a sector $V_\delta = \{z \in \mathbb{C} \mid |\arg z| \le \frac{\pi}{2} - \delta\}$ for $|\xi| \ge 1$, some $0 < \delta < \frac{\pi}{2}$; equivalently, $\operatorname{Re} p_0(x,\xi) \ge c_0|\xi|^d$ for $|\xi| \ge 1$, with $c_0 > 0$. (Recall that we take p_0 to be homogeneous in ξ for $|\xi| \ge 1$, and C^∞.) It is not hard to choose the extension of the homogeneous function into $|\xi| \le 1$ to keep satisfying $p_0(x,\xi) \in V_\delta$, with $\min_{|\xi|\le 1} \operatorname{Re} p_0(x,\xi) > 0$. Then $p_0(x,\xi) + i\tau$ satisfies $|p_0(x,\xi) + i\tau| \ge c(\langle\xi\rangle^d + |\tau|)$ with $c > 0$. The operators $P + \partial_t$ and symbols $p + i\tau$ are called *parabolic* in this case. See also the discussion in [15,18], Definition 1.5.3 ff.

In the parabolic case there is a parametrix symbol (symbol of an approximate inverse):

Lemma 2.6. *Let $h(x,\xi,\tau) = p(x,\xi) + i\tau$, where $p(x,\xi)$ is a classical strongly elliptic ψdo symbol of order $d \in \mathbb{R}_+$ on $\mathbb{R}^n$. Then there is a parametrix symbol $k(x,\xi,\tau)$ such that $(p + i\tau)\#k(x,\xi,\tau) - 1$ and $k(x,\xi,\tau)\#(p + i\tau) - 1$ are in $\bigcap_{k\in\mathbb{N}_0} S^{-k,d-k}_{1,0}(\mathbb{R}^{2n+1})$ (i.e., they are 0 modulo regularity d).*

Proof. As noted in Remark 2.5, p has a principal symbol p_0 that is nonvanishing and takes values in a closed subsector of $\{\operatorname{Re} z > 0\}$, whereby $p_0+i\tau$ and its inverse $(p_0+i\tau)^{-1}$

are as in Lemma 2.2 3°. Here $k_0 = (p_0 + i\tau)^{-1}$ is the principal term in a parametrix symbol k for h. The construction of the remaining terms in a full parametrix symbol is a standard construction similar to the proof of [15,18], Th. 2.1.22. □

Now we shall account for the mapping properties of such operators between anisotropic Bessel-potential spaces (2.3). For isotropic standard ψdo's this relies on Mihlin's multiplier theorem, but for the present operators we need instead a Lizorkin-type multiplier theorem allowing separate estimates in different groups of coordinates; here we shall use the criterion of Yamazaki [47] (see the account of the various criteria in [23], Sect. 1.3):

Lemma 2.7 *([47]). Let $n' \in \mathbb{N}$. When $a(y,\eta)$ on $\mathbb{R}^{n'} \times \mathbb{R}^{n'}$ satisfies*

$$|\partial_y^\beta \eta_j^{\alpha_j} \partial_{\eta_j}^{\alpha_j} a(y,\eta)| \le C_{\beta,j}, \quad j = 1, \dots, n',\ \alpha_j \le n'+1,\ |\beta| \le 1, \tag{2.11}$$

then $\mathrm{OP}(a)$ *is bounded in* $L_p(\mathbb{R}^{n'})$ *for* $1 < p < \infty$.

It is a space-dependent variant of Lizorkin's criterion. It will be used with $n' = n+1$.

Theorem 2.8. *Let $h(x,\xi,\tau) \in S^{m,\nu}_{1,0}(\mathbb{R}^{2n+1})$ with $\nu \ge 0$. Then $H = \mathrm{OP}(h(x,\xi,\tau))$ is continuous:*

$$\mathrm{OP}(h(x,\xi,\tau))\colon H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) \to H_p^{(s-m,(s-m)/d)}(\mathbb{R}^n \times \mathbb{R}),\ \textit{for all } s \in \mathbb{R}. \tag{2.12}$$

Proof. By Lemma 2.3,

$$\{\xi,\tau\}^{s-m} \# h(x,\xi,\tau) \# \{\xi,\tau\}^{-s} \in S^{0,\min\{\nu,2d\}}_{1,0}(\mathbb{R}^{2n+1}). \tag{2.13}$$

In view of (2.5), it satisfies (2.11) with $n' = n+1$, $y = (x,t)$ and $\eta = (\xi,\tau)$, hence

$$H_1 = \Theta^{s-m} H \Theta^{-s}$$

is bounded in $L_p(\mathbb{R}^{n+1})$. By the definition of the spaces in (2.3), it follows that (2.12) holds. □

The proof is of course simpler for x-independent operators, where (2.13) is just a product.

As a corollary, we have:

Corollary 2.9. *Under the hypotheses in Theorem 2.8, H also maps*

$$\mathrm{OP}(h(x,\xi,\tau))\colon B_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) \to B_p^{(s-m,(s-m)/d)}(\mathbb{R}^n \times \mathbb{R}),\ \textit{for all } s \in \mathbb{R}. \tag{2.14}$$

Proof. This follows from (2.12) by use of real interpolation as in the third line of (A.7). □

Example 2.10. Consider the operator $H_0 = (-\Delta)^a + \partial_t = \mathrm{OP}(|\xi|^{2a} + i\tau)$. Introducing the smooth positive modification $[\xi]$ of $|\xi|$:

$$[\xi] \text{ is } C^\infty \text{ and } \geq 1/2 \text{ on } \mathbb{R}^n,\ [\xi] = |\xi| \text{ for } |\xi| \geq 1, \tag{2.15}$$

and setting $r(\xi) = |\xi|^{2a} - [\xi]^{2a}$ (supported for $|\xi| \leq 1$) we can write H_0 as

$$H_0 = H_0' + \mathcal{R}, \quad H_0' = \mathrm{OP}([\xi]^{2a} + i\tau),\ \mathcal{R} = \mathrm{OP}(r(\xi)). \tag{2.16}$$

Here H_0' satisfies 3° of Lemma 2.2 with $d = 2a$, and $\mathcal{R}$ is smoothing in $\mathbb{R}^n$, since its symbol is $O(\langle\xi\rangle^{-N})$ for all N. Then

$$H_0'\colon H_p^{(s,s/(2a))}(\mathbb{R}^n \times \mathbb{R}) \to H_p^{(s-2a,s/(2a)-1)}(\mathbb{R}^n \times \mathbb{R}),\ \text{for all } s \in \mathbb{R}, \tag{2.17}$$

by Theorem 2.8. For $\mathcal{R}$, we note that

$$|\{\xi,\tau\}^{s-2a} r(\xi)\hat{u}(\xi,\tau)| \leq C|\{\xi,\tau\}^s \hat{u}(\xi,\tau)|,$$

since $2a > 0$, so $\mathcal{R}$ also has the continuity in (2.17) for all s. It follows that

$$H_0\colon H_p^{(s,s/(2a))}(\mathbb{R}^n \times \mathbb{R}) \to H_p^{(s-2a,s/(2a)-1)}(\mathbb{R}^n \times \mathbb{R}),\ \text{for all } s \in \mathbb{R}. \tag{2.18}$$

There is the parametrix $K_0' = \mathrm{OP}(([\xi]^{2a} + i\tau)^{-1})$; it clearly maps continuously in the opposite direction of (2.18).

The mapping properties also hold with H-spaces replaced by B-spaces throughout.

A similar proof works for symbols $p(\xi)$ that are positive for $\xi \neq 0$ and homogeneous of degree $2a$.

Example 2.11. Here are some other simple examples, to which the theory applies: $P = (-\Delta + b(x) \cdot \nabla + c(x))^a$ of order $2a$ (fractional powers of a perturbed Laplacian), and $P = (-\Delta)^{\frac{1}{2}} + b(x) \cdot \nabla$ of order 1 (the square-root Laplacian with drift). The coefficients $b(x) = (b_1(x), \dots, b_n(x))$ and $c(x)$ are taken smooth, real and bounded with bounded derivatives. The symbol $|\xi| + ib(x) \cdot \xi$ is complex, with real part $|\xi|$, hence strongly elliptic, and $|[\xi] + ib(x) \cdot \xi + i\tau| \geq c(\langle\xi\rangle + |\tau|)$ with $c > 0$. Another example is $P = -\Delta + (-\Delta)^{\frac{1}{2}}$; here $d = 2$.

Remark 2.12. There is an important work of Yamazaki [48] dealing with quasi-homogeneous ψdo's (and para-differential generalizations), acting in associated quasi-homogeneous variants of the Triebel–Lizorkin spaces $F^s_{p,q}$ and Besov spaces $B^s_{p,q}$ with $0 < p, q \leq \infty$; the spaces are defined by refined techniques involving dyadic decompositions in ξ-space (see also Triebel [46] for accounts of function spaces, and Schmeisser and Triebel [41] for anisotropic variants). However, it seems that the ψdo's in [48] do not include our cases, since the symbol spaces require that high derivatives of the symbols are dominated by $\langle(\xi,\tau)\rangle^{-N}$ where any N is reached, in contrast to our estimates (2.4). (Cf. [48], Definition, pp. 157–158.)

Much of what is said above could also be carried through in cases where P is allowed to depend moreover on t (allowing a treatment of t-dependent heat equations); in particular, Lemma 2.6 could easily be generalized. We have kept P t-independent here in order to draw directly on the proofs in [18], and leave the t-dependent case for future investigations.

3. Parabolic regularity

Now we can show regularity and local regularity of solutions to parabolic heat equations.

Theorem 3.1. *Consider a classical strongly elliptic ψdo $P = \mathrm{OP}(p(x,\xi))$ of order $d \in \mathbb{R}_+$ on $\mathbb{R}^n$. Let $s \in \mathbb{R}$.*

1° *If $u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$ satisfies*

$$(P + \partial_t)u = f \text{ with } f \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}), \tag{3.1}$$

then

$$u \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R}). \tag{3.2}$$

2° *The implication from (3.1) to (3.2) also holds if u merely satisfies $u \in H_p^{(r,r/d)}(\mathbb{R}^n \times \mathbb{R})$ for some large negative r (this holds in particular if $u \in \mathcal{E}'(\mathbb{R}^{n+1})$ or e.g. $u \in L_p(\mathbb{R}; \mathcal{E}'(\mathbb{R}^n))$).*

Similar statements hold with H_p replaced by B_p throughout.

Proof. Let $k(x,\xi,\tau)$ be a parametrix symbol according to Lemma 2.6. Then $H = P + \partial_t$ and $K = \mathrm{OP}(k(x,\xi,\tau))$ satisfy

$$KH = I + \mathcal{R}_1, \text{ where } \mathcal{R}_1 = \mathrm{OP}(r_1),\ r_1(x,\xi,\tau) \in \bigcap\nolimits_k S_{1,0}^{-k,d-k}(\mathbb{R}^{2n+1}).$$

With the notation (2.6), we have that $r_1(x,\xi,\tau)$ satisfies the estimates

$$\begin{aligned}|\partial_\xi^\alpha \partial_\tau^j r_1| &\le C(\sigma^{d-k-|\alpha|} + \kappa^{d-k-|\alpha|})\kappa^{-k-(d-k)-dj} = C(\sigma^{d-k-|\alpha|}\kappa^{-d-dj} + \kappa^{-k-|\alpha|-d-dj}) \\ &\le C'\sigma^{d-k-|\alpha|}\kappa^{-d-dj} \le C'\sigma^{-|\alpha|}\kappa^{-d-dj} \text{ when } k \ge d.\end{aligned}$$

Hence $r_1 \in S_{1,0}^{-d,0}(\mathbb{R}^{2n+1})$ (the lowest order we can assign with a nonnegative regularity number).

Now if u and f are given as in 1°, then

$$KHu = u + \mathcal{R}_1 u,$$

where $KHu = Kf \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R})$ since K is of order $-d$ with regularity number d, and $\mathcal{R}_1 u \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R})$ since $\mathcal{R}_1$ is of order $-d$ with regularity number 0, cf. Theorem 2.8. It follows that $u \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R})$. This shows 1°.

2°. Recall that a distribution in $\mathcal{E}'(\mathbb{R}^{n+1})$, i.e., with compact support in $\mathbb{R}^{n+1}$, is of finite order, in particular lies in $H_p^{-M}(\mathbb{R}^{n+1})$ for some $M \geq 0$. A distribution in $\mathcal{E}'(\mathbb{R}^n)$ lies in $H_p^{-M}(\mathbb{R}^n)$ for some $M \geq 0$. Note moreover that

$$H_p^{-M}(\mathbb{R}^{n+1}) \subset H_p^{(r,r/d)}(\mathbb{R}^n \times \mathbb{R}),$$

with $r = -M$ if $d \leq 1$, $r = -M/d$ if $d \geq 1$. Also, $L_p(\mathbb{R}; H_p^{-M}(\mathbb{R}^n)) \subset H_p^{(-M,-M/d)}(\mathbb{R}^n \times \mathbb{R})$. So we can assume $u \in H_p^{(r,r/d)}(\mathbb{R}^n \times \mathbb{R})$.

We use a bootstrap method, iterating applications of 1°, as follows: If $r \geq s$, the statement is covered by 1°. If $r < s$, we observe that a fortiori $f \in H_p^{(r,r/d)}(\mathbb{R}^n \times \mathbb{R})$. An application of 1° then gives the conclusion $u \in H_p^{(r+d,r/d+1)}(\mathbb{R}^n \times \mathbb{R})$. Here if $r_1 = r+d \geq s$, we need only apply 1° to reach the desired conclusion. If $r_1 < s$, we repeat the argument, concluding that $u \in H_p^{(r_2,r_2/d)}(\mathbb{R}^n \times \mathbb{R})$ for $r_2 = r + 2d$. The argument is repeated until $r_k = r + kd \geq s$.

The proofs in the scale of B_p-spaces follow by replacing H_p by B_p throughout. □

The conclusion is best possible, in view of the forward mapping properties in Theorem 2.8 and Corollary 2.9.

We can also show a local regularity result.

Theorem 3.2. *Let P be as in Theorem 3.1. Let $s \in \mathbb{R}$, and let Σ be an open subset of $\mathbb{R}^{n+1}$. If $u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$ satisfies*

$$(P + \partial_t)u|_\Sigma \in H_{p,\mathrm{loc}}^{(s,s/d)}(\Sigma), \tag{3.3}$$

then $u|_\Sigma \in H_{p,\mathrm{loc}}^{(s+d,s/d+1)}(\Sigma)$.

Proof. Notation as in Theorem 3.1 will be used. We have to show that for any $(x_0, t_0) \in \Sigma$, there is a function $\psi \in C_0^\infty(\Sigma)$ that is 1 on a neighborhood of (x_0, t_0) such that $\psi u \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R})$.

Let $(x_0, t_0) \in \Sigma$ and let $B_j^n = \{x \in \mathbb{R}^n \mid |x - x_0| < r/j\}$ and $B_j^1 = \{t \in \mathbb{R} \mid |t - t_0| < r/j\}$ for $j = 1, 2, \dots$, with $r > 0$ so small that $\overline{B_1^n \times B_1^1} \subset \Sigma$. For $j \in \mathbb{N}$, define functions ψ_j such that

$$\begin{aligned}&\psi_j(x,t) = \varphi_j(x)\varrho_j(t) \in C_0^\infty(\Sigma) \text{ with}\\ &\operatorname{supp}\varphi_j \subset B_j^n,\ \varphi_j(x) = 1 \text{ on } B_{j+1}^n,\ \operatorname{supp}\varrho_j \subset B_j^1,\ \varrho_j(t) = 1 \text{ on } B_{j+1}^1.\end{aligned} \tag{3.4}$$

It is given that $\psi_1 u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$ and $\psi_1 Hu \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$. Now

$$H\psi_1 u = \psi_1 Hu + [H, \psi_1]u. \tag{3.5}$$

The commutator $[H, \psi_1] = H\psi_1 - \psi_1 H$ satisfies

$$[H, \psi_1]u = \varrho_1(t)[P, \varphi_1(x)]u + \varphi_1(x)\partial_t \varrho_1(t)u,$$

where $[P, \varphi_1]$ is a ψdo in x of order $d-1$; in the calculus on $\mathbb{R}^{n+1}$ it counts as an operator with symbol in $S_{1,0}^{d-1,d-1}$, cf. Lemma 2.2 2°.

If $d \leq 1$, $[P, \varphi_1]$ also satisfies the estimates for an operator with symbol in $S_{1,0}^{0,0}$. Then both terms in the right-hand side of (3.5) are in $H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$, and an application of K to (3.5) shows that $\psi_1 u \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R})$, as in the proof of Theorem 3.1.

If $d > 1$, $[H, \psi_1]$ is of positive order (with symbol in $S_{1,0}^{d-1,d-1}$), sending u into $H_p^{(s-d+1,s/d-1/d+1)}(\mathbb{R}^n \times \mathbb{R})$. Then we can only conclude by application of K that $\psi_1 u \in H_p^{(s+1,s/d+1/d)}(\mathbb{R}^n \times \mathbb{R})$. Here we need to make an extra effort. Take $\psi_2 = \varphi_2(x)\varrho_2(t)$ as in (3.4). Now we can write, since $\psi_1\psi_2 = \psi_2$,

$$H\psi_2 u = H\psi_1\psi_2 u = \psi_2 H\psi_1 u + [H, \psi_2]\psi_1 u = \psi_2 Hu + \psi_2 H(\psi_1 - 1)u + [H, \psi_2]\psi_1 u. \tag{3.6}$$

In the final expression, the first term is in $H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$. In the second term, since $\psi_2(\psi_1 - 1) = 0$,

$$\psi_2 H(\psi_1 - 1) = \varphi_2(x)\varrho_2(t)P(\varphi_1(x)\varrho_1(t) - \varphi_1(x) + \varphi_1(x) - 1) = \varphi_2(x)\varrho_2(t)P(\varphi_1(x) - 1),$$

where $\varphi_2 P(\varphi_1 - 1)$ is a ψdo in x of order $-\infty$ so we can regard it as an operator with symbol in $S_{1,0}^{0,0}$. Then the term lies in $H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$. For the third term, $[H, \psi_2] = \varrho_2[P, \varphi_2] + \varphi_2\partial_t\varrho_2$, where $[P, \varphi_2]$ enters as an operator with symbol in $S_{1,0}^{d-1,d-1}$, and when this is applied to $\psi_1 u \in H_p^{(s+1,s/d+1/d)}(\mathbb{R}^n \times \mathbb{R})$, we get a term in $H_p^{(s+2-d,(s+2-d)/d)}(\mathbb{R}^n \times \mathbb{R})$. If $d \leq 2$, we see altogether that $H\psi_2 u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$, and an application of K as in Theorem 3.1 shows that $\psi_2 u \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R})$, ending the proof.

If $d > 2$, we repeat the argument, first using that $\psi_3 = \psi_3\psi_2$, leading to the information that $H\psi_3 u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) + H_p^{(s+3-d,(s+3-d)/d)}(\mathbb{R}^n \times \mathbb{R})$, next $\psi_4\psi_3 = \psi_4$, and so on, until $j \geq d$, so that $H\psi_j u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$, and we can conclude that $\psi_j u \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R})$. □

By use of embedding theorems, we can also obtain a local regularity result in anisotropic Hölder spaces:

Theorem 3.3. *Let P be as in Theorem 3.1. Let $s > 0$, and let $u \in C^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) \cap \mathcal{E}'(\mathbb{R}^n \times \mathbb{R})$. If, for a bounded open subset $\Omega \times I$,*

$$(P+\partial_t)u|_{\Omega\times I}\in C^{(s,s/d)}_{\mathrm{loc}}(\Omega\times I),\tag{3.7}$$

then for small $\varepsilon>0$, $u|_{\Omega\times I}\in C^{(s+d-\varepsilon,(s-\varepsilon)/d+1)}_{\mathrm{loc}}(\Omega\times I)$.

Proof. Choose p so large that $s-n/p>0$. In view of the first embedding in (A.15), (3.7) implies that (3.3) holds for $\Sigma=\Omega\times I$ with s replaced by $s-\varepsilon_1$ for $\varepsilon_1>0$. Moreover, $u\in H_p^{(s-\varepsilon_1,(s-\varepsilon_1)/d)}(\mathbb{R}^n\times\mathbb{R})$. Then Theorem 3.2 shows that $u|_{\Omega\times I}\in H^{(s-\varepsilon_1+d,(s-\varepsilon_1)/d+1)}_{p,\mathrm{loc}}(\Omega\times I)$. Taking $\varepsilon_1<s-n/p$, we can use the second embedding in (A.15) to see that u is locally in the anisotropic Hölder space $C^{(s-n/p+d-\varepsilon_1-\varepsilon_2,(s-n/p-\varepsilon_1-\varepsilon_2)/d+1)}_{\mathrm{loc}}(\Omega\times I)$, for $0<\varepsilon_2<s-n/p+d-\varepsilon_1$. Since p can be taken arbitrarily large, the statement in the theorem follows. □

For the fractional Laplacian $(-\Delta)^a$ and other related singular integral operators, Fernandez-Real and Ros-Oton showed in [10] a result comparable to Theorem 3.3, in cases where s and $s/(2a)<1$:

$$u\in\overline{C}^{(s,s/(2a))}(\mathbb{R}^n\times I), f\in\overline{C}^{(s,s/(2a))}(\Omega\times I)\implies u\in C^{(s+2a,s/(2a)+1)}_{\mathrm{loc}}(\Omega\times I).\tag{3.8}$$

This is sharper by avoiding the subtraction by ε. On the other hand, our result expands the knowledge in many other directions, including that it allows not just $s,s/(2a)<1$ but the values of s up to ∞, and it allows variable-coefficient operators, and does not require the symmetries entering in the definition of the fractional Laplacian, but just assumes that $P+\partial_t$ is parabolic. We think that a removal of ε would be possible in Theorem 3.3 too — by extending the action of our anisotropic ψdo's (with symbols with finite regularity numbers) to anisotropic Hölder–Zygmund spaces.

[10] has in Cor. 3.8 and Rem. 6.4 some information on high spatial regularity of u when f has high spatial regularity, assuming that the integral operator kernel has a correspondingly high regularity.

To our knowledge, the regularity results obtained above are new in several ways: by including all classical strongly elliptic ψdo's P of positive real orders down to zero, and by including all $p\in\,]1,\infty[$, and all $s\in\mathbb{R}$.

Observe the special cases:

Corollary 3.4. *Let P and u be as in Theorem 3.1 2°.*

If u satisfies

$$(P+\partial_t)u=f\ \text{with}\ f\in L_p(\mathbb{R}^n\times\mathbb{R}),\tag{3.9}$$

then $u\in H_p^{(d,1)}(\mathbb{R}^n\times\mathbb{R})=L_p(\mathbb{R};H_p^d(\mathbb{R}^n))\cap H_p^1(\mathbb{R};L_p(\mathbb{R}^n))$. *Conversely,* $u\in H_p^{(d,1)}(\mathbb{R}^n\times\mathbb{R})$ *implies (3.9).*

If $u\in L_p(\mathbb{R}^n\times\mathbb{R})$ *and* Σ *is an open subset of* $\mathbb{R}^{n+1}$, *then*

$$(P+\partial_t)u|_\Sigma\in L_{p,\mathrm{loc}}(\Sigma)\implies u|_\Sigma\in H^{(d,1)}_{p,\mathrm{loc}}(\Sigma).\tag{3.10}$$

Note that in view of the embeddings (1.6), (A.8), we also have that (3.9) implies $u \in L_p(\mathbb{R}; B_p^d(\mathbb{R}^n)) \cap H_p^1(\mathbb{R}; L_p(\mathbb{R}^n))$ if $p \geq 2$; but this cannot be inferred when $p < 2$. There are similar observations for (3.10).

4. A global estimate for the fractional Dirichlet heat equation

We shall not in this paper abord the question of possible extensions of the boundary value calculations of [17] to fractional-order situations. We will just turn to a basic global L_p-result for the fractional heat equation $Pu + \partial_t u = f$ on $\Omega \times I$, with P equal to $(-\Delta)^a$ (0.4) or the generalization (0.5) with smooth $k(y)$, and provided with a homogeneous Dirichlet condition. It will be obtained by combination of a functional analysis method put forward in [3] with detailed information from our earlier studies.

Recall first some notation from [21]:

The following subsets of $\mathbb{R}^n$ will be considered: $\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n \mid x_n \gtrless 0\}$ (where $(x_1, \dots, x_{n-1}) = x'$), and bounded C^∞-subsets Ω with boundary $\partial\Omega$, and their complements. Restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$ (or from $\mathbb{R}^n$ to Ω resp. $\complement\overline{\Omega}$) is denoted $r^\pm$, extension by zero from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$ (or from Ω resp. $\complement\overline{\Omega}$ to $\mathbb{R}^n$) is denoted $e^\pm$. Restriction from $\overline{\mathbb{R}}^n_+$ or $\overline{\Omega}$ to $\partial\mathbb{R}^n_+$ resp. $\partial\Omega$ is denoted γ_0.

We denote by $d(x)$ a function of the form $d(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$, x near $\partial\Omega$, extended to a smooth positive function on Ω; $d(x) = x_n$ in the case of $\mathbb{R}^n_+$.

Along with the spaces $H_p^s(\mathbb{R}^n)$ defined in (1.4), we have the two scales of spaces associated with Ω for $s \in \mathbb{R}$:

$$\begin{aligned} \dot{H}_p^s(\overline{\Omega}) &= \{u \in H_p^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}, \\ \overline{H}_p^s(\Omega) &= \{u \in \mathcal{D}'(\Omega) \mid u = r^+ U \text{ for some } U \in H_p^s(\mathbb{R}^n)\}; \end{aligned} \tag{4.1}$$

here $\operatorname{supp} u$ denotes the support of u. The definition is also used with $\Omega = \mathbb{R}^n_+$. In most current texts, $\overline{H}_p^s(\Omega)$ is denoted $H_p^s(\Omega)$ without the overline (that was introduced along with the notation $\dot{H}_p$ in [27,26]), but we prefer to use it, since it is makes the notation more clear in formulas where both types occur. We recall that $\overline{H}_p^s(\Omega)$ and $\dot{H}_{p'}^{-s}(\overline{\Omega})$ are dual spaces with respect to a sesquilinear duality extending the $L_2(\Omega)$-scalar product; $\frac{1}{p} + \frac{1}{p'} = 1$.

A special role in the theory is played by the *order-reducing operators*. There is a simple definition of operators $\Xi^t_\pm$ on $\mathbb{R}^n$, $t \in \mathbb{R}$,

$$\Xi^t_\pm = \mathrm{OP}(\chi^t_\pm), \quad \chi^t_\pm = (\langle \xi' \rangle \pm i\xi_n)^t; \tag{4.2}$$

they preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively. The functions $(\langle \xi' \rangle \pm i\xi_n)^t$ do not satisfy all the estimates for $S^t_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, but lie in a space as in Definition 2.1 with $d = 1$, $\nu = 1$, $m = t$, with (ξ, τ) replaced by (ξ', ξ_n). There is a more refined choice $\Lambda^t_\pm$ [16,21], with symbols $\lambda^t_\pm(\xi)$ that do satisfy all the estimates for $S^t_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$; here $\overline{\lambda^t_+} = \lambda^t_-$. The symbols have holomorphic extensions in ξ_n to the complex halfspaces $\mathbb{C}_\mp = \{z \in \mathbb{C} \mid \operatorname{Im} z \lessgtr 0\}$; it is for this reason that the operators preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively. Operators with that property are called "plus" resp. "minus" operators. There is also a pseudodifferential definition $\Lambda^{(t)}_\pm$ adapted to the situation of a smooth domain Ω, cf. [21].

It is elementary to see by the definition of the spaces $H^s_p(\mathbb{R}^n)$ in terms of Fourier transformation, that the operators define homeomorphisms for all s: $\Xi^t_\pm\colon H^s_p(\mathbb{R}^n) \overset{\sim}{\to} H^{s-t}_p(\mathbb{R}^n)$, $\Lambda^t_\pm\colon H^s_p(\mathbb{R}^n) \overset{\sim}{\to} H^{s-t}_p(\mathbb{R}^n)$. The special interest is that the "plus"/"minus" operators also define homeomorphisms related to $\overline{\mathbb{R}}^n_+$ and $\overline{\Omega}$, for all $s \in \mathbb{R}$: $\Xi^t_+\colon \dot{H}^s_p(\overline{\mathbb{R}}^n_+) \overset{\sim}{\to} \dot{H}^{s-t}_p(\overline{\mathbb{R}}^n_+)$, $r^+\Xi^t_- e^+\colon \overline{H}^s_p(\mathbb{R}^n_+) \overset{\sim}{\to} \overline{H}^{s-t}_p(\mathbb{R}^n_+)$, with similar statements for $\Lambda^t_\pm$, and for $\Lambda^{(t)}_\pm$ relative to Ω. Moreover, the operators Ξ^t_+ and $r^+\Xi^t_- e^+$ identify with each other's adjoints over $\overline{\mathbb{R}}^n_+$, because of the support preserving properties. There is a similar statement for Λ^t_+ and $r^+\Lambda^t_- e^+$, and for $\Lambda^{(t)}_+$ and $r^+\Lambda^{(t)}_- e^+$ relative to the set Ω.

The special *μ-transmission spaces* were introduced by Hörmander [27] for $p = 2$, cf. [21]; we shall just use them here for $\mu = a$:

$$\begin{aligned} H^{a(s)}_p(\overline{\mathbb{R}}^n_+) &= \Xi^{-a}_+ e^+ \overline{H}^{s-a}_p(\mathbb{R}^n_+) = \Lambda^{-a}_+ e^+ \overline{H}^{s-a}_p(\mathbb{R}^n_+), \quad s > a - 1/p', \\ H^{a(s)}_p(\overline{\Omega}) &= \Lambda^{(-a)}_+ e^+ \overline{H}^{s-a}_p(\Omega), \quad s > a - 1/p'; \end{aligned} \tag{4.3}$$

they are the appropriate solution spaces for homogeneous Dirichlet problems for elliptic operators P having the a-transmission property (cf. [21]). Note that in (4.3), Ξ^{-a}_+ is applied to functions with a jump at $x_n = 0$ (when $s > a + 1/p$), this results in a singularity at $x_n = 0$.

The ψdo P can be applied to functions in the spaces in (4.1) when they are extended by zero to all of $\mathbb{R}^n$. This is already understood for the spaces $\dot{H}^s_p(\overline{\Omega})$, but should be mentioned explicitly (by an indication with e^+) for the spaces $\overline{H}^s_p(\Omega)$. Also, when $u \in \dot{H}^s_p(\overline{\Omega})$ and Pu is considered on Ω, it is most correct to indicate this by writing r^+Pu. The indications e^+ and r^+ can be left out as an "abuse of notation", when they are understood from the context; note however the importance of e^+ in (4.3).

Recall from [21], Theorems 4.4 and 5.4:

Theorem 4.1. *Let Ω be an open bounded smooth subset of $\mathbb{R}^n$, let P be a ψdo on $\mathbb{R}^n$ of the form (0.5) with $k \in C^\infty(S^{n-1})$ (in particular, P can be equal to $(-\Delta)^a$) for an $a > 0$. Let $1 < p < \infty$, and let $P_{\mathrm{Dir},p}$ stand for the L_p-Dirichlet realization on Ω, acting like r^+P and with domain*

$$D(P_{\mathrm{Dir},p}) = \{u \in \dot{H}^a_p(\overline{\Omega}) \mid r^+Pu \in L_p(\Omega)\}; \tag{4.4}$$

the operators are consistent for different p. Then

$$D(P_{\mathrm{Dir},p}) = H^{a(2a)}_p(\overline{\Omega}) = \Lambda^{(-a)}_+ e^+ \overline{H}^a_p(\Omega). \tag{4.5}$$

It satisfies (for any $\varepsilon > 0$):

$$H_p^{a(2a)}(\overline{\Omega}) \begin{cases} = \dot{H}_p^{2a}(\overline{\Omega}), & \text{if } a < 1/p, \\ \subset \dot{H}_p^{2a-\varepsilon}(\overline{\Omega}), & \text{if } a = 1/p, \\ \subset e^+ d^a \overline{H}_p^a(\Omega) + \dot{H}_p^{2a}(\overline{\Omega}), & \text{if } a > 1/p,\ a - 1/p \notin \mathbb{N}, \\ \subset e^+ d^a \overline{H}_p^a(\Omega) + \dot{H}_p^{2a-\varepsilon}(\overline{\Omega}), & \text{if } a > 1/p,\ a - 1/p \in \mathbb{N}. \end{cases} \tag{4.6}$$

More precisely, in the case $a \in]0,1[$, the functions have in local coordinates where Ω is replaced by $\mathbb{R}^n_+$ the following structure when $a > 1/p$:

$$u = w + x_n^a K_0 \varphi, \tag{4.7}$$

where w and φ run through $\dot{H}_p^{2a}(\overline{\mathbb{R}}^n_+)$ and $B_p^{a-1/p}(\mathbb{R}^{n-1})$, and K_0 is the Poisson operator $K_0\varphi = \mathcal{F}^{-1}_{\xi' \to x'}[\hat{\varphi}(\xi') e^+ r^+ e^{-\langle \xi' \rangle x_n}]$.

Proof. When P is defined by (0.5) with a smooth k, it is the ψdo with symbol $p(\xi)$ equal to the Fourier transform of the kernel function $k(y/|y|)|y|^{-n-2a}$. (For $P = (-\Delta)^a$, k equals the constant $c_{n,a}$.) The symbol is a function homogeneous of degree $2a$, smooth positive for $\xi \neq 0$, and *even*: $p(-\xi) = p(\xi)$. The L_2-Dirichlet realization $P_{\mathrm{Dir},2}$ can be defined variationally from the sesquilinear form

$$Q(u,v) = \tfrac{1}{2} \int\limits_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(\bar{v}(x) - \bar{v}(y)) k((x-y)/|x-y|)}{|x-y|^{n+2a}}\, dxdy \tag{4.8}$$

considered for $u, v \in \dot{H}^a(\Omega)$. As accounted for in Ros-Oton [37], it satisfies a Poincaré inequality over Ω so that the selfadjoint operator in $L_2(\Omega)$ induced by the Lax–Milgram lemma is a bijection from its domain to $L_2(\Omega)$. The explanation in [37] is formulated for real functions, but the operator defined in $L_2(\Omega, \mathbb{C})$ is *real* in the sense that it maps real functions to real functions, and it can be retrieved from the definition on $L_2(\Omega, \mathbb{R})$ by linear extension.

Since $p(\xi)$ is even and homogeneous of degree $2a$, it satisfies the condition in [21] for having the a-transmission property, and since it is positive, it has factorization index a (since $a_+ = a_-$ in [21], (3.3)–(3.4)). Then, considering its action in L_p-spaces, the description of the domain in (4.5) follows from [21], Th. 4.4 with $m = 2a$, $\mu_0 = a$, $s = 2a$. The consistency for various p holds as a general property of pseudodifferential operators. The statement (4.6) is from [21], Th. 5.4, with $\mu = a$, $s = 2a$.

The information (4.7) is a consequence of the proof given there; let us give a direct explanation here: It is well-known that when $1/p < a < 1+1/p$, the functions $v \in \overline{H}_p^a(\mathbb{R}^n_+)$ have a first trace $\gamma_0 v \in B_p^{a-1/p}(\mathbb{R}^{n-1})$, and that $v - K\gamma_0 v \in \dot{H}_p^a(\overline{\mathbb{R}}^n_+)$, when K is a continuous right inverse of γ_0. In fact, the functions $v \in \overline{H}_p^a(\mathbb{R}^n_+)$ are exactly the functions of the form $v = g + K\varphi$, where g runs through $\dot{H}_p^a(\overline{\mathbb{R}}^n_+)$ and φ runs through $B_p^{a-1/p}(\mathbb{R}^{n-1})$. Take as K the Poisson operator K_0,

$$K_0\varphi = \mathcal{F}^{-1}_{\xi' \to x'}[\hat{\varphi}(\xi') e^+ r^+ e^{-\langle \xi' \rangle x_n}] = \mathcal{F}^{-1}_{\xi \to x}[\hat{\varphi}(\xi')(\langle \xi' \rangle + i\xi_n)^{-1}]. \tag{4.9}$$

Now to describe $H_p^{a(2a)}(\overline{\mathbb{R}}^n_+) = \Xi_+^{-a} e^+ \overline{H}_p^a(\mathbb{R}^n_+)$, we use that

$$\Xi_+^{-a} K_0\varphi = \mathcal{F}^{-1}_{\xi\to x}[(\langle\xi'\rangle + i\xi_n)^{-a}\hat{\varphi}(\xi')(\langle\xi'\rangle + i\xi_n)^{-1}] = c_a x_n^a K_0\varphi,$$

cf. [21], (2.5) and (5.16). Moreover, $\Xi_+^{-a}\dot{H}_p^a(\overline{\mathbb{R}}^n_+) = \dot{H}_p^{2a}(\overline{\mathbb{R}}^n_+)$. This shows the representation in (4.7). (We are using the formulas from [21] with $[\xi']$ replaced by $\langle\xi'\rangle$, which works equally well, as shown there.) $\square$

We underline that for $a > 1/p$, $D(P_{\mathrm{Dir},p})$ contains not only $\dot{H}_p^{2a}(\overline{\Omega})$, but also functions of the form $e^+d^a z$ with $z \in \overline{H}_p^a(\Omega)$, not in $\overline{H}_p^s(\Omega)$ for $s > a$. (On the interior, the functions are in $H^{2a}_{p,\mathrm{loc}}(\Omega)$, by elliptic regularity.)

The operators $P_{\mathrm{Dir},p}$ are bijective for all p; for $p = 2$ this holds since the sesquilinear form defining $P_{\mathrm{Dir},2}$ has positive lower bound by the Poincaré inequality, and for the other p it follows in view of [20], Th. 3.5, on the stability of kernels and cokernels when spaces change.

Next, we will describe $P_{\mathrm{Dir},p}$ from a functional analysis point of view. We already have the definition of $P_{\mathrm{Dir},2}$ from the sesquilinear form (4.8). The associated quadratic form $Q(u,u)$ is denoted $Q(u)$,

$$Q(u) = \tfrac{1}{2}\int\limits_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2 k((x-y)/|x-y|)}{|x-y|^{n+2a}}\, dxdy \text{ on } \dot{H}_p^a(\overline{\Omega}). \tag{4.10}$$

In view of the positivity, $-P_{\mathrm{Dir},2}$ generates a strongly continuous semigroup $e^{-tP_{\mathrm{Dir},2}}$ of contractions in $L_2(\Omega,\mathbb{R})$.

We shall now see that the form $Q(u,v)$ is a so-called Dirichlet form, as defined in Davies [8] and Fukushima, Oshima and Takeda [12]. This is observed e.g. in Bogdan, Burdzy and Chen [4] for a related form defining the regional fractional Laplacian. It means that Q has the Markovian property (cf. [12], pp. 4–5):

Definition 4.2. A closed nonnegative symmetric form $E(u,v)$ with domain $D(E) \subset L_2(\Omega,\mathbb{R})$ is said to be Markovian, if for any $\varepsilon > 0$ there exists a function φ_ε on $\mathbb{R}$ taking values in $[-\varepsilon, 1+\varepsilon]$ with $\varphi_\varepsilon(t) = t$ on $[0,1]$ and $0 \le \varphi_\varepsilon(t) - \varphi_\varepsilon(s) \le t - s$ when $t > s$, such that

$$u \in D(E) \implies \varphi_\varepsilon \circ u \in D(E) \text{ and } E(\varphi_\varepsilon u, \varphi_\varepsilon u) \le E(u,u). \tag{4.11}$$

There is an equivalent definition with $\varphi_\varepsilon \circ u$ in (4.11) replaced by $\min\{\max\{u,0\},1\}$.

We can choose φ_ε to be C^∞ on $\mathbb{R}$ (but not in $C_0^\infty(\mathbb{R})$ as written in [4]), then it is clear that (4.11) holds for $E = Q$.

The interest of the Markovian property here is that then the semigroup $e^{-tP_{\mathrm{Dir},2}}$ extends for each $p \in\,]1,\infty[$ to a semigroup $T_p(t)$ that is contractive in L_p-norm and bounded holomorphic ([12] Th. 1.4.1 and [8] Th. 1.4.1); with infinitesimal generators that are consistent for varying p. The generator of $T_p(t)$ is in fact $-P_{\mathrm{Dir},p}$, since the latter is bijective from its domain to $L_p(\Omega)$, and the operators $-P_{\mathrm{Dir},p}$ are consistent for varying p.

As pointed out in [3], one can use the existence of these extensions to L_p to apply a theorem of Lamberton [31] giving information on the heat equation solvability.

Theorem 4.3. *Let P and Ω be as in Theorem 4.1, with $0 < a < 1$, and let $I =]0, T[$ for some $T > 0$. The Dirichlet evolution problem*

$$\begin{aligned} Pu + \partial_t u &= f \ \text{on } \Omega \times I, \\ u &= 0 \ \text{on } (\mathbb{R}^n \setminus \Omega) \times I, \\ u &= 0 \ \text{for } t = 0, \end{aligned} \tag{4.12}$$

has for any $f \in L_p(\Omega \times I)$ a unique solution $u(x,t) \in C^0(\overline{I}; L_p(\Omega))$, which satisfies:

$$u \in L_p(I; H_p^{a(2a)}(\overline{\Omega})) \cap H_p^1(I; L_p(\Omega)). \tag{4.13}$$

Here $H_p^{a(2a)}(\overline{\Omega})$ is the domain of $P_{\mathrm{Dir},p}$, as described in detail in Theorem 4.1.

Proof. In (4.12) it is tacitly understood that u identifies with a function on $\mathbb{R}^n \times I$ vanishing for $x \in \mathbb{R}^n \setminus \Omega$, in order for the ψdo to be defined on u.

As accounted for above, the operator $P_{\mathrm{Dir},2}$ satisfies the hypotheses for the operator $-A$ studied in [31], namely that A generates a bounded holomorphic semigroup for $p = 2$, and induces bounded holomorphic semigroups $T_p(t)$ in $L_p(\Omega)$ that are contractions for all $p \in]1, \infty[$, and are consistent with the case $p = 2$. Then, according to [31] Th. 1, the problem

$$(\partial_t - A)u = f \text{ on } I, \quad u|_{t=0} = 0, \tag{4.14}$$

has for any $f \in L_p(\Omega \times I)$ a solution $u(x,t) \in C^0(\overline{I}; L_p(\Omega))$ such that

$$\|\partial_t u\|_{L_p(\Omega\times I)} + \|Au\|_{L_p(\Omega\times I)} \le C\|f\|_{L_p(\Omega\times I)}. \tag{4.15}$$

The bound on the first term shows that $u \in H_p^1(I; L_p(\Omega))$. Since $\|Pv\|_{L_p(\Omega)} \ge C'\|v\|_{H_p^{a(2a)}(\overline{\Omega})}$ for all $v(x) \in D(P_{\mathrm{Dir},p})$, the bound on the second term shows that $u \in L_p(I; H_p^{a(2a)}(\overline{\Omega}))$.

The uniqueness of the solution is accounted for e.g. in [32]. $\square$

The regularity in (4.13) is optimal for $f \in L_p(\Omega \times I)$.

In view of the general rules (1.6), the theorem implies that the solution also satisfies $u \in L_p(I; B_p^{a(2a)}(\overline{\Omega}))$ when $p \ge 2$, but hits a larger space than $L_p(I; B_p^{a(2a)}(\overline{\Omega}))$ when $p < 2$.

For $P=(-\Delta)^a$, Biccari, Warma and Zuazua [3] used [31] to show semi-local variants of (4.13): $u\in L_p(I;W^{2a,p}_{\mathrm{loc}}(\Omega))$ with $\partial_t u\in L_p(I\times\Omega)$ when $p\ge 2$ or $a=\frac12$; and it holds with $W^{2a,p}_{\mathrm{loc}}(\Omega)$ replaced by $B^{2a}_{p,2,\mathrm{loc}}(\Omega)$ when $p<2$, $a\ne\frac12$ (there is an embedding $H^s_p\subset B^s_{p,2}$ for such p).

When f has a higher regularity, we can use Theorem 3.2 to get a local result:

Theorem 4.4. *Let u be the solution of (4.12) defined in Theorem 4.3, and let $r=2a$ if $a<1/p$, $r=a+1/p-\varepsilon$ if $a\ge 1/p$ (for some small $\varepsilon>0$). Then $u\in\overline{H}_p^{(r,r/(2a))}(\mathbb{R}^n\times I)$, vanishing for $x\notin\overline{\Omega}$.*

If $f\in H^{(s,s/(2a))}_{p,\mathrm{loc}}(\Omega\times I)$ for some $0<s\le r$, then $u\in H^{(s+2a,s/(2a)+1)}_{p,\mathrm{loc}}(\Omega\times I)$.

Proof. Since $e^+\overline{H}^a_p(\Omega)=\dot H^a_p(\overline{\Omega})$ for $a<1/p$, and is contained in $\dot H^{1/p-\varepsilon}_p(\overline{\Omega})$ when $a\ge 1/p$, $H^{a(2a)}_p(\overline{\Omega})=\Lambda^{-a}_+e^+\overline{H}^a_p(\Omega)\subset\dot H^r_p(\overline{\Omega})$, where r is as defined in the theorem. Note that $r\le 2a$, and that $H^{a(2a)}_p(\overline{\Omega})\subset\dot H^{a+1/p-\varepsilon}_p(\overline{\Omega})$ in any case.

Thus, when u is a solution as in Theorem 4.3, $u\in\overline{H}_p^{(r,r/(2a))}(\mathbb{R}^n\times I)$, vanishing for $x\notin\overline{\Omega}$.

In view of the initial condition $u|_{t=0}=0$, the extension by zero for $t<0$ lies in $\overline{H}_p^{(r,r/(2a))}(\mathbb{R}^n\times\,]-\infty,T[\,)$. We can moreover extend our function for $t\ge T$ to a function in $H_p^{(r,r/(2a))}(\mathbb{R}^n\times\mathbb{R})$, and denote the fully extended function $\tilde u$. Observe that the values of $\tilde f=(P+\partial_t)\tilde u$ are consistent with the values of f on $\Omega\times I$, since P acts only in the x-direction and ∂_t is local.

Now if $f\in H^{(s,s/(2a))}_{p,\mathrm{loc}}(\Omega\times I)$ for some $0<s\le r$, we can apply Theorem 3.2, and conclude that $u\in H^{(s+2a,s/(2a)+1)}_{p,\mathrm{loc}}(\Omega\times I)$. $\square$

For higher values of s, Theorem 3.2 gives that if $u\in H_p^{(s,s/(2a))}(\mathbb{R}^n\times I)$ and $f\in H^{(s,s/(2a))}_{p,\mathrm{loc}}(\Omega\times I)$, then $u\in H^{(s+2a,s/(2a)+1)}_{p,\mathrm{loc}}(\Omega\times I)$, but the global prerequisite on u may not be easy to obtain.

Remark 4.5. In comparison with the result of [10], Cor. 1.6, in anisotropic Hölder spaces, our study in $H_p^{(s,s/(2a))}$-spaces has the advantage that the regularity in t of the solution of the Dirichlet problem is lifted by a full step 1 in Theorem 4.3, whereas [10] obtains a $C^{1-\varepsilon}$-estimate in t.

Also for the interior regularity we observe a better lifting in t-derivatives, namely that when $a<1/p$, t-derivatives of order 2 are controlled in Theorem 4.4 (taking $s=2a$). Since $0<a<1$, this holds for p sufficiently close to 1. For the Hölder estimates in [10], Th. 1.1, second t-derivatives are not reached (cf. (3.8) above).

Remark 4.6. Theorem 4.1 is also valid for x-dependent strongly elliptic ψdo's P of order $2a$ with even symbol. Then $P_{\mathrm{Dir},2}$ is defined from the sesquilinear form $(Pu,v)_{L_2(\Omega)}$ on $C_0^\infty(\Omega)$, extended by closure to $\dot H^a_2(\overline{\Omega})$, and the $P_{\mathrm{Dir},p}$ are consistent with this by (4.4).

A variant of Theorem 4.3 can be shown for $p = 2$ by techniques from Lions and Magenes [33] vol. 2 (we shall explain details elsewhere), but for $p \neq 2$, other methods are needed.

Appendix A. Anisotropic Bessel-potential and Besov spaces

In this Appendix we present the appropriate anisotropic generalizations of Bessel-potential and Besov spaces. We just give a summary of the essentially well-known facts that are needed for the parabolic operator $P + \partial_t$ on $\mathbb{R}^n \times \mathbb{R}$ with a ψdo P on $\mathbb{R}^n$ of positive order. Let $d \in \mathbb{R}_+$ and $p \in \,]1, \infty[\,$. This material is taken from the appendix of [17], with small modifications and less focus on cases where d is integer.

For $m, d \in \mathbb{N}$, the anisotropic Sobolev spaces $W_p^{(dm,m)}(\mathbb{R}^n \times \mathbb{R})$ are defined by

$$\begin{aligned} W_p^{(dm,m)}(\mathbb{R}^n \times \mathbb{R}) &= L_p\big(\mathbb{R}; H_p^{dm}(\mathbb{R}^n)\big) \cap H_p^m\big(\mathbb{R}; L_p(\mathbb{R}^n)\big) \\ &= \{\, u(x,t) \in L_p(\mathbb{R}^{n+1}) \mid \mathcal{F}^{-1}_{(\xi,\tau)\to(x,t)}(\langle\xi\rangle^{dm} + \langle\tau\rangle^m)\hat{u}) \in L_p(\mathbb{R}^{n+1}) \,\} \\ &= \{\, u(x,t) \in L_p(\mathbb{R}^{n+1}) \mid D_x^\alpha D_t^j u \in L_p(\mathbb{R}^{n+1}) \text{ for } |\alpha| + dj \le dm \,\}. \end{aligned} \tag{A.1}$$

For the generalization to noninteger and negative values of the Sobolev exponents, we observe that if we define $\{\xi, \tau\}$ and $\Theta^s = \mathrm{OP}_{x,t}(\{\xi,\tau\}^s)$ as in (2.1), (2.2), then

$$W_p^{(dm,m)}(\mathbb{R}^n \times \mathbb{R}) = \Theta^{-dm} L_p(\mathbb{R}^{n+1}). \tag{A.2}$$

This is seen by use of Lizorkin's criterion [35], cf. (2.11) above, applied to the operators with symbol $(\langle\xi\rangle^{dm} + \langle\tau\rangle^m)\{\xi,\tau\}^{-dm}$ and $\{\xi,\tau\}^{dm}(\langle\xi\rangle^{dm} + \langle\tau\rangle^m)^{-1}$.

Clearly, $\Theta^s\Theta^t = \Theta^{s+t}$ for $s, t \in \mathbb{R}$.

More generally, one can now define with $d \in \mathbb{R}_+$, for any $s \in \mathbb{R}$,

$$H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) = \Theta^{-s} L_p(\mathbb{R}^{n+1}); \tag{A.3}$$

it is an anisotropic generalization of the Bessel-potential spaces $H_p^s(\mathbb{R}^n)$, and clearly $H_p^{(dm,m)}(\mathbb{R}^n \times \mathbb{R}) = W_p^{(dm,m)}(\mathbb{R}^n \times \mathbb{R})$ for $m, d \in \mathbb{N}$. Here we follow the notation of Schmeisser and Triebel, cf. e.g. [41] and its references, deviating from another extensively used notation $L_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$ (as in Lizorkin [34,35], Nikolskiĭ [36], Besov, Il'in and Nikolskiĭ [2], …), where the spaces are often called Liouville spaces. See also Sadosky and Cotlar [38]. These spaces fit together in *complex interpolation* (by an anisotropic variant of the proof for $H_p^s(\mathbb{R}^{n+1})$ spaces, cf. Calderón [6], Schechter [39], and Schmeisser and Triebel [40], Rem. 4):

$$\begin{aligned} &[H_p^{(s_0,s_0/d)}(\mathbb{R}^n \times \mathbb{R}), H_p^{(s_1,s_1/d)}(\mathbb{R}^n \times \mathbb{R})]_\theta = H_p^{(s_2,s_2/d)}(\mathbb{R}^n \times \mathbb{R}), \\ &s_2 = (1-\theta)s_0 + \theta s_1, \quad \text{for } s_0, s_1 \in \mathbb{R},\ \theta \in \,]0,1[\,. \end{aligned} \tag{A.4}$$

Another generalization of the $W_p^{(dm,m)}$ spaces is the scale of anisotropic Besov spaces $B_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$, that can be defined as follows:

$$
\begin{aligned}
&\|u\|^p_{B_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R})} \\
&\quad = \|u\|^p_{L_p} + \int\limits_{\mathbb{R}^{2n+2}} \Big(\frac{|u(x,t)-u(y,t)|^p}{|x-y|^{n+ps}} + \frac{|u(x,t)-u(x,t')|^p}{|t-t'|^{1+ps/d}}\Big)\, dxdydtdt' \\
&\qquad \text{for } s\in\,]0,1[\,; \\
&B_p^{(s+r,(s+r)/d)}(\mathbb{R}^n\times\mathbb{R}) = \Theta^{-r}B_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}),\ \text{for } r\in\mathbb{R},\ s\in\,]0,1[\,.
\end{aligned} \tag{A.5}
$$

The norm can also be described in terms of dyadic decompositions (see e.g. [41]), or, for $s>0$, by a generalization of the integral formula in (A.5) involving higher order differences as in (1.5) (see e.g. Besov [1], Solonnikov [43], and [30], [2]). These spaces arise from the $W_p^{(dm,m)}(\mathbb{R}^n\times\mathbb{R})$ spaces by *real interpolation* (cf. Grisvard [13, I.9]), when $m,d\in\mathbb{N}$:

$$
\big(W_p^{(dm,m)}(\mathbb{R}^n\times\mathbb{R}), L_p(\mathbb{R}^{n+1})\big)_{\theta,p} = B_p^{((1-\theta)dm,(1-\theta)m)}(\mathbb{R}^n\times\mathbb{R});\ \theta\in\,]0,1[\,. \tag{A.6}
$$

Moreover, one has for all $d\in\mathbb{R}_+$, all $s_0,s_1,s\in\mathbb{R}$ with $s_0\neq s_1$, all $p_0,p_1,p\in\,]1,\infty[$ with $p_0\neq p_1$, all $\theta\in\,]0,1[$, setting $s_2=(1-\theta)s_0+\theta s_1$, $\frac{1}{p_2}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ (cf. [13] and [41, 3.2]):

$$
\begin{aligned}
\big(B_p^{(s_0,s_0/d)}(\mathbb{R}^n\times\mathbb{R}), B_p^{(s_1,s_1/d)}(\mathbb{R}^n\times\mathbb{R})\big)_{\theta,p} &= B_p^{(s_2,s_2/d)}(\mathbb{R}^n\times\mathbb{R}),\\
[B_p^{(s_0,s_0/d)}(\mathbb{R}^n\times\mathbb{R}), B_p^{(s_1,s_1/d)}(\mathbb{R}^n\times\mathbb{R})]_{\theta} &= B_p^{(s_2,s_2/d)}(\mathbb{R}^n\times\mathbb{R}),\\
\big(H_p^{(s_0,s_0/d)}(\mathbb{R}^n\times\mathbb{R}), H_p^{(s_1,s_1/d)}(\mathbb{R}^n\times\mathbb{R})\big)_{\theta,p} &= B_p^{(s_2,s_2/d)}(\mathbb{R}^n\times\mathbb{R}),\\
\big(H_{p_0}^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}), H_{p_1}^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R})\big)_{\theta,p_2} &= H_{p_2}^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}).
\end{aligned} \tag{A.7}
$$

The Bessel-potential spaces and Besov spaces are interrelated by

$$
\begin{aligned}
B_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}) \subset H_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}) \subset B_p^{(s-\varepsilon,(s-\varepsilon)/d)}(\mathbb{R}^n\times\mathbb{R}) \text{ for } p\le 2;\\
H_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}) \subset B_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}) \subset H_p^{(s-\varepsilon,(s-\varepsilon)/d)}(\mathbb{R}^n\times\mathbb{R}) \text{ for } p\ge 2;
\end{aligned} \tag{A.8}
$$

for $s\in\mathbb{R}$, any $\varepsilon>0$; and $B_p^{(s,s/d)}\neq H_p^{(s,s/d)}$ when $p\neq 2$.

For both types, one has the identification of *dual spaces* (with $\frac{1}{p}+\frac{1}{p'}=1$ as usual):

$$
\begin{aligned}
&H_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R})^* \simeq H_{p'}^{(-s,-s/d)}(\mathbb{R}^n\times\mathbb{R}) \quad \text{and}\\
&B_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R})^* \simeq B_{p'}^{(-s,-s/d)}(\mathbb{R}^n\times\mathbb{R}), \quad \text{for } s\in\mathbb{R}.
\end{aligned} \tag{A.9}
$$

For positive s, the spaces can moreover be described in the following way:

$$
\begin{aligned}
&\text{(i)}\quad H_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}) = L_p\big(\mathbb{R}; H_p^s(\mathbb{R}^n)\big)\cap H_p^{s/d}\big(\mathbb{R}; L_p(\mathbb{R}^n)\big),\ s\ge 0;\\
&\text{(ii)}\quad B_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}) = L_p\big(\mathbb{R}; B_p^s(\mathbb{R}^n)\big)\cap B_p^{s/d}\big(\mathbb{R}; L_p(\mathbb{R}^n)\big),\ s> 0;
\end{aligned} \tag{A.10}
$$

cf. Grisvard [13,14]. The anisotropic Sobolev–Slobodetskiĭ spaces $W_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R})$ are defined from the isotropic ones by (cf. e.g. Solonnikov [44])

$$\begin{aligned} W_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}) &= L_p\big(\mathbb{R};W_p^s(\mathbb{R}^n)\big)\cap W_p^{s/d}\big(\mathbb{R};L_p(\mathbb{R}^n)\big),\ s\ge 0;\\ \text{thus } W_p^{(s,s/d)} &= H_p^{(s,s/d)} \text{ when } s \text{ and } s/d\in\mathbb{N},\\ W_p^{(s,s/d)} &= B_p^{(s,s/d)} \text{ when } s \text{ and } s/d\in\mathbb{R}_+\setminus\mathbb{N}. \end{aligned} \tag{A.11}$$

(Note that the $W_p^{(s,s/d)}$ spaces do not always interpolate well. Moreover, when e.g. $s\in\mathbb{N}$, $s/d\notin\mathbb{N}$, one has a space where the x-regularity is H_p^s-type, the t-regularity is $B_p^{s/d}$-type; for $p\neq 2$ it is not one of the spaces in (A.10), cf. also (1.6).)

Observe that differential operators have the effect, for any $s\in\mathbb{R}$,

$$\begin{aligned} &D_x^\alpha D_t^j\, H_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}) \to H_p^{(s-m,(s-m)/d)}(\mathbb{R}^n\times\mathbb{R}) \text{ and}\\ &D_x^\alpha D_t^j\, B_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}) \to B_p^{(s-m,(s-m)/d)}(\mathbb{R}^n\times\mathbb{R}),\ \text{for } |\alpha|+dj\le m. \end{aligned} \tag{A.12}$$

This follows for the Bessel-potential spaces by Lizorkin's criterion (cf. (2.11)) applied to $\xi^\alpha\tau^j\{\xi,\tau\}^{-m}$; it is seen for the Besov spaces e.g. from the definition by difference norms, or by interpolation.

The spaces are defined over open subsets of $\mathbb{R}^{n+1}$ by restriction; here the cylindrical subsets $\Sigma=\Omega\times I$ with Ω open $\subset\mathbb{R}^n$ and I an open interval of $\mathbb{R}$ are particularly interesting. We use the notation

$$\begin{aligned} &\overline{H}_p^{(s,s/d)}(\Omega\times I) = r_{\Omega\times I}H_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}),\quad \overline{B}_p^{(s,s/d)}(\Omega\times I) = r_{\Omega\times I}B_p^{(s,s/d)}(\mathbb{R}^n\times\mathbb{R}),\\ &H_{p,\mathrm{loc}}^{(s,s/d)}(\Omega\times I) = \{u\in\mathcal{D}'(\Omega\times I)\mid \psi u\in\overline{H}_p^{(s,s/d)}(\Omega\times I) \text{ for any } \psi\in C_0^\infty(\Omega\times I)\}, \end{aligned} \tag{A.13}$$

and similar spaces $B_{p,\mathrm{loc}}^{(s,s/d)}(\Omega\times I)$, $\overline{W}_p^{(s,s/d)}(\Omega\times I)$, $W_{p,\mathrm{loc}}^{(s,s/d)}(\Omega\times I)$. Much of the above information, e.g. (A.7), (A.8), (A.10), carries over to the scales in the first line.

Let us also define anisotropic Hölder spaces. For $k\in\mathbb{N}_0$, $0<\sigma<1$, the usual Hölder space of order $s=k+\sigma$ over $\Omega\subset\mathbb{R}^n$, in our notation $\overline{C}^s(\Omega)$, is provided with the norm

$$\|u\|_{\overline{C}^s(\Omega)} = \sum\nolimits_{|\alpha|\le k}\|D^\alpha u\|_{L_\infty} + \sum\nolimits_{|\alpha|=k}\sup\nolimits_{x,x'\in\Omega}\frac{|D^\alpha u(x)-D^a u(x')|}{|x-x'|^\sigma}.$$

Integer values of s will be included in the scale by defining $\overline{C}^k(\Omega)$ for $k\in\mathbb{N}_0$ as the space of bounded continuous functions with bounded continuous derivatives up to order k on $\overline{\Omega}$. (Then we have $\overline{C}^s(\Omega)\subset\overline{C}^{s_1}(\Omega)$ whenever $s>s_1\ge 0$. There is a more refined choice of slightly larger spaces for $k\in\mathbb{N}_0$, the Hölder–Zygmund spaces, that fits better with interpolation theory, but which we shall not need here, since we only show results "with a loss of ε".)

We then define the anisotropic Hölder space over $\Omega\times I$ ($I\subset\mathbb{R}$) for $s>0$ by:

$$\overline{C}^{(s,s/d)}(\Omega\times I) = L_\infty(I;\overline{C}^s(\Omega))\cap\overline{C}^{s/d}(I;L_\infty(\Omega)),\ \text{when } s>0. \tag{A.14}$$

(Note that $\overline{C}^{s/d}(I;L_\infty(\Omega)) = L_\infty(\Omega;\overline{C}^{s/d}(I))$.) In view of the well-known embedding properties for isotropic spaces, we have with $\varepsilon > 0$:

$$\begin{aligned}
&\overline{C}^{(s,s/d)}(\Omega\times I) \subset \overline{H}_p^{(s-\varepsilon,(s-\varepsilon)/d)}(\Omega\times I) \text{ and } \overline{B}_p^{(s-\varepsilon,(s-\varepsilon)/d)}(\Omega\times I),\\
&\qquad\text{when } s>0,\\
&\overline{H}_p^{(s,s/d)}(\Omega\times I) \text{ and } \overline{B}_p^{(s,s/d)}(\Omega\times I) \subset \overline{C}^{(s-n/p-\varepsilon,(s-n/p-\varepsilon)/d)}(\Omega\times I),\\
&\qquad\text{if } s-n/p-\varepsilon>0;
\end{aligned}\tag{A.15}$$

in the first inclusion we assume $\Omega\times I$ to be bounded. Both inclusions are shown by comparing the spaces via (A.10) and (A.14). The first inclusion follows immediately from the isotropic case. For the second inclusion we can use that for small positive $\varepsilon_1 < s$:

$$\begin{aligned}
\overline{H}_p^{(s,s/d)}(\Omega\times I) &\subset \overline{H}_p^{\varepsilon_1/d}(I;\overline{H}_p^{s-\varepsilon_1}(\Omega)) \cap \overline{H}_p^{(s-\varepsilon_1)/d}(I;\overline{H}_p^{\varepsilon_1}(\Omega))\\
&\subset L_\infty(I;\overline{H}_p^{s-\varepsilon_1}(\Omega)) \cap \overline{H}_p^{(s-\varepsilon_1)/d}(I;L_\infty(\Omega)),
\end{aligned}$$

in order to relate to (A.14) (and similarly for B-spaces); this is allowed since $\langle\tau\rangle^{\varepsilon_1/d}\langle\xi\rangle^{s-\varepsilon_1}$ and $\langle\tau\rangle^{(s-\varepsilon_1)/d}\langle\xi\rangle^{\varepsilon_1}$ are $\le c(\langle\xi\rangle^{2d}+\tau^2)^{s/(2d)}$.

There is also the local version:

$$C_{\mathrm{loc}}^{(s,s/d)}(\Omega\times I) = \{u\in\mathcal{D}'(\Omega\times I) \mid \psi u\in\overline{C}^{(s,s/d)}(\Omega\times I) \text{ for any } \psi\in C_0^\infty(\Omega\times I)\}. \tag{A.16}$$

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