ON COERCIVENESS AND SEMIBOUNDEDNESS OF GENERAL BOUNDARY PROBLEMS

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ABSTRACT

The paper treats coerciveness inequalities (of the form $\mathrm{Re}(Au, u) \geqq c \|u\|_s^2 - \lambda \|u\|_0^2$, $c > 0$, $\lambda \in R$) and semiboundedness inequalities (of the form $\mathrm{Re}(Au, u) \geqq -\lambda \||u\||^2$) for the general boundary problems associated with an elliptic $2m$-order differential operator A in a compact n-dimensional manifold with boundary. In particular, we study the normal pseudo-differential boundary conditions, for which we determine necessary and sufficient conditions for coerciveness with $s = m$, and for semiboundedness with $\||u\|| = \|u\|_m$, in explicit form.

1. Introduction

Let A be a properly elliptic $2m$ order linear differential operator with C^∞ coefficients on an n-dimensional compact Riemannian manifold $\bar{\Omega}$ with boundary Γ ($\bar{\Omega} \setminus \Gamma$ denoted Ω). With $A_1 = A$ defined on $\{u \in L^2(\Omega) \mid Au \in L^2(\Omega)\}$; and $A_0 =$ the closure, as an operator in $L^2(\Omega)$, of A defined on $\mathscr{D}(\Omega)$, we call the linear operators $\tilde{A}$ in $L^2(\Omega)$ with $A_0 \subset \tilde{A} \subset A_1$ the realizations of A. In part of the paper we shall assume that the realization defined by the Dirichlet problem is bijective, which permits application of [11].

When $s \geqq 0$ we shall say that $\tilde{A}$ is s-coercive[1] if there exist $c > 0$, $\lambda \in \mathbb{R}$ such that (with the $L^2(\Omega)$ Sobolev norms)

$$\mathrm{Re}(Au, u) \geqq c\| u \|_s^2 - \lambda\| u \|_0^2, \text{ for all } u \in D(\tilde{A}) \tag{1.1}$$

(the case $s = 0$ is included for convenience). More generally, we say that $\tilde{A}$ satisfies a semiboundedness estimate if, for some norm $|||\cdot|||$, one has the estimate

1 This terminology is inspired by [20, Def. 2.9.2]; we use the term *regularity* for estimates like $\|u\|_s \leqq c(\|Au\|_t + \|u\|_{s-1})$, and *coerciveness* for estimates like (1.1).

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(1.2) $$\operatorname{Re}(Au,u) \geqq -\lambda\, |||u|||^2, \text{ for all } u \in D(\tilde{A});$$

it will in particular be studied for $|||u||| = \|u\|_m$, and for

$$|||u||| = (\|u\|_0^2 + \|A'u\|_{-m}^2)^{1/2}.$$

In Chapter 2, we introduce notations and collect the known results that our theory builds on.

In Chapter 3, we discuss (1.1) and (1.2) within the general framework of [11]. It was shown there how the set of closed realizations $\tilde{A}$ is in 1–1 correspondence with the set of closed, densely defined operators $L\colon X \to Y'$, where X and Y denote closed subspaces of $\prod_{j=0}^{m-1} H^{-j-1/2}(\Gamma)$; in such a way that each $\tilde{A}$ represents a specific boundary condition described in terms of the corresponding $L\colon X \to Y'$. Here $X = \overline{\gamma D(\tilde{A})}$ and $Y = \overline{\gamma D(\tilde{A}^*)}$, where γ denotes the Dirichlet boundary operator $\gamma = \{\gamma_0, \cdots, \gamma_{m-1}\}$, with $\gamma_j = (i^{-1}\partial/\partial n)^j$. Under the assumption that A equals its formal adjoint A' it was shown, in [11] for $s = 0$, and in [12] for $s \in [0, m]$, how (1.1) is related to a similar property of L. In the present paper we permit $A \neq A'$, and then we treat (1.1) and (1.2) in general only on

(1.3) $$D(\tilde{A}) \cap \{u \in L^2(\Omega) \mid A'u \in H^{-m}(\Omega)\},$$

since we need the Dirichlet problem of A' as well as of A to have a sense. We find, roughly speaking, that (1.2) with $|||u||| = \|u\|_m$ is characterized by: $X \subset Y$ and a related semiboundedness estimate for L; and that (1.1), when A is strongly elliptic, again requires $X \subset Y$, but now depends on the validity of a related coerciveness estimate for $L + Q$, where Q is a certain non-positive pseudo-differential operator in Γ, defined from A. ($Q = 0$ when $A = A'$.) (Theorem 3.4 and 3.6 give these statements with the relevant modifications; other estimates are also treated.)

In Chapter 4 the results are applied to realizations defined by normal boundary conditions:

(1.4) $$\gamma_j u - \sum_{k \in K,\, k<j} F_{jk}\gamma_k u = 0, \qquad j \in J;$$

here J and K denote complementing subsets of $\{0, 1, \cdots, 2m-1\}$, each consisting of m elements; and the F_{jk} denote pseudo-differential operators in Γ of orders $j - k$. We set

(1.5) $$D(\tilde{A}) = \{u \in D(A_1) \mid (1.4) \text{ holds}\}.$$

The main explicit results are here:

THEOREM I. (*Cf. Theorems* 4.1, 5.2.) *When A is properly elliptic, and $\tilde{A}$ is determined by* (1.5), *the following statements* (1.6)–(1.8) *are equivalent:*

$$\exists \lambda \in \mathbb{R} \text{ s.th. } \mathrm{Re}(Au,u) \geqq -\lambda \| u \|_m^2, \text{ for all } u \in D(\tilde{A}) \cap H^m(\Omega), \tag{1.6}$$

$$\gamma D(\tilde{A}) \subset \overline{\gamma D(\tilde{A}^*)}, \tag{1.7}$$

$$\begin{cases} J = \{ j \mid 2m - j - 1 \in K \}, \text{ and the matrix } (F_{jk})_{j \geqq m, k \geqq m} \text{ is a certain explicit} \\ \text{function of the matrix } (F_{jk})_{j<m,k<m} \ (cf. (4.66)). \end{cases} \tag{1.8}$$

In the course of the proof one finds that (1.6)–(1.8) are also equivalent with: $|(Au,v)| \leqq c \| u \|_m \| v \|_m$ on $D(\tilde{A}) \cap H^m(\Omega)$, and with: $\overline{\gamma D(\tilde{A})} = \overline{\gamma D(\tilde{A}^*)}$.

THEOREM II. (*Cf. Theorem* 4.3.) *When furthermore A is strongly elliptic, $\tilde{A}$ is m-coercive if and only if*

(i) *the equivalent conditions in Theorem I hold,*

(ii) *a certain matrix-valued function on $S(\Gamma)$, formed of the principal symbols on Γ of A and the F_{jk}, is positive definite.*

The function in (ii) is $\sigma^0(\mathscr{K})$, where $\mathscr{K}$ is a pseudo-differential operator in Γ, which in a sense represents the real part of $L + Q$. The restriction indicated in (1.3) is eliminated in these theorems by easy density arguments.

Theorem II solves completely the old problem of characterizing m-coerciveness of normal boundary problems; in Chapter 5 we compare this with previous results. It was solved by Agmon in [1] for the case where $\tilde{A}$ is associated with an integro-differential sesquilinear form $a(u,v)$ in such a way that

$$a(u,v) = (Au,v), \text{ for all } u,v \in D(\tilde{A}) \cap H^{2m}(\Omega); \tag{1.9}$$

however, the problem of expressing *when* (1.9) may be obtained was left unsolved. Another characterization, not using sesquilinear forms, of *selfadjoint* $\tilde{A}$ was given by Agmon in [2]. Recently Shimakura [26], Shimakura-Fujiwara [27] and Grubb [12] characterized, without the use of sesquilinear forms, the m-coercive realizations (1.5) where

$$J = \{0, 1, \cdots, m-p-1, \ m, m+1, \cdots, m+p-1\} \tag{1.10}$$

for some $p \in [0, m]$. ((1.10) is necessary for the stability of (1.1) w.r.t. perturbations of A.) All these partial characterizations concern principal symbols; the remarkable aspect of Theorem II is the condition (i), which concerns the full operators F_{jk} (not even just their symbols). It is trivially satisfied when (1.10) holds. For the case where the F_{jk} are *differential* operators, we investigate in

Chapter 5 the connection between (i) and Agmon's sesquilinear forms, and find that indeed (i) is necessary and sufficient for the existence of a form $a(u,v)$ fitting together with $\tilde{A}$ in (1.9).

The Appendix (Chapter 6) elaborates a statement in Chapter 2 about certain operators P in Γ ; a simple proof is given there that they are pseudo-differential operators, together with some explicit formulae concerning their principal symbols. (These were already used in [12], the proofs being deferred to a later paper.)

Theorem II and a weaker version of Theorem I were previously announced, for the case $A = A'$ in [13], and for general A in [14].

Chapter 5 was written after, and inspired by, a correspondence with Professor S. Agmon, to whom the author would like to express her gratitude.

2. Notations and preparatory theorems

2.1. *Spaces*. Throughout this paper we assume:

ASSUMPTION 2.1. $\bar{\Omega}$ is an infinitely differentiable n-dimensional compact Riemannian manifold with boundary Γ; $\bar{\Omega}\backslash\Gamma$ is denoted by Ω.

As it will sometimes be convenient, one may regard Ω as an open subset of a compact Riemannian manifold Σ without boundary, in which Ω has the C^∞ boundary Γ and the closure $\bar{\Omega}$. The generic points in Σ resp. Γ will be denoted x resp. y. In a neighborhood Σ_ε of Γ, the points may be represented in tangential and normal coordinates: $x = (y,t)$, where x denotes the point at the distance t from Γ on the geodesic through y (we take $t > 0$ in Ω, $t < 0$ in $\bar{\Sigma}\backslash\Omega$,

$$\Sigma_\varepsilon = \{(y,t) \mid y \in \Gamma,\ |t| < \varepsilon\}$$

for a suitable $\varepsilon > 0$). Thereby is defined a first order differential operator $D_t = i^{-1}\partial/\partial t$ in Σ_ε, which we call the normal derivative.

For a manifold Ξ, we denote by $\mathscr{D}(\Xi)$ the space of C^∞ functions on Ξ with compact support in Ξ. When $u \in \mathscr{D}(\Sigma_\varepsilon)$ or $\mathscr{D}(\bar{\Omega} \cap \Sigma_\varepsilon)$, we denote by $\gamma_0 u$ its restriction to Γ, $\gamma_0 u = u|_\Gamma \in \mathscr{D}(\Gamma)$; and by $\gamma_j u$ (j integer > 0) the function $\gamma_j u = \gamma_0(D_t^j u) \in \mathscr{D}(\Gamma)$.

The cotangent bundle of Σ will be denoted by $T^*(\Sigma)$, the subbundle obtained by suppressing the zero section by $T^*_\cdot(\Sigma)$, and the subbundle obtained by replacing the fibres by their unit spheres by $S(\Sigma)$. The restrictions to $\bar{\Omega}$ resp. Ω are denoted $T^*(\bar{\Omega})$ resp. $T^*(\Omega)$, etc., and the analogous bundles for Γ are denoted $T^*(\Gamma)$, $T^*_\cdot(\Gamma)$, $S(\Gamma)$. The generic element of $T^*(\Sigma)$ is (x,ξ), where ξ denotes a covector at the point x; analogously the generic element of $T^*(\Gamma)$ is (y,η).

The space $L^2(\bar{\Omega}) = L^2(\Omega)$ consists of the (equivalence classes of) complex valued square integrable functions on Ω w.r.t. the measure dx defined by the Riemannian metric, it is a Hilbert space with inner product and norm

$$(u,v) = \int_\Omega u\bar{v}\,dx \quad \text{resp.} \quad \| u \|_0 = (u,u)^{\frac{1}{2}}.$$

$L^2(\Gamma)$ is the analogous space for Γ, provided with the measure $d\sigma$ induced on Γ by the metric on $\bar{\Omega}$.

By the help of local coordinates one defines the Sobolev spaces $H^s(\Omega)$ and $H^s(\Gamma)$ for $s \in \mathbb{R}$, and $H_0^s(\Omega)$ for $s \geqq 0$ (cf. e.g. Lions-Magenes [20]); they are Hilbert spaces with norms denoted $\| u \|_s$, and for $s = 0$ we identify them with the Hilbert spaces $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. For $s \neq 0$ we prefer not to fix on beforehand the choice of norm (since, as is well known, there are various equally sensible ways of defining these norms), but recall that, for $s > 0$, the anti-duality between $H_0^s(\Omega)$ and $H^{-s}(\Omega)$, and the antiduality between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ (usually written with sharp brackets $\langle , \rangle$) coincide with the inner products in $L^2(\Omega)$ resp. $L^2(\Gamma)$, when they are applied to elements that also lie in $L^2(\Omega)$ resp. $L^2(\Gamma)$.

2.2. *Vector- and matrix-notation.* Throughout this paper we assume that m is a fixed positive integer. We denote by M, M_0 and M_1 the following ordered sets of integers

(2.1) $$M = \{0,1,\cdots,2m-1\},\ M_0 = \{0,\cdots,m-1\} \text{ and } M_1 = \{m,\cdots,2m-1\}$$

(so $M = M_0 \cup M_1$). When $N \subset M$, we denote

(2.2) $$N \cap M_0 = N_0,\ N \cap M_1 = N_1 \text{ and } \{j \,|\, 2m-1-j \in N\} = N'$$

(N' again considered as an ordered subset of M). The number of elements in N will be denoted $|N|$.

Let $J \subset M$ and $K \subset M$. A matrix $E = (E_{jk})_{j\in J,\ k\in K}$ indexed by $J \times K$ will be called a $J \times K$-matrix, and a vector $\phi = \{\phi_j\}_{j\in J}$ indexed by J will be called a J-vector.

Let $L \subset J \subset M$ and $N \subset K \subset M$. When E is a $J \times K$-matrix, and $\{j,k\} \in J \times K$, E_{jk} denotes the $\{j,k\}$th entry in E, and E_{LN} denotes the minor

(2.3) $$E_{LN} = (E_{jk})_{j\in L, k\in N}.$$

Similarly, when ϕ is a J-vector, ϕ_j is the jth entry, and ϕ_L is the vector

(2.4) $$\phi_L = \{\phi_j\}_{j\in L}.$$

We denote by I and 0 the unit resp, zero $M \times M$-matrices

(2.5) $$I = (\delta_{jk})_{j,k\in M}, \quad 0 = (0)_{j,k\in M}$$

(Then, in (2.4), $\phi_L = I_{LJ}\phi$.)

The vector notation will primarily be used in the following connection: Let $J \subset M$ and let $\{s_j\}_{j\in J}$ be a vector of real numbers. Then the elements ϕ of $\prod_{j\in J} H^{s_j}(\Gamma)$ are J-vectors with $\phi_j \in H^{s_j}(\Gamma)$. When a norm $\|\cdot\|_{s_j}$ in $H^{s_j}(\Gamma)$ is chosen for each s_j, the expression $(\sum_{j\in J}\|\phi_j\|^2_{s_j})^{\frac{1}{2}}$ defines a Hilbert space norm in the product space $\prod_{j\in J} H^{s_j}(\Gamma)$. Such a norm, and *any* Hilbert space norm in $\prod_{j\in J} H^{s_j}(\Gamma)$ equivalent with it, will be denoted

(2.6) $$\|\phi\|_{\{s_j\},j\in J},$$

where the "$j \in J$" may be omitted if it is understood from the text. The duality between $\prod_{j\in J} H^{s_j}(\Gamma)$ and $\prod_{j\in J} H^{-s_j}(\Gamma)$ will be denoted $<_{\{s_j\},\{-s_j\}}>$ or just $\langle\,,\rangle$.

Certain vectors of the boundary operators γ_j defined in Section 2.1 will be given special names:

(2.7) $$\rho = \{\gamma_j\}_{j\in M}, \ \gamma = \{\gamma_j\}_{j\in M_0}, \ \nu = \{\gamma_j\}_{j\in M_1},$$

here $\gamma = \rho_{M_0}$, $\nu = \rho_{M_1}$. We recall the classical "trace-theorem" (cf. Lions-Magenes [20])

PROPOSITION 2.1. *γ, defined on $\mathscr{D}(\bar{\Omega})$, extends by continuity to a mapping, also denoted by γ, which sends $H^s(\Omega)$ continuously onto $\prod_{j\in M_0} H^{s-j-\frac{1}{2}}(\Gamma)$ for all $s > m - \frac{1}{2}$; here $\{u \in H^s(\Omega) \,|\, \gamma u = 0\}$ equals $H_0^s(\Omega)$, when $m - \frac{1}{2} < s \leqq m$, and $H_0^m(\Omega) \cap H^s(\Omega)$, when $s \geqq m$.*

The matrix-notation will be used mainly on pseudo-differential operators in Γ and their symbols. We shall use the "classical" pseudo-differential operators (from now on abbreviated to ps.d.o.'s) introduced in Kohn-Nirenberg [18], Hörmander [16], [17], Seeley [25], to which we refer for details. Here, when P is a ps.d.o. in Γ, its symbol $\sigma(P)(y,\eta)$ is, in local coordinates, a formal series of functions on $T^*_\cdot(\Gamma)$,

$$\sigma(P)(y,\eta) = \sum_{l=0}^{\infty} p^l(y,\eta),$$

each p^l being C^∞ in y and homogeneous in η of degree r_l, the r_l forming a sequence of real numbers strictly decreasing towards $-\infty$. The principal symbol is $\sigma^0(P)(y,\eta) = p^0(y,\eta)$, also denoted $\sigma_{r_0}(P)(y,\eta)$ if one wants to emphasize the degree of homogeneity. Note that it is determined by its value on $S(\Gamma)$ and r_0. P has order r_0, i.e., is continuous from $H^{s+r_0}(\Gamma)$ into $H^s(\Gamma)$, all $s \in \mathbb{R}$.

DEFINITION 2.1. Let $J \subset M$, $K \subset M$, and let $\{t_j\}_{j\in J}$ and $\{s_k\}_{k\in K}$ be vectors of real numbers. Furthermore, let $P = (P_{jk})_{j\in J,k\in K}$ denote a $J \times K$-matrix of ps.d.o.'s P_{jk}. Then P will be said to be of *type* $(s_k, t_j)_{j\in J,k\in K}$ if each P_{jk} is of *order* $s_k - t_j$, for $\{j,k\} \in J \times K$. When this understood to hold, the principal symbol of P is defined as the $J \times K$-matrix

$$\sigma^0(P) = (\sigma_{s_k - t_j}(P_{jk}))_{j\in J,k\in K} \;\; [= (\sigma^0(P_{jk}))_{j\in J,k\in K}]. \tag{2.8}$$

We note that P being of type $(s_k, t_j)_{j\in J,\, k\in K}$ means that P is continuous from $\prod_{k\in K} H^{s_k+r}(\Gamma)$ into $\prod_{j\in J} H^{t_j+r}(\Gamma)$, for all $r \in \mathbb{R}$. Then P is also of type $(s_k + r,\, t_j + r)_{j\in J,\, k\in K}$, any $r \in \mathbb{R}$. Now, the adjoint of P is the $K \times J$-matrix of ps.d.o.'s

$$P^* = (Q_{jk})_{j\in K,\, k\in J}, \text{ where } Q_{jk} = P^*_{kj}, \tag{2.9}$$

P^* is of type $(-t_k, -s_j)_{j\in K,k\in J}$, and its principal symbol is $\sigma^0(P)^*$ (the conjugate transpose of $\sigma^0(P)$). When in particular $K = J$, and $s_j = -t_j$, each $j \in J$, then P^* is of the same type as P, and we define the "real" and "imaginary" parts of P by

$$\operatorname{Re} P = \tfrac{1}{2}(P + P^*), \quad \operatorname{Im} P = \frac{1}{2i}(P - P^*), \tag{2.10}$$

a similar notation will be used for the symbols.

Recall that P is said to be elliptic if $\sigma^0(P)(y,\eta)$ is injective at each point $(y,\eta) \in S(\Gamma)$. Let us further mention the following results on positive semidefiniteness and definiteness:

PROPOSITION 2.2. *Let $J \subset M$, and let P be a $J \times J$-matrix of ps.d.o.'s in Γ, of type $(s_k, -s_j)_{j,k\in J}$, where $\{s_j\}_{j\in J}$ is any real J-vector.*

(i) *If*

$$\operatorname{Re}\langle P\phi, \phi\rangle \geqq 0, \textit{ all } \phi \in \prod_{j\in J} \mathscr{D}(\Gamma), \tag{2.11}$$

then

$$\operatorname{Re}\sigma^0(y,\eta) \geqq 0 \textit{ for all } (y,\eta) \in S(\Gamma) \tag{2.12}$$

(*i.e.,* $\operatorname{Re}\sigma^0(y,\eta)$ *is positive semidefinite*).

(ii) *Let $r > 0$. In order that there exist $c > 0$, $\lambda \in \mathbb{R}$ such that*

$$\operatorname{Re}\langle P\phi, \phi\rangle \geqq c\|\phi\|^2_{\{s_j\}} - \lambda\|\phi\|^2_{\{s_j - r\}}, \textit{ all } \phi \in \prod_{\in J} \mathscr{D}(\Gamma), \tag{2.13}$$

it is necessary and sufficient that

$$\mathrm{Re}\,\sigma^0(P)(y,\eta) > 0 \text{ for all } (y,\eta) \in S(\Gamma) \tag{2.14}$$

(*i.e., for some* $c' > 0$, Re $\sigma^0(P)(y,\eta) - c' I_{JJ} \geqq 0$ *on* $S(\Gamma)$).

These statements are well known or are at least elementary consequences of well known theorems, see e.g. [12], Appendix.

We shall finally compose the boundary operators γ_j with ps.d.o.'s in Γ: *A (pseudo-) differential boundary operator* β is a composite operator

$$\beta = \sum_{l=0}^{k} B_l \gamma_l,$$

where the B_l are (pseudo-) differential operators in Γ. The order is the largest of the numbers $l +$ order of B_l. When $J \subset M$, *a normal system of (pseudo-) differential boundary operators of orders* $m_j, j \in J$, is a J-vector of boundary operators $\{\beta_j\}_{j \in J}$, where each β_j is of the form

$$\beta_j = b_j \gamma_{m_j} + \sum_{k < m_j} B_{jk} \gamma_k$$

with b_j and $1/b_j \in \mathscr{D}(\Gamma)$, and the B_{jk} denoting (pseudo-) differential operators in Γ of orders $m_j - k$, respectively, the m_j being distinct. (In contrast with previous papers we denote all boundary operators by small Greek letters.)

2.3 *The elliptic operator A.* Once and for all we assume

ASSUMPTION 2.2. *A is a 2m-order uniformly properly elliptic operator with C^∞ coefficients on $\bar{\Omega}$. Its symbol is $\sigma(A)(x,\xi)$; the principal symbol $\sigma^0(A)(x,\xi)$ will also be noted $a(x,\xi)$.*

With A are associated the following operators in $L^2(\Omega)$[2]:
the maximal operator A_1: A defined on the domain
$D(A_1) = \{u \in L^2(\Omega) \,|\, Au \in L^2(\Omega)$ in the distribution sense$\}$,
the minimal operator A_0: the closure of A defined on $\mathscr{D}(\Omega)$,
the realizations of A: all linear operators $\tilde{A}$ in $L^2(\Omega)$ satisfying

$$A_0 \subset \tilde{A} \subset A_1.$$

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[2] When S is an operator in a Hilbert space H, we denote by $D(S)$, $R(S)$ and $Z(S)$ its domain range and null-space, respectively. Moreover, we denote its numerical range

$$\nu(S) = \{(Su, u) \,|\, u \in D(S),\ \|u\|_H = 1\},$$

and its lower bound

$$m(S) = \inf \mathrm{Re}\, \nu(S).$$

The adjoint of S in H is denoted S^*.

The (formal) adjoint of A will be denoted A', the "real" part $\frac{1}{2}(A + A') = A^r$. Recall that, with A_1' and A_0' denoting the maximal resp. the minimal operator for A', one has

$$A_1' = A_0^*, \quad A_0' = A_1^*, \tag{2.15}$$

and therefore, that adjoints $\tilde{A}^*$ of realizations $\tilde{A}$ of A are realizations of A'. We also recall that, because of the ellipticity, $D(A_0) = D(A_0') = H_0^{2m}(\Omega)$, and $D(A_1) \subset H_{loc}^{2m}(\Omega)$. A special realization of A is the Dirichlet realization A_γ defined by

$$D(A_\gamma) = H_0^m(\Omega) \cap H^{2m}(\Omega).$$

A well known regularity theorem assures that the realization A_γ' of A' with the same domain satisfies

$$A_\gamma' = (A_\gamma)^*.$$

DEFINITION 2.2. A will be said to have uniquely solvable Dirichlet problem if A_γ is a bijection of $H_0^m(\Omega) \cap H^{2m}(\Omega)$ onto $L^2(\Omega)$.

Most of the results in the following will presume the validity of Definition 2.2, which makes the theory of [11] applicable. In general, A_γ has only finite dimensional kernel and cokernel; we have not made the effort to include this case in our general theory, but it is possible that it may be done by use of the technique of Lions and Magenes [20] and others, factoring out finite dimensional subspaces. Anyway, the main aim of the present paper is a discussion of inequalities that require at least semidefiniteness of $\sigma^0(A)$, in which case Definition 2.2 is satisfied after the addition of a constant to A. More precisely, we recall

PROPOSITION 2.3.

(i) (Gårding [15]) *In order that, with some* $c > 0$, $\lambda \in \mathbb{R}$,

$$\operatorname{Re}(Au, u) \geqq c\| u \|_m^2 - \lambda \| u \|_0^2, \text{ all } u \in H_0^m(\Omega) \cap H^{2m}(\Omega), \tag{2.17}$$

it is necessary and sufficient that $\operatorname{Re} a(x,\xi) > 0$ *for all* $(x,\xi) \in S(\bar{\Omega})$ (*i.e., A is strongly elliptic*).

(ii) (Agmon [3]) *If there exists* $\theta \in [0, 2\pi]$ *such that*

$$\frac{a(x,\xi)}{|a(x,\xi)|} \neq e^{i\theta}, \text{ all } (x,\xi) \in S(\bar{\Omega}) \tag{2.18}$$

then $A - r\, e^{i\theta}$ has uniquely solvable Dirichlet problem for sufficiently large $r > 0$.

(iii) *If there exists* $\lambda \in \mathbb{R}$ *such that*

$$\operatorname{Re}(Au, u) \geqq -\lambda \| u \|_0^2, \text{ all } u \in \mathscr{D}(\Omega), \tag{2.19}$$

then $\operatorname{Re} a(x, \xi) \geqq 0$ *on* $S(\bar{\Omega})$.

The last statement is related to and derivable from Proposition 2.2 (i). Note that Proposition 2.3 (ii) is *not* its converse, since (ii) does not give information on the numerical range.

We shall need one more observation on differential operators in $\bar{\Omega}$:

When C is a $2m$-order differential operator with C^∞ coefficients in $\bar{\Omega}$, then for $u \in H^{2m}(\Omega)$, $v \in H_0^m(\Omega)$,

$$|(Cu, v)| = |\langle Cu, v\rangle| \leqq \| Cu \|_{-m} \| v \|_m \leqq \text{const.} \| u \|_m \| v \|_m, \tag{2.20}$$

the sharp brackets denoting the duality between $H^{-m}(\Omega)$ and $H_0^m(\Omega)$.

2.4. *General trace- and decomposition theorems.*

Define, for each $s \in \mathbb{R}$, $t \in \mathbb{R}$ the spaces

$$\mathscr{H}_A^{s,t}(\Omega) = \{u \in H^s(\Omega) \mid Au \in H^t(\Omega)\}$$

and

$$Z_A^s(\Omega) = \{u \in H^s(\Omega) \mid Au = 0 \text{ in } \Omega\},$$

Au always taken in the distribution sense. Provided with the graph-norms

$$\| u \|_{\mathscr{H}_A^{s,t}(\Omega)} = (\| u \|_s^2 + \| Au \|_t^2)^{\frac{1}{2}}$$

resp.

$$\| u \|_{Z_A^s(\Omega)} = \| u \|_s,$$

the spaces are easily seen to be Hilbert spaces. Moreover we note that, when $s \geqq t + 2m$,

$$\mathscr{H}_A^{s,t}(\Omega) = H^s(\Omega) = \mathscr{H}_A^{s,s-2m}(\Omega),$$

with equivalent norms. Note also that $Z_A^s(\Omega)$ is a closed subspace of $\mathscr{H}_A^{s,t}(\Omega)$ for any t.

Lions and Magenes proved in [19, II and V] and [20]

Proposition 2.4. *Let* $J \subset M$ *and let* $\beta = \{\beta_j\}_{j \in J}$ *be a normal system of differential boundary operators of orders* j, $j \in J$. *Then* β, *originally defined on* $\mathscr{D}(\bar{\Omega})$, *extends by continuity to an operator, also denoted by* β, *which maps* $\mathscr{H}_A^{s,0}(\Omega)$ *continuously into* $\prod_{j \in J} H^{s-j-1/2}(\Gamma)$, *for each* $s \in \mathbb{R}$.

The operator $\gamma = \{\gamma_j\}_{j \in M_0}$ *may be extended even further, to an operator,* *also called* γ, *that maps* $\mathscr{H}_A^{s,-m}(\Omega)$ *continuously into* $\prod_{j\in M_0} H^{s-j-\frac{1}{2}}(\Gamma)$, *for each* $s \in \mathbb{R}$.

With this extended definition of γ, we shall present a general version of the fundamental existence, uniqueness and regularity theorem, due mainly to Nirenberg, for $s \geqq 2m$, and to Lions and Magenes, for $s \leqq 2m$.

THEOREM 2.1. *Assume that* A *satisfies Definition* 2.2.

(a) *For all pairs of real numbers* $\{s,t\}$, *where* $t \geqq -m$, $t \neq -\frac{1}{2}, -3/2, \cdots, -m+\frac{1}{2}$, *and* $s \leqq t+2m$, *the mapping* $\{A,\gamma\}$ *is an isomorphism of*

$$\mathscr{H}_A^{s,t}(\Omega) \text{ onto } H^t(\Omega) \times \prod_{j\in M_0} H^{s-j-\frac{1}{2}}(\Gamma).$$

(b) *For* $\{s,t\}$ *as in* (a), *and* $u \in \mathscr{H}_A^{s,t}(\Omega)$, *let* u_γ *denote the solution of*

$$Au_\gamma = Au, \quad \gamma u_\gamma = 0, \tag{2.21}$$

and let $u_\zeta = u - u_\gamma$, *then the decomposition*

$$u = u_\gamma + u_\zeta \tag{2.22}$$

decomposes $\mathscr{H}_A^{s,t}(\Omega)$ *into the topological direct sum*

$$\mathscr{H}_A^{s,t}(\Omega) = [H_0^m(\Omega) \cap H^{t+2m}(\Omega)] \dotplus Z_A^s(\Omega) \tag{2.23}$$

(*here*, $H_0^m(\Omega) \cap H^{t+2m}(\Omega)$ *is provided with the norm in* $H^{t+2m}(\Omega)$).

(c) *For all* $s \in \mathbb{R}$, γ *maps* $Z_A^s(\Omega)$ *isomorphically onto* $\prod_{j\in M_0} H^{s-j-\frac{1}{2}}(\Gamma)$.

PROOFS AND REFERENCES FOR THEOREM 2.1.

(b) and (c) are easy corollaries of (a); however, we shall use (b) and (c) in our explanation of the proof of (a).

For $s = t+2m$, t integer $\geqq 0$, (a) is a consequence of the regularity theory initiated by Nirenberg [21] (see also Schechter [23], Agmon-Douglis-Nirenberg [5]) stating that a distribution solution u of the problem

$$Au = f, \; \gamma u = \phi,$$

with $f \in H^t(\Omega)$, $\phi \in \prod_{j\in M_0} H^{t+2m-j-\frac{1}{2}}(\Gamma)$, is necessarily in $H^{t+2m}(\Omega)$. (They also showed that the problem satisfies the Fredholm alternative; then it is uniquely solvable when we assume Definition 2.2.) Their result was extended to integer $t \geqq -m$ by Peetre [22], and to non-integer t (excepting the values $-\frac{1}{2}, \cdots, -m+\frac{1}{2}$) by interpolation by Lions and Magenes (cf.[20]), this gives (a) for $s = t+2m$ with t real $\geqq -m$ avoiding certain values.

Next, (a) was proved for $t = 0$, $2m \geqq s > -\infty$ by Lions and Magenes, see

[19, V] and [20] (the values $s = \frac{1}{2}$ + integer were excepted in [19, V], but may be included by an application of the results in [20]).

Now, the validity of (a) for $t \geqq 0$, $s = t + 2m$, and for $t = 0$, $s \leqq 2m$, suffices to imply (c). After this, (b) is proved as follows:

Let $\{s, t\}$ be a pair satisfying the assumptions in the theorem. Then, evidently each summand in the right side of (2.23) is contained in $\mathscr{H}_A^{s,t}(\Omega)$. Conversely, when $u \in \mathscr{H}_A^{s,t}(\Omega)$, then $Au \in H^t(\Omega)$, which implies $u_\gamma \in H^{t+2m}(\Omega) \cap H_0^m(\Omega)$ (note that $t + 2m \geqq m$) by the abovementioned regularity theory. Since $s \leqq t + 2m$, $u_\zeta = u - u_\gamma \in H^s(\Omega)$; then, since $Au_\zeta = Au - Au_\zeta = 0$, $u_\zeta \in Z_A^s(\Omega)$. This shows the desired decomposition, which is unique because of (c). To complete the proof of (b) it remains to show that the decomposition is continuous both ways; this is easy and will be omitted.

Finally, one obtains the remaining part of (a) by combining (b) and (c) with the fact that, by the already proven part of (a), A maps $H_0^m(\Omega) \cap H^{2m+t}(\Omega)$ isomorphically onto $H^t(\Omega)$, for $t \geqq -m$, $t \neq -\frac{1}{2}, \cdots, -m + \frac{1}{2}$.

In connection with this theorem, we shall introduce some further notation.

DEFINITION 2.3. Let $\{s, t\}$ be as in Theorem 2.1. The projections $u \to u_\gamma$ and $u \to u_\zeta$, defined for $u \in \mathscr{H}_A^{s,t}(\Omega)$, will be denoted pr_γ resp. pr_ζ. The inverse of the isomorphism $\gamma \colon Z_A^s(\Omega) \to \prod_{j \in M_0} H^{s-j-\frac{1}{2}}(\Gamma)$ will be denoted γ_A^{-1}. When convenient, we indicate the dependence on A by writing instead pr_γ^A, pr_ζ^A and $(\gamma_Z^A)^{-1}$. However, when A is replaced by A' or $A^r = \frac{1}{2}(A + A')$ (then assumed elliptic etc.), we write

$$u = u_{\gamma'} + u_{\zeta'} \text{ resp. } u = u_{\gamma^r} + u_{\zeta^r}$$

and we denote the corresponding mapping pr'_γ, pr'_ζ and $(\gamma'_Z)^{-1}$, resp. pr^r_γ, pr^r_ζ and $(\gamma^r_Z)^{-1}$.

We use here tacitly that the definition of each of these operators is consistent for varying $\{s, t\}$, One may show furthermore, that γ is consistently defined for varying A:

LEMMA 2.1. *Let A and B be two properly elliptic operators of order $2m$, and let $s \leqq m$. Let γ^A and γ^B be the extensions of the classical operator γ, defined on $\mathscr{H}_A^{s,-m}(\Omega)$ resp. $\mathscr{H}_B^{s,-m}(\Omega)$ by Proposition 2.4. Then $\gamma^A u = \gamma^B u$ for $u \in \mathscr{H}_A^{s,-m}(\Omega) \cap \mathscr{H}_B^{s,-m}(\Omega)$.*

PROOF. Let $u \in \mathscr{H}_A^{s,-m}(\Omega) \cap \mathscr{H}_B^{s,-m}(\Omega)$. Then, since $Z_A^s(\Omega)$ and $Z_B^s(\Omega)$ lie in $C^\infty(\Omega)$, $u \in H_{loc}^m(\Omega)$. On the surfaces Γ_ε parallel to Γ in the distance ε, "γu" is therefore defined as an element $\gamma_{(\varepsilon)} u \in \prod_{j \in M_0} H^{m-j-\frac{1}{2}}(\Gamma_\varepsilon)$, by Proposition 2.1.

By theorem 2.8.1 in [20] (p. 207), $\gamma_{(\varepsilon)}u \to \gamma^A u$ as well as $\gamma^B u$ in $\prod_{j\in M_0} H^{s-j-\frac{1}{2}}(\Gamma)$ (with a suitable identification between Γ and Γ_ε, cf. [20]); thus $\gamma^A u = \gamma^B u$.

We now note that pr_ζ satisfies

$$pr_\zeta = \gamma_Z^{-1} \circ \gamma. \tag{2.24}$$

By this formula, pr_ζ may actually be defined on any space $\mathscr{H}_B^{s,t}(\Omega)$ as in Lemma 2.1. We shall show

PROPOSITION 2.5. *Let A and B be properly elliptic, of order $2m$, satisfying Definition 2.2. Let $s \in \mathbb{R}$. Then pr_ζ^A may be defined on $\mathscr{H}_B^{s,-m}(\Omega)$ by*

$$pr_\zeta^A = (\gamma_Z^A)^{-1} \circ \gamma;$$

this coincides on $\mathscr{H}_A^{s,-m}(\Omega) \cap \mathscr{H}_B^{s,-m}(\Omega)$ with the original definition. Moreover, pr_ζ^A is continuous from $\mathscr{H}_B^{s,-m}$ into $Z_A^s(\Omega)$, and it maps $Z_B^s(\Omega)$ isomorphically onto $Z_A^s(\Omega)$. Finally, when C is $2m$ order properly elliptic, satisfying Definition 2.2, then

$$pr_\zeta^A\, pr_\zeta^B u = pr_\zeta^A u, \text{ all } u \in \mathscr{H}^{s,-m}_C(\Omega). \tag{2.25}$$

PROOF. The first statements follow immediately by use of Lemma 2.1 and the properties of γ and $(\gamma_Z^A)^{-1}$ stated in Proposition 2.4 resp. Theorem 2.1 (c). For the last statement we note that

$$pr_\zeta^A pr_\zeta^B u = (\gamma_Z^A)^{-1}\gamma(\gamma_Z^B)^{-1}\gamma u = (\gamma_Z^A)^{-1}\gamma u = pr_\zeta^A u,$$

since $\gamma(\gamma_Z^B)^{-1}$ is the identity on $\prod_{j\in M_0} H^{s-j-\frac{1}{2}}(\Gamma)$.

(2.25) will later be used with B replaced by A' or A^{r}.

2.5. *Green's formulae, the operators P and μ.* Near Γ, we may write A in the form

$$A = \sum_{l=0}^{2m} A_l(t) D_t^l, \tag{2.26}$$

where each $A_l(t)$ is a differential operator of order $2m - l$ in Γ_t, the parallel surface to Γ in the distance t.

In particular, $A_{2m}(t)$ is a function, nonvanishing because of the ellipticity of A. We denote $A_l(0) = A_l$. Now $\sigma^0(A)$ may at points $y \in \Gamma$ be written

$$a(y,\eta,\tau) = \sum_{l=0}^{2m} a_l(y,\eta)\tau^l, \tag{2.27}$$

where $a_l(y,\eta) = \sigma^0(A_l)(y,\eta)$, and $\tau \in \mathbb{R}$. For each fixed $(y,\eta) \in T_*^*(\Gamma)$, the polynomial $a(y,\eta,\tau)$ in $\tau \in \mathbb{C}$ has, by the assumption of proper ellipticity, m roots $\{\tau_i^+(y,\eta)\}_{i=1}^m$ in $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda > 0\}$ and m roots $\{\tau_i^-(y,\eta)\}_{i=1}^m$ in $\mathbb{C}_- = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda > 0\}$. Let

$$a^+(y,\eta,\tau) = \prod_{i=1}^m (\tau - \tau_i^+(y,\eta)),\ a^-(y,\eta,\tau) = \prod_{i=1}^m (\tau - \tau_i^-(y,\eta)), \tag{2.28}$$

then

$$a(y,\eta,\tau) = A_{2m}(y)\, a^+(y,\eta,\tau)\, a^-(y,\eta,\tau). \tag{2.29}$$

The coefficients in a^+ resp. a^- will be denoted s_l^+resp. s_l^-:

$$a^+(y,\eta,\tau) = \sum_{l=0}^m s_l^+(y,\eta)\tau^l, \quad a^-(y,\eta,\tau) = \sum_{l=0}^m s_l^-(y,\eta)\tau^l. \tag{2.30}$$

Following Seeley [24] we find from (2.26) Green's formula

$$(Au, v) - (u, A'v) = \int_\Gamma \mathscr{A}\rho u \cdot \overline{\rho v}\, d\sigma, \quad u, v \in H^{2m}(\Omega), \tag{2.31}$$

where $\mathscr{A} = (\mathscr{A}_{jk})_{j,k \in M}$ is an $M \times M$-matrix of differential operators in Γ of the form

$$\mathscr{A}_{jk} = \begin{cases} iA_{j+k+1} + S_{jk} & \text{when } j+k+1 < 2m, \\ iA_{2m} & \text{when } j+k+1 = 2m, \\ 0 & \text{when } j+k+1 > 2m, \end{cases} \tag{2.32}$$

here the S_{jk} denote differential operators of orders $< 2m - (j+k+1)$. Note that the matrix $(iA_{2m})^{-1}\mathscr{A}$ is skew-triangular with ones in the second diagonal and zeroes below it. Therefore, $\mathscr{A}$ is invertible with its inverse $\mathscr{A}^{-1}$ again a differential operator, now having $(iA_{2m})^{-1}$ in its second diagonal and zeroes above it. A similar statement holds for any minor of the form $\mathscr{A}_{KK'}$, $K \subset M$; in particular it holds for $\mathscr{A}_{M_0 M_1}$.

Denoting the corresponding operators associated with A' resp. A^r (when it is elliptic) by $\mathscr{A}'$ resp. $\mathscr{A}^r$, we note (cf. (2.31))

$$\mathscr{A}' = -\mathscr{A}^*, \quad \mathscr{A}^r = \tfrac{1}{2}(\mathscr{A} + \mathscr{A}') \quad [= \tfrac{1}{2}(\mathscr{A} - \mathscr{A}^*) = i \operatorname{Im} \mathscr{A}!]. \tag{2.33}$$

We now introduce the particular boundary differential operators

$$\begin{cases} \chi = \mathscr{A}_{M_0 M_1} \nu + \tfrac{1}{2}\mathscr{A}_{M_0 M_0} \gamma \\ \chi' = \mathscr{A}'_{M_0 M_1} \nu + \tfrac{1}{2}\mathscr{A}'_{M_0 M_0} \gamma \quad [= -\mathscr{A}^*_{M_1 M_0} \nu - \tfrac{1}{2}\mathscr{A}^*_{M_0 M_0} \gamma]. \end{cases} \tag{2.34}$$

Then in view of (2.32), the formula (2.31) may be written in the form

$$(Au, v) - (u, A'v) = \langle \chi u, \gamma v \rangle - \langle \gamma u, \chi' v \rangle, \quad u, v \in H^{2m}(\Omega) \tag{2.35}$$

(where $\langle \ , \ \rangle$ denotes the inner product in $\prod_{j \in M_0} L^2(\Gamma)$ or suitable extensions). The choice (2.34) of χ and χ' is not so special, for we observe

LEMMA 2.2. *The pairs of normal systems of boundary differential operators* $\kappa = \{\kappa_j\}_{j \in M_0}$, $\kappa' = \{\kappa'_j\}_{j \in M_0}$, κ_j *and* κ'_j *of orders* $2m - j - 1$, $j \in M_0$, *with which*

$$(Au, v) - (u, A'v) = \langle \kappa u, \gamma v \rangle - \langle \gamma u, \kappa' v \rangle, \quad u, v \in H^{2m}(\Omega), \tag{2.36}$$

are exactly those of the form

$$\kappa = \chi + S\gamma, \ \kappa' = \chi' + S^*\gamma, \tag{2.37}$$

where S *runs through all differential operators in* Γ *of type*

$$(-k, -2m + j + 1)_{j,k \in M_0}.$$

The proof of this elementary fact amounts to a comparison of (2.36) with (2.35) for all $u, v \in H^{2m}(\Omega)$; details will be omitted.

It was noted by Lions and Magenes in [19, V] that the formula (2.35), with the extensions of definitions of γ and χ given in Proposition 2.4, extends as far as the orders of the boundary operators permit (and not further, cf. [11, Remark I.3.3]): When $s \in [0, 2m]$, (2.35) is valid for $u \in \mathscr{H}^{s,0}_A(\Omega)$ and $v \in \mathscr{H}^{2m-s,0}_{A'}(\Omega)$ (with the relevant interpretations of the sharp brackets). In order to have a Green's formula valid for $u \in \mathscr{H}^{0,0}_A(\Omega)$ and $v \in \mathscr{H}^{0,0}_{A'}(\Omega)$, we shall introduce an additional device.

DEFINITION 2.4. When A satisfies Definition 2.2, we denote by $P_{\gamma,\nu}$ the composite operator

$$P_{\gamma,\nu} = \nu \circ \gamma_Z^{-1}, \tag{2.38}$$

it maps $\prod_{k \in M_0} H^{s-k}(\Gamma)$ continuously into $\prod_{j \in M_1} H^{s-j}(\Gamma)$ for all $s \in \mathbb{R}$. To emphasize the connection with A we may write $P^A_{\gamma,\nu}$ instead of $P_{\gamma,\nu}$. However, we usually write, for the operators associated with A' and A^r (when it is elliptic etc.)

$$P'_{\gamma,\nu} = \nu(\gamma'_Z)^{-1}, \quad P^r_{\gamma,\nu} = \nu(\gamma^r_Z)^{-1}. \tag{2.39}$$

About such operators one has

PROPOSITION 2.6. $P^A_{\gamma,\nu}$ *is a ps.d.o. in* Γ *of type* $(-k, -j)_{j \in M_1, k \in M_0}$, *with principal symbol* $(p_{jk})_{j \in M_1, k \in M_0}$ *consisting, at each* $(y, \eta) \in T^*_\cdot(\Gamma)$, *of the coefficients in the rest polynomials*

(2.40) $$\sum_{k\in M_0} p_{jk}(y,\eta)\tau^k \equiv \tau^j \pmod{a^+(y,\eta,\tau)}, \quad j\in M_1.$$

This result is a consequence of the work of Boutet de Monvel [6], and was also proved explicitly by Vajnberg and Grusin [28], in both cases by means of a composition rule for boundary operators (like ν) and Poisson integral operators (like γ_Z^{-1}). Let us however also mention the observation, that it may be shown as an elementary consequence of the conceptually simpler result on the "Calderón-Seeley-projector" (Calderón [7], Seeley [24], Hörmander [17]). We describe this[2'] in the Appendix, which gives us an opportunity to derive some useful explicit formulae.

We also introduce

DEFINITION 2.5. When A satisfies Definition 2.2, and β is a normal system of pseudo-differential boundary operators of orders m_j, $j\in J$ (the $m_j\in M$), we denote

(2.41) $$P_{\gamma,\beta} = \beta \circ \gamma_Z^{-1};$$

it is a ps.d.o. in Γ of type $(-k, -m_j)_{j\in J, k\in M_0}$. In particular, we denote

(2.42) $$P_{\gamma,\chi} = \chi\gamma_Z^{-1}, \quad P'_{\gamma,\chi'} = \chi'(\gamma'_Z)^{-1}, \quad P^r_{\gamma,\chi^r} = \chi^r(\gamma^r_Z)^{-1};$$

the last definition requires A^r elliptic etc., and then $\chi^r = \frac{1}{2}(\chi+\chi')$.

Note that by (2.34)

(2.43) $$P_{\gamma,\chi} = \mathscr{A}_{M_0M_1}P_{\gamma,\nu} + \tfrac{1}{2}\mathscr{A}_{M_0M_0},$$

with analogous formulae with "$'$" and "r".

DEFINITION 2.6. Let A be a $2m$ order properly elliptic operator. Then we define the (non-normal) pseudo-differential boundary operators μ, μ' and μ^r on $\mathscr{H}^{s,0}_A(\Omega)$ $(s\in\mathbb{R})$, when the respective ps.d.o.'s $P_{\gamma,\chi}$, $P'_{\gamma,\chi'}$ or P^r_{γ,χ^r} are defined:

(2.44) $$\mu = \chi - P_{\gamma,\chi}\gamma, \quad \mu' = \chi' - P'_{\gamma,\chi'}\gamma, \quad \mu^r = \chi^r - P^r_{\gamma,\chi^r}\gamma.$$

Note the formula, easily seen from (2.34) and (2.43)

(2.45) $$\mu = \mathscr{A}_{M_0M_1}(\nu - P_{\gamma,\nu}\gamma).$$

The fundamental properties of μ are expressed in the following statement, proved in [11, theorem III 1.2] (where μ was called M):

PROPOSITION 2.7. *Assume that A satisfies Definition 2.2. Consider μ, restricted*

[2'] The proof was constructed at a time where we needed the result but could not find it in the literature. We have later become aware that related ideas have been known for some time.

to $D(A_1) = \mathscr{H}_A^{0,0}(\Omega)$. *It maps* $D(A_1)$ *continuously onto* $\prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma)$, *and may alternatively be defined by*

$$\mu u = \chi \; pr_\gamma u, \tag{2.46}$$

or

$$\langle \underset{\{j+\frac{1}{2}\}}{\mu u}, \underset{\{-j-\frac{1}{2}\}}{\phi} \rangle = (Au, (\gamma_Z')^{-1}\phi), \text{ all } \phi \in \prod_{j \in M_0} H^{-j-\frac{1}{2}}(\Gamma). \tag{2.47}$$

The kernel of μ: $D(A_1) \to \prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma)$ *is* $D(A_0) + Z(A_1)$. *One has the general Green's formula*

$$(Au, v) - (u, A'v) = \langle \mu u, \gamma v \rangle - \langle \gamma u, \mu' v \rangle, \quad \text{all } u \in D(A_1), v \in D(A_1'). \tag{2.48}$$

Comparison of (2.48) with (2.35) for all $u, v \in H^{2m}(\Omega)$ gives

COROLLARY 2.7. *For* $P_{\gamma,\chi}$ *and* $P'_{\gamma,\chi'}$ *defined* in *Definition* 2.5,

$$P'_{\gamma,\chi'} = P^*_{\gamma,\chi}.$$

We remark however, that P^r_{γ,χ^r} is in general different from Re $P_{\gamma,\chi} = \frac{1}{2}(P_{\gamma,\chi} + P^*_{\gamma,\chi})$

3. General theory

3.1. *Resumé of old results.* We assume throughout this chapter:

ASSUMPTION 3.1. A has uniquely solvable Dirichlet problem (cf. Definition 2.2).

With A_0, A_γ and A_1 defined as in Section 2.3, Theorem 2.1(b) for $s = t = 0$ may be expressed as follows:

LEMMA 3.1. *By the projections* pr_γ *and* pr_ζ *defined in Definition* 2.3, $D(A_1)$ *is decomposed into the topological direct sum (with respect to the graph-topologies)*

$$D(A_1) = D(A_\gamma) \dotplus Z(A_1). \tag{3.1}$$

pr'_γ *and* pr'_ζ *decompose* $D(A_1')$ *similarly*:

$$D(A_1') = D(A_\gamma^*) \dotplus Z(A_1'). \tag{3.2}$$

With this as basis, the author showed in [11]

PROPOSITION 3.1. *Let* $\tilde{A}$ *be a closed realization of* A. *Let*

$$V = \overline{pr_\zeta D(\tilde{A})}, \quad W = \overline{pr'_\zeta D(\tilde{A}^*)}; \tag{3.3}$$

they are closed subspaces of $Z(A_1)$ *resp.* $Z(A_1')$. *Then there exists a uniquely determined closed, densely defined operator* T: $V \to W$ *such that*

(3.4) $$D(\tilde{A}) = \{u \in D(A_1) \,|\, u_\zeta \in D(T),\ (Au, w) = (Tu_\zeta, w) \textit{ for all } w \in W\},$$

here $D(T) = pr_\zeta D(\tilde{A})$.

Conversely, if V *and* W *are any closed subspaces of* $Z(A_1)$ *resp.* $Z(A_1')$, *and* T *is any closed, densely defined operator from* V *into* W, *then* (3.4) *determines a unique closed realization* $\tilde{A}$ *of* A, *such that* $T\colon V \to W$ *is exactly the operator derived from* $\tilde{A}$ *in the above fashion.*

When $\tilde{A}$ *corresponds to* T *in this way, the general element of* $D(\tilde{A})$ *is decomposed uniquely as*

(3.5) $$u = v + A_\gamma^{-1}(Tz + f) + z,$$

where $[v, z, f]$ *runs through* $D(A_0) \times D(T) \times (Z(A_1') \ominus W)$. *Moreover, the realization* $\tilde{A}^*$ *of* A' *then corresponds to the adjoint* $T^*\colon W \to V$ *by*

(3.6) $$D(\tilde{A}^*) = \{u \in D(A_1') \,|\, u_{\zeta'} \in D(T^*), (A'u, v) = (T^*u_{\zeta'}, v), \forall v \in V\}.$$

The correspondence introduced above carries numerous properties, —dimension of nullspace, closedness of range, codimension of range —to mention a few (cf. [11]). The property we are interested in, "s-coerciveness", was treated in [12] for the case where $A' = A$:

PROPOSITION 3.2. *Assume that* A *is strongly elliptic, with* $A' = A$ *and*

(3.7) $$(Au, u) \geqq c_m \| u \|_m^2, \quad c_m > 0, \quad \textit{all } u \in H_0^{2m}(\Omega)$$

(then also $m(A_0) > 0$, *cf.*^{2}*). Let* $\tilde{A}$ *be a closed realization of* A, *corresponding by Proposition* 3.1 *to* $T\colon V \to W$. *With a real number* $s \in [0, m]$, *we consider the two assertions*

(3.8) $$\exists\, c > 0, \lambda \in \mathbb{R} \textit{ s.th. } \operatorname{Re}\,(Au, u) \geqq c\| u \|_s^2 - \lambda \| u \|_0^2, \quad \forall u \in D(\tilde{A}).$$

(3.9) $$\begin{cases} \text{(i)} \ V \subset W \\ \text{(ii)} \ \exists c' > 0,\ \lambda' \in \mathbb{R} \textit{ s.th. } \operatorname{Re}\,(Tz, z) \geqq c'\| z \|_s^2 - \lambda' \| z \|_0^2,\ \forall z \in D(T). \end{cases}$$

Here, (3.8) *implies* (3.9) *for all* $s \in [0, m]$, *and* (3.9) *implies* (3.8) *for all* $s \in\,]m - \frac{1}{2}, m]$. *When* $s \in [0, m - \frac{1}{2}]$, (3.9) *implies* (3.8) *if furthermore* $\lambda' < m(A_0)$.

This result was proved for $s = 0$ in [11], and for $s \in [0, m]$ in [12] (cf. the proof of Proposition 2.7 there). The direction (3.8) $\Rightarrow$ (3.9) uses that when $u \in D(\tilde{A})$, $u = u_\gamma + u_\zeta$ where $u_\zeta \in D(T)$ and u_γ may be brought to converge to 0 in $H_0^m(\Omega)$, since $D(\tilde{A}) \supset D(A_0) = H_0^{2m}(\Omega)$ which is dense in $H_0^m(\Omega)$. The converse direction uses a splitting of $(\tilde{A}u, u)$ that holds when $V \subset W$.

Let us point out that it is the very natural property $V \subset W$, necessary for the validity of (3.8), that leads to the seemingly previously unnoticed "global" condition for semiboundedness of realizations of normal boundary problems, as explained in Chapter 4.

3.2. *New results, formulated for the correspondence between $\tilde{A}$ and T.*

The main aim of the present chapter is to generalize Proposition 3.2 to non-selfadjoint A. As it will be seen, the splitting we use in the general case does not work on the full domain $D(A_1)$. On the other hand, we exploit the technique of the proof of (3.8) $\Rightarrow$ (3.9) much further, to show how a condition generalizing $V \subset W$ is necessary even for very weak kinds of semiboundedness, that do not require (semi-) definiteness of $\sigma^0(A)$.

In the rest of this chapter, we shall always assume:

ASSUMPTION 3.2. $\tilde{A}$ is a closed realization of A, corresponding to $T\colon V \to W$ by Proposition 3.1.

LEMMA 3.2. *Let* $u \in D(\tilde{A}) \cap \mathscr{H}_A^{0,-m}(\Omega)$. *With* $u = v + A_\gamma^{-1}(Tz+f) + z$ *according to* (3.5) *in Proposition* 3.1, *and*

$$u = u_{\gamma'} + u_{\zeta'}, \text{ where } u_{\zeta'} = pr'_\zeta z, \tag{3.10}$$

according to Definition 2.3 *and Proposition* 2.5, *one has*

$$(Au,u) = (Au,u_{\gamma'}) + (Tz, pr'_\zeta z) + (f, pr'_\zeta z). \tag{3.11}$$

PROOF.

$$\begin{aligned}(Au,u) &= (Au,u_{\gamma'}) + (Au,u_{\zeta'}) \\ &= (Au,u_{\gamma'}) + (Av + Tz + f,\ pr'_\zeta z) \\ &= (Au,u_{\gamma'}) + (Tz, pr'_\zeta z) + (f, pr'_\zeta z),\end{aligned}$$

where $(Av, pr'_\zeta z) = 0$ since $Av \in R(A_0) \perp Z(A'_1)$, cf. (2.15).

THEOREM 3.1. *Let U be a linear space with $H^{2m}(\Omega) \subset U \subset H^m(\Omega)$. Then the following statements* (3.12) *and* (3.13) *are equivalent*:

$$\exists \lambda \in \mathbb{R} \text{ s.th. } \operatorname{Re}(Au,u) \geqq -\lambda \| u \|_m^2, \quad \forall u \in D(\tilde{A}) \cap U. \tag{3.12}$$

$$\begin{cases} \text{(i)} & pr'_\zeta(D(T) \cap U) \subset W \\ \text{(ii)} & \exists \lambda' \in \mathbb{R} \text{ s.th. } \operatorname{Re}(Tz, pr'_\zeta z) \geqq -\lambda' \| z \|_m^2, \quad \forall z \in D(T) \cap U. \end{cases} \tag{3.13}$$

PROOF. We use the decomposition

$$u = v + A_\gamma^{-1}(Tz+f) + z. \tag{3.5}$$

Note first of all that u runs through $D(\tilde{A}) \cap U$ exactly when $[v, z, f]$ runs through

$D(A_0) \times (D(T) \cap U) \times (Z(A_1) \ominus W)$, since $v + A_\gamma^{-1}(Tz + f) \in D(A_\gamma) \subset H^{2m}(\Omega) \subset U$.

1° (3.12) $\Rightarrow$ (3.13). Let $u \in D(\tilde{A}) \cap U$, decomposed by (3.5). Let $\{w^n\}_{n\in N}$ be a sequence in $D(A_0) = H_0^{2m}(\Omega)$, converging to $-v - A_\gamma^{-1}(Tz+f)$ in $H_0^m(\Omega)$. Then $u^n = u + w^n$ belongs to $D(\tilde{A}) \cap U$, and

$$\begin{aligned} u_\gamma^n &= u_\gamma + w^n \to 0 \text{ in } H_0^m(\Omega), \\ u_\zeta^n &= z \in U \subset H^m(\Omega), \\ u_{\zeta'}^n &= pr'_\zeta z \in H^m(\Omega) \text{ (cf. (2.25))}, \end{aligned}$$

and $u_{\gamma'}^n = u_\gamma^n + pr'_\gamma z \to pr'_\gamma z$ in $H_0^m(\Omega)$. (3.12) and Lemma 3.2 give

$$\text{(3.14)} \qquad \mathrm{Re}[(Au^n, u_{\gamma'}^n) + (Tz, pr'_\zeta z) + (f, pr'_\zeta z)] \geqq -\lambda \| u^n \|_m^2.$$

Here $|(Au^n, u_{\gamma'}^n)| = |(Au_\gamma^n, u_{\gamma'}^n)| \leqq c \| u_\gamma^n \|_m \; \| u_{\gamma'}^n \|_m$ (cf. (2.20)), and therefore goes to zero as $n \to \infty$. Thus, using that $u^n \to z$ in $H^m(\Omega)$,

$$\text{(3.15)} \qquad \mathrm{Re}[(Tz, pr'_\zeta z) + (f, pr'_\zeta z)] \geqq -\lambda \| z \|_m^2.$$

For each fixed pair $[z, f] \in (D(T) \cap U) \times (Z(A'_1) \ominus W)$ we find by inserting kf, $k \in \mathbb{C}$, in (3.15), that one must have

$$\text{(3.16)} \qquad (f, pr'_\zeta z) = 0,$$

i.e., since $pr'_\zeta(D(T) \cap U) \subset Z(A'_1)$,

$$\text{(3.17)} \qquad pr'_\zeta(D(T) \cap U) \subset W.$$

Inserting (3.16) in (3.15) we now also have

$$\text{(3.18)} \qquad \mathrm{Re}\,(Tz, pr'_\zeta z) \geqq -\lambda \| z \|_m^2, \quad z \in D(T) \cap U.$$

2° (3.13) $\Rightarrow$ (3.12). When (3.13) holds, we have, by Lemma 3.2, for $u \in D(\tilde{A}) \cap U$

$$\begin{aligned} \mathrm{Re}\,(Au, u) &= \mathrm{Re}\,(Au_\gamma, u_{\gamma'}) + \mathrm{Re}\,(Tz, pr'_\zeta z) \\ &\geqq -c_1 \| u_\gamma \|_m \| u_{\gamma'} \|_m - \lambda' \| z \|_m^2, \end{aligned}$$

cf. (2.20). Here, $\| u_\gamma \|_m \leqq c_2 \| u \|_m$, $\| u_{\gamma'} \|_m \leqq c_3 \| u \|_m$ and $\| z \|_m \leqq c_4 \| u \|_m$ for $u \in H^m(\Omega)$, by various applications of Theorem 2.1(b) (with $s = m, t = -m$), so that finally

$$\mathrm{Re}\,(Au, u) \geqq -c_5 \| u \|_m^2, \text{ when } u \in D(\tilde{A}) \cap U,$$

for some $c_5 \in \mathbb{R}$. This proves the theorem.

REMARK 3.1. Note, for one thing, that (3.12) does not require strong ellipticity of A, in fact it holds on $H^{2m}(\Omega) \cap H_0^m(\Omega)$ for any smooth $2m$-order operator, cf. (2.20). Secondly, for the normal boundary problems considered in Chapter 4, (3.13) (ii) is automatically satisfied. Then the above theorem singles out the exact role of (3.13) (i), as a necessary and sufficient condition for the rather weak inequality (3.12).

COROLLARY 3.1. *Let U be as in Theorem 3.1; then, when T belongs to the class of operators satisfying $|(Tz, pr'_\zeta z)| \leqq c\| z \|_m^2$ on $D(T) \cap U$ (some $c > 0$),*

$$(3.19) \qquad |(Au, v)| \leqq c' \| u \|_m \| v \|_m, \qquad \forall u, v \in D(\tilde{A}) \cap U$$

is equivalent with

$$pr'_\zeta(D(T) \cap U) \subset W.$$

PROOF. Apply Theorem 3.1 to $e^{i\theta}\tilde{A}$, all $\theta \in [0, 2\pi]$, noting the equivalence between (3.19) and

$$|(Au, u)| \leqq c' \| u \|_m^2, \quad \forall u \in D(\tilde{A}) \cap U.$$

The next inequality is also independent of requirements on $\sigma^0(A)$, however, it gives a nontrivial condition on T, also when $\tilde{A}$ is as in Chapter 4 ((3.21) (ii)′ below).

THEOREM 3.2. *Let U be a linear space with $H^{2m}(\Omega) \subset U \subset \mathscr{H}_{A'}^{0,-m}(\Omega)$. If there exists $\lambda \in \mathbb{R}$ such that*

$$(3.20) \qquad \mathrm{Re}\,(Au, u) \geqq -\lambda(\| u \|_0^2 + \| A'u \|_{-m}^2) \text{ on } D(\tilde{A}) \cap U,$$

then

$$(3.21) \qquad \begin{cases} \text{(i)} \quad pr'_\zeta(D(T) \cap U) \subset W; \\ \text{(ii)} \quad \mathrm{Re}\,(Tz, pr'_\zeta z) \geqq -\lambda(\| z \|_0^2 + \| A'z \|_{-m}^2), \ \forall z \in D(T) \cap U; \\ \text{(ii)}' \ \mathrm{Re}\,(Tz, pr'_\zeta z) \geqq -\lambda \| pr'_\zeta z \|_0^2 \geqq -\lambda c \| z \|_0^2, \ \forall z \in D(T) \cap U, \end{cases}$$

for a certain $c > 0$.

PROOF. In analogy with (3.14) we now have

$$(3.22) \qquad \mathrm{Re}\,[(Au^n, u^n_{\gamma'}) + (Tz, pr'_\zeta z) + (f, pr'_\zeta z)] \geqq -\lambda(\| u^n \|_0^2 + \| A'u^n \|_{-m}^2),$$

where $u \in D(\tilde{A}) \cap U$, and $u^n = u + w^n$, $w^n \in D(A_0)$. Letting $w^n \to -u_\gamma = -v - A_\gamma^{-1}(Tz + f)$ in $H_0^m(\Omega)$, we have that $u^n_\gamma \to 0$ in $H_0^m(\Omega)$, $u^n_{\gamma'} = u^n_\gamma + pr'_\gamma z \to pr'_\gamma z$ in $H_0^m(\Omega)$ (since $z \in \mathscr{H}_{A'}^{0,-m}(\Omega)$); and $u^n \to z$ in $H^0(\Omega)$, with $A'u^n = A'u^n_{\gamma'} \to A' \ pr'_\gamma z = A'z$ in $H^{-m}(\Omega)$. Altogether, this gives

$$\mathrm{Re}\,[(Tz, pr'_\zeta z) + (f, pr'_\zeta z)] \geqq -\lambda(\| z \|_0^2 + \| A'z \|_{-m}^2),$$

for $[z,f] \in (D(T) \cap U) \times (Z(A_1') \ominus W)$, from which (i) and (ii) follow as in Theorem 3.1.

To obtain (ii)′, we let instead $w^n \to -u_{\gamma'} = -u_\gamma - pr_\gamma' z$ in $H_0^m(\Omega)$. Then $u_{\gamma'}^n \to 0$ in $H_0^m(\Omega)$, $u_\gamma^n \to -pr_\gamma' z$ in $H_0^m(\Omega)$, $u^n \to pr_\zeta' z$ in $H^0(\Omega)$ and $A'u^n = A'u_{\gamma'}^n \to 0$ in $H^{-m}(\Omega)$. Altogether, (3.22) gives by passing to the limit (and using (i))

$$\operatorname{Re}\,(Tz, pr_\zeta' z) \geqq -\lambda \| pr_\zeta' z \|_0^2.$$

The remaining part follows by using that pr_ζ' is continuous from $Z_A^0(\Omega)$ to $Z_{A'}^0(\Omega)$, cf. Proposition 2.5.

We now turn to inequalities of the kind treated in Proposition 3.2.

DEFINITION 3.1. Let $s \geqq 0$, and let $U \subset L^2(\Omega)$. An operator S in $L^2(\Omega)$ will be said to be s-coercive on U if there exist $c > 0$, $\lambda \in \mathbb{R}$ such that

$$\operatorname{Re}\,(Su, u) \geqq c\| u \|_s^2 - \lambda \| u \|_0^2, \; u \in D(S) \cap U. \tag{3.23}$$

The case $s = 0$, where the terminology is somewhat unjustified, has been included for convenience. Note that (3.23) in particular means that $D(S) \cap U \subset H^s(\Omega)$.

By Proposition 2.3 (iii), s-coerciveness of a realization $\tilde{A}$ requires at least semi-definiteness of $\sigma^0(A)$, and then Assumption 3.1 becomes trivial, cf. Proposition 2.3(ii). But $\operatorname{Re}\sigma^0(A) \geqq 0$ does not (to our knowledge) imply even 0-coerciveness of A_0. (According to a theorem of Hörmander [17], $\operatorname{Re}\sigma_{2m}(C) \geqq 0$ is necessary and sufficient for the inequality $\operatorname{Re}\,(Cu,u) \geqq -\lambda \| u \|_{m-\frac{1}{2}}^2$ on $\mathscr{D}(\Omega')$, each $\bar{\Omega}' \subset \Omega$, when C is any $2m$ order operator.) In our search for simultaneously necessary and sufficient conditions for s-coerciveness of *realizations*, we shall let this aspect lie and simply assume strong ellipticity of A. We shall also assume a sufficiently large constant added to A such that

$$\operatorname{Re}\,(Au, u) \geqq c_m \| u \|_m^2, \quad c_m > 0, \quad \forall u \in D(A_0). \tag{3.24}$$

Let $H^{2m}(\Omega) \subset U \subset \mathscr{H}_{A'}^{0,-m}(\Omega)$, then when $pr_\zeta'(D(T) \cap U) \subset W$, we have found that for $u \in D(\tilde{A}) \cap U$

$$\begin{aligned} (Au, u) &= (Au_\gamma, u_{\gamma'}) + (Tz, pr_\zeta' z) \\ &= (Au_\gamma, u_\gamma) + (Au_\gamma, pr_\gamma' z) + (Tz, pr_\zeta' z). \end{aligned} \tag{3.25}$$

In the case $A = A'$, $pr_\gamma' z = 0$, so (Au, u) is split by (3.25) into a quadratic form in u_γ and a quadratic form in z; this led to Proposition 3.2 by use of (3.24). However, when $A \neq A'$, the mixed term $(Au_\gamma,\ pr_\gamma' z)$ prevents us from getting truly necessary and sufficient condition for s-coerciveness of $\tilde{A}$ on U in terms of similar estimates on $(Tz,\ pr'_\zeta z)$. This necessitates the following development:

Recall that $A^r = \frac{1}{2}(A + A')$ is strongly elliptic and satisfies (3.24) when A does, and recall Definition 2.3. Note that

$$D(A_1) \cap \mathscr{H}^{0,-m}_{A'}(\Omega) = D(A_1) \cap \mathscr{H}^{0,-m}_{A^r}(\Omega). \tag{3.26}$$

LEMMA 3.3. *Let A be strongly elliptic satisfying (3.24). For $u \in D(A_1) \cap \mathscr{H}^{0,-m}_{A'}(\Omega)$,*

$$\operatorname{Re}\,(Au, u_{\gamma'}) = \operatorname{Re}\langle A\,u_{\gamma^r},\, u_{\gamma^r}\rangle + \operatorname{Re}\langle A\,u_{\zeta^r}, pr'_\gamma u_{\zeta^r}\rangle, \tag{3.27}$$

the sharp brackets denoting the duality between $H^{-m}(\Omega)$ and $H^m_0(\Omega)$.

PROOF. Set $u_{\gamma^r} = x$, $u_{\zeta^r} = y$, then $x \in H^m_0(\Omega)$ and $y \in Z^0_{A^r}(\Omega) \cap \mathscr{H}^{0,-m}_{A'}(\Omega)$, by Theorem 2.1(b). Then

$$u_\gamma = pr_\gamma x + pr_\gamma y = x + pr_\gamma y$$
$$u_{\gamma'} = pr'_\gamma x + pr'_\gamma y = x + pr'_\gamma y,$$

where also $pr_\gamma y$ and $pr'_\gamma y$ lie in $H^m_0(\Omega)$; and thus

$$\begin{aligned}\operatorname{Re}(Au_\gamma, u_{\gamma'}) &= \operatorname{Re}\,(A(x + pr_\gamma y),\, x + pr'_\gamma y)\\ &= \operatorname{Re}\,\langle Ax + A\,pr_\gamma y, x + pr'_\gamma y\rangle\\ &= \operatorname{Re}\,\langle Ax, x\rangle + \operatorname{Re}\,[\langle Ax, pr'_\gamma y\rangle + \langle A\,pr_\gamma y, x\rangle]\\ &\quad + \operatorname{Re}\,\langle A\,pr_\gamma y,\, pr'_\gamma y\rangle.\end{aligned}$$

We now observe that $A\,pr_\gamma y = Ay$ ($\in H^{-m}(\Omega)$) and that the term in [ ] equals

$$\begin{aligned}\operatorname{Re}[\langle Ax, pr'_\gamma y\rangle + \langle Ay, x\rangle] &= \operatorname{Re}[\overline{\langle A' pr'_\gamma y, x\rangle} + \langle Ay, x\rangle]\\ &= \operatorname{Re}[\langle A'y, x\rangle + \langle Ay, x\rangle] = \operatorname{Re}\langle 2A^r y, x\rangle = 0,\end{aligned}$$

since $y \in Z^0_{A^r}(\Omega)$. Thus finally

$$\operatorname{Re}\,(Au_\gamma, u_{\gamma'}) = \operatorname{Re}\langle Ax, x\rangle + \operatorname{Re}\langle Ay, pr'_\gamma y\rangle,$$

as was to be shown.

By use of Lemmas 3.2 and 3.3 and the fact that $pr^r_\zeta u = pr^r_\zeta\, pr_\zeta u = pr^r_\zeta z$ (cf. Proposition 2.5) we then obtain

PROPOSITION 3.3. *Let A be strongly elliptic satisfying (3.24). For $u \in D(\tilde{A}) \cap \mathscr{H}^{0,-m}_{A'}(\Omega)$, with $u = v + A_\gamma^{-1}(Tz + f) + z$ according to Proposition 3.1,*

$$\text{(3.28)} \qquad \begin{aligned} \operatorname{Re}(Au,u) &= \operatorname{Re}\langle Au_{\gamma r}, u_{\gamma r}\rangle + \operatorname{Re}(f, pr'_\zeta z) \\ &\quad + \operatorname{Re}\left[(Tz, pr'_\zeta z) + \langle A\, pr^r_\zeta z, pr'_\gamma pr^r_\zeta z\rangle\right] \end{aligned}$$

(the sharp brackets denoting the duality between $H^{-m}(\Omega)$ and $H_0^m(\Omega)$).

REMARK 3.2. Whereas Lemma 3.2 extends easily to (Au,v), for u and v different elements of $D(\tilde{A})$, it is essential in Proposition 3.3 that we have u on both places and take the real part.

THEOREM 3.3. *Assume that A is strongly elliptic, satisfying* (3.24). *With $s\in[0,m]$ and U a linear space satisfying $H^{2m}(\Omega) \subset U \subset \mathscr{H}_A^{0,-m}(\Omega)$, we consider the two statements*

$$\text{(3.29)} \qquad \exists\, c>0, \lambda\in\mathbb{R} \text{ s.th. } \operatorname{Re}(Au,u) \geqq c\|u\|_s^2 - \lambda\|u\|_0^2,\ \forall u\in D(\tilde{A})\cap U.$$

$$\text{(3.30)} \qquad \begin{cases} \text{(i) } pr'_\zeta(D(T)\cap U) \subset W \\ \text{(ii) } \exists\, c'>0,\ \lambda'\in\mathbb{R} \text{ s.th. for all } z\in D(T)\cap U, \\ \quad \operatorname{Re}\left[(Tz, pr'_\zeta z) + \langle A\, pr^r_\zeta z, pr'_\gamma pr^r_\zeta z\rangle\right] \geqq c'\|z\|_s^2 - \lambda'\|z\|_0^2. \end{cases}$$

Here, (3.29) *implies* (3.30) *for all $s\in[0,m]$, and* (3.30) *implies* (3.29) *when $s\in\,]m-\frac{1}{2},m]$. When $s\in[0,m-\frac{1}{2}]$,* (3.30) *implies* (3.29) *if furthermore $\lambda'\alpha < m(A_0)$, where*

$$\text{(3.31)} \qquad \alpha = \sup\{\|z\|_0^2 : z\in D(T)\cap U \text{ with } \|pr^r_\zeta z\|_0 = 1\}.$$

(Here $-\lambda>0$ or $-\lambda\geqq 0$ implies $-\lambda'>0$ resp. $-\lambda'\geqq 0$ and vice versa.)

PROOF. We use the decompositions $u = v + A_\gamma^{-1}(Tz+f) + z$ and $u = u_{\gamma r} + u_{\zeta r}$, where $u_{\zeta r} = pr^r_\zeta z$, as in Proposition 3.3. It is already known from Theorem 3.2, that (3.29) implies (3.30) (i). To obtain (ii), let $z\in D(T)\cap U$ and let

$$u^n = w^n + A_\gamma^{-1}\, Tz + z,$$

where w^n is a sequence in $D(A_0)$ converging to $-A_\gamma^{-1}Tz - pr^r_\gamma z$ in $H_0^m(\Omega)$. Then $u^n_{\gamma r} = w^n + A_\gamma^{-1}\, Tz + pr^r_\gamma z \to 0$ in $H_0^m(\Omega)$, and, since $u^n\in H^s(\Omega)$, $u^n\to pr^r_\zeta z$ in $H^s(\Omega)$. By use of (3.28), the inequality (3.29) applied to u^n then gives by passage to the limit

$$\operatorname{Re}\left[(Tz, pr'_\zeta z) + \langle A\, pr^r_\zeta z, pr'_\gamma pr^r_\zeta z\rangle\right] \geqq c\|pr^r_\zeta z\|_s^2 - \lambda\|pr^r_\zeta z\|_0^2.$$

In view of Proposition 2.5, that pr^r_ζ is an isomorphism of $Z_A^t(\Omega)$ onto $Z_{A^r}^t(\Omega)$, all t, this implies (3.30) (ii).

In the converse direction, we have that (3.30) (i)–(ii) imply

$$\text{(3.32)} \qquad \operatorname{Re}(Au,u) \geqq \operatorname{Re}\langle Au_{\gamma r}, u_{\gamma r}\rangle + c'\|z\|_s^2 - \lambda'\|z\|_0^2,$$

for $u \in D(\tilde{A}) \cap U$, decomposed by (3.5), hence by use of Proposition 2.5 and an easy extension of (3.24)

$$\operatorname{Re}(Au,u) \geqq c_m \| u_{\gamma r} \|_m^2 + c'' \| u_{\zeta r} \|_s^2 - \alpha\lambda' \| u_{\zeta r} \|_0^2$$

(cf. (3.31)). If $-\alpha\lambda' \geqq 0$, we are through (and the last statement of the theorem is ensured). Otherwise, we proceed as in [12, Proposition 2.7]:

1°. $s \in]m - \frac{1}{2}, m]$. Choose $t \in]m - \frac{1}{2}, s[$, then

$$\begin{aligned} \| u_{\zeta r} \|_0^2 &\leqq c_1 \| \gamma u \|_{\{-j-\frac{1}{2}\}}^2 \leqq c_2 \| \gamma u \|_{\{t-j-\frac{1}{2}\}}^2 \leqq c_3 \| u \|_t^2 \\ &\leqq \varepsilon \| u \|_s^2 + C(\varepsilon) \| u \|_0^2, \text{ for any given } \varepsilon > 0; \end{aligned}$$

here we used Theorem 2.1 and Proposition 2.1 and a well known inequality. Now

$$\begin{aligned} \operatorname{Re}(Au,u) &\geqq \tfrac{1}{2}\min(c_m, c'') \| u \|_s^2 - \alpha\lambda' \| u_{\zeta r} \|_0^2 \\ &\geqq \tfrac{1}{4}\min(c_m, c'') \| u \|_s^2 - \alpha\lambda' C(\varepsilon) \| u \|_0^2, \end{aligned}$$

when we choose $\varepsilon = (4\alpha\lambda')^{-1}\min(c_m, c'')$.

2°. $s \in [0, m - \frac{1}{2}], \alpha\lambda' < m(A_0)$. Let $h \in]0,1[$. Then since we also have, besides (3.24), that

$$\operatorname{Re}\langle Au_{\gamma r}, u_{\gamma r}\rangle \geqq m(A_0) \| u_{\gamma r} \|_0^2,$$

(3.32) also leads to

$$\operatorname{Re}(Au,u) \geqq h\, c_m \| u_{\gamma r} \|_m^2 + (1-h)\, m(A_0) \| u_{\gamma r} \|_0^2 + c'' \| u_{\zeta r} \|_s^2 - \alpha\lambda' \| u_{\zeta r} \|_0^2,$$

whence by use of the inequality

$$(1+\delta) \| x \|^2 - \| y \|^2 \geqq -(1+\delta^{-1}) \| x + y \|^2, \ \forall \delta > 0,$$

follows that

$$\operatorname{Re}(Au,u) \geqq \tfrac{1}{2}\min(hc_m, c'') \| u \|_s^2 - C(h) \| u \|_0^2,$$

when h is chosen such that $\alpha\lambda' < (1-h)\, m(A_0)$.

REMARK 3.3. It is still an open question to the author (cf. [12, Remark 2.9]) whether the bound on λ' for $s \in [0, m - \frac{1}{2}]$ may be removed in general, as it may be in certain cases of constant coefficients, $\Omega = \mathbb{R}^n_+$. In Fujiwara [9], [10] (which concerns a class of normal boundary problems), this difficulty is circumvented by use of a technique (related to a device in Agmon [3]) of introducing an extra variable. Fujiwara studied the case $s = m - \frac{1}{2}$; however, his method seems likely to work in other cases where $s > 0$.

The above study can of course be continued in several directions. For one thing, one may study the numerical range of $\tilde{A}$ by applying Theorem 3.3 to rotations $e^{i\theta}\tilde{A}$ of $\tilde{A}$, as in [11]. Secondly, one may on the basis of Proposition 3.3 investigate inequalities like (3.12) with other norms appearing on the right, e.g. the norms in $H^{m-\frac{1}{2}}(\Omega)$, or in more general $H^s(\Omega)$, or in $\mathscr{H}^{0,-m}_{A^r}(\Omega)$; which give interesting results. We shall not go further into this in the present paper.

Let us conclude this section with the following observation: As might be expected by comparing the methods of Theorem 3.2 and 3.3, the "new" term in (3.30) (ii) is always non-positive;

PROPOSITION 3.4. *When* $y \in Z^0_{A^r}(\Omega) \cap \mathscr{H}^{0,-m}_{A^r}(\Omega)$,

$$\operatorname{Re}\langle Ay, pr'_\gamma y\rangle \leqq 0. \tag{3.33}$$

PROOF. Since $(A + A')y = 0$,

$$\begin{aligned}\operatorname{Re}\langle Ay, pr'_\gamma y\rangle &= -\operatorname{Re}\langle A'y, pr'_\gamma y\rangle \\ &= -\operatorname{Re}\langle A' pr'_\gamma y, pr'_\gamma y\rangle \leqq -c_m\| pr'_\gamma y\|^2_m \leqq 0,\end{aligned}$$

by a simple extension of (3.24).

3.3. *The new results formulated for general boundary problems.*
We recall from [11, Chapter III] the definition

DEFINITION 3.2. Let $V \subset Z(A_1)$, $W \subset Z(A_1)$, closed subspaces, and let T be an operator with $D(T) \subset V$, $R(T) \subset W$. We denote $\gamma(V)$ by X, $\gamma(W)$ by Y, and by γ_V resp. γ_W the isomorphisms from V to X resp. from W to Y obtained by restriction of γ. Identifying the spaces V and W with their duals, and denoting the dual spaces of X and Y by X' resp. Y', we introduce the adjoint isomorphisms $\gamma_V^*: X' \to V$ and $\gamma_W^*: Y' \to W$. Then we denote by L the operator from X to Y' defined by

$$D(L) = \gamma D(T),\ L = (\gamma_W^*)^{-1} T \gamma_V^{-1}. \tag{3.34}$$

Here, L may equivalently be defined as the operator $L: X \to Y'$ for which

$$\langle L\gamma v, \gamma w\rangle = (Tv, w)_W,\ \text{all } v \in D(T),\ w \in W, \tag{3.35}$$

where the sharp brackets denote the duality between Y' and Y.

REMARK 3.4. As it stands, Y', the (strong anti-) dual of the Hilbert space $Y \subset \prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$, has a somewhat abstract character; however, as soon as we choose a fixed norm in $\prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$, we have therewith an isometry E of $\prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$ onto its dual space $\prod_{j\in M_0} H^{j+\frac{1}{2}}(\Gamma)$ (so $\|\phi\|_{\{-j-\frac{1}{2}\}} = \langle E\phi, \phi\rangle^{\frac{1}{2}}$), which places Y' as the subspace EY of $\prod_{j\in M_0} H^{j+\frac{1}{2}}(\Gamma)$. For instance, when $\prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$ is given the norm with which $\gamma: Z(A_1') \to \prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$ is an *isometry*, the associated isometry from $\prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$ to $\prod_{j\in M_0} H^{j+\frac{1}{2}}(\Gamma)$ is the ps.d.o. R' described in Example 6.3; then we identify Y' with $R'Y$. This identification has some advantages (e.g. when one wants to define the numerical range of L) but on the other hand the disadvantage that, when Y is a "product" subspace $Y = \prod_{j\in J_0} H^{-j-\frac{1}{2}}(\Gamma)$ $(J_0 \subset M_0)$, $R'Y$ is generally not a similar product subspace of $\prod_{j\in M_0} H^{j+\frac{1}{2}}(\Gamma)$; this could be obtained by a different choice of norm. We shall simply refrain from fixing a norm on beforehand.

With a slight abuse of notation, we introduce

DEFINITION 3.3. The adjoint i_Y^* of the injection i_Y: $Y \hookrightarrow \prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$ will be denoted $pr_{Y'}$, it maps $\prod_{j\in M_0} H^{j+\frac{1}{2}}(\Gamma)$ onto Y'. Similar definitions of i_X and $pr_{X'}$.

Here, as soon as $\prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$ *is* provided with a Hilbert space norm etc., $pr_{Y'}$ (resp. $pr_{X'}$) becomes a true projection.

Since γ_V and γ_W are isomorphisms, Definition 3.2 introduces a 1–1 correspondence between all operators T: $V \to W$ with closed $V \subset Z(A_1)$, $W \subset Z(A_1)$, and all operators L: $X \to Y'$ with X and Y closed subspaces of $\prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$. The correspondence translates in a straightforward way all the properties we shall be concerned with; let us just note the following:

When $H^{2m}(\Omega) \subset U \subset \mathscr{H}_{A'}^{0,-m}(\Omega)$, then

$$
\begin{aligned}
\gamma\, pr'_\zeta(D(T) \cap U) &= \gamma(\gamma'_Z)^{-1}\gamma(D(T) \cap U) \\
&= \gamma(D(T) \cap [U \cap Z(A_1)]) = \gamma D(T) \cap \gamma[U \cap Z(A_1)] \\
&= D(L) \cap \gamma[U \cap D(A_1)].
\end{aligned} \tag{3.36}
$$

Therefore,

$$
pr'_\zeta(D(T) \cap U) \subset W \iff D(L) \cap \gamma[U \cap D(A_1)] \subset Y. \tag{3.37}
$$

When this holds, one has for $z \in D(T)$, $w \in D(T) \cap U$

$$
(Tz, pr'_\zeta w) = \langle L\gamma z, \gamma w \rangle, \tag{3.38}
$$

the sharp brackets denoting the duality between Y' and Y.

Now Proposition 3.1 may be translated as follows (see [11, III§2] for the proof; cf. also section 2.5 for changes in notation):

PROPOSITION 3.5. *There is a 1–1 correspondence between all closed realizations $\tilde{A}$ of A and all operators L: $X \to Y'$, where X and Y denote closed subspaces of $\prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$, and L is densely defined in X and closed; the correspondence being determined by*

$$D(\tilde{A}) = \{u \in D(A_1) \mid \gamma u \in D(L), L\gamma u = pr_{Y'}\mu u\}. \tag{3.39}$$

In this correspondence, $D(L) = \gamma D(\tilde{A})$ *and* $X = \overline{\gamma D(\tilde{A})}$; *moreover, the realization* $\tilde{A}^*$ *of* A' *corresponds to* $L^*: Y \to X'$ *by*

$$D(\tilde{A}^*) = \{u \in D(A_1') \mid \gamma u \in D(L^*),\ L^*\gamma u = pr_{X'}\mu' u\}, \tag{3.40}$$

and $D(L^*) = \gamma D(\tilde{A}^*)$, $Y = \overline{\gamma D(\tilde{A}^*)}$.

REMARK 3.5. It is the surjectiveness of $\{\gamma, \mu\}: D(A_1) \to \prod_{j \in M_0} H^{-j-\frac{1}{2}}(\Gamma) \times \prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma)$ that permits us to characterize $\tilde{A}$ and L by each other in this way, cf. the discussion in [11, III §2].

In the rest of this chapter we assume (in addition to Assumption 3.2)

ASSUMPTION 3.3. $\tilde{A}$ corresponds to $L: X \to Y'$ by Proposition 3.5.

Now Theorems 3.1–3.2 translate into the following results, by use of the isomorphism in Theorem 2.1(c) together with the above remarks:

THEOREM 3.4. *(No particular assumption on* $\sigma^0(A)$*.) Let* U *be a linear space with* $H^{2m}(\Omega) \subset U \subset H^m(\Omega)$. *Then* (3.41) *and* (3.42) *are equivalent:*

$$\exists \lambda \in \mathbb{R}\ s.th.\ \operatorname{Re}(Au, u) \geqq -\lambda \| u \|_m^2, \quad \forall u \in D(\tilde{A}) \cap U; \tag{3.41}$$

$$\begin{cases} \text{(i)} \quad D(L) \cap \gamma[U \cap D(A_1)] \subset Y \\ \text{(ii)} \quad \exists \lambda' \in \mathbb{R}\ s.th.\ \operatorname{Re}\langle L\phi, \phi \rangle \geqq -\lambda' \| \phi \|_{\{m-j-\frac{1}{2}\}}^2, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \phi \in D(L) \cap \gamma[U \cap D(A_1)]. \end{cases} \tag{3.42}$$

COROLLARY 3.4. *Assumptions of Theorem* 3.4. *Identify* Y' *with a subspace of* $\prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma)$ *as in Remark* 3.4. *If* L *has the property*: $|\langle L\phi, \phi \rangle| \leqq c \| \phi \|_{\{m-j-\frac{1}{2}\}}^2$ *on* $D(L) \cap \gamma[U \cap D(A_1)]$, *then*

$$|(Au, v)| \leqq c' \| u \|_m \| v \|_m \ \textit{for}\ u, v \in D(\tilde{A}) \cap U$$

is equivalent with

$$D(L) \cap \gamma[U \cap D(A_1)] \subset Y.$$

THEOREM 3.5. *(No particular assumption on* $\sigma^0(A)$*.) Let* $H^{2m}(\Omega) \subset U \subset \mathscr{H}_{A'}^{0,-m}(\Omega)$. *If there exists* $\lambda \in \mathbb{R}$ *such that*

$$\operatorname{Re}\ (Au, u) \geqq -\lambda(\| u \|_0^2 + \| A'u \|_{-m}^2),\ u \in D(\tilde{A}) \cap U, \tag{3.43}$$

then

$$(3.44)\quad \begin{cases} \text{(i)} & D(L) \cap \gamma[U \cap D(A_1)] \subset Y \\ \text{(ii)} & \exists \lambda' \in \mathbb{R} \text{ s.th. } \operatorname{Re}\langle L\phi, \phi\rangle \geqq -\lambda' \| \phi \|^2_{\{-j-\frac{1}{2}\}}, \\ & \qquad \forall \phi \in D(L) \cap \gamma[U \cap D(A_1)]. \end{cases}$$

Remark 3.6. One may actually define X, Y and L *without* the assumption that A has a uniquely solvable Dirichlet problem, and prove corresponding versions of Theorems 3.4 and 3.5, and Corollary 3.4 on the basis of the techniques of Chapter 5. To limit the article we shall not reproduce it here.

Remark 3.7. Recall that $D(L)$ is dense in X; then of course, when $D(L) \cap \gamma[U \cap D(A_1)]$ is also dense in X, (i) means that

$$X \subset Y.$$

Before translating Theorem 3.3 we shall look more closely at the "new" term.

Definition 3.4. Assume that A is strongly elliptic satisfying (3.24). For

$$(3.45)\qquad \phi \in \gamma[\mathscr{H}^{0,-m}_A(\Omega) \cap \mathscr{H}^{0,-m}_{A'}(\Omega)]$$

we define the quadratic form $q(\phi, \phi)$ by

$$(3.46)\qquad q(\phi,\phi) = \operatorname{Re}\langle A(\gamma^r_Z)^{-1}\phi,\ pr'_\gamma(\gamma^r_Z)^{-1}\phi\rangle,$$

the sharp brackets denoting the duality between $H^{-m}(\Omega)$ and $H^m_0(\Omega)$.

The expression (3.46) is well defined when (3.45) holds, since

$$(3.47)\qquad \begin{aligned} \gamma[\mathscr{H}^{0,-m}_A(\Omega) \cap \mathscr{H}^{0,-m}_{A'}(\Omega)] &= \gamma[\mathscr{H}^{0,-m}_{A'}(\Omega) \cap \mathscr{H}^{0,-m}_A(\Omega)] \\ &= \gamma[Z^0_{A'}(\Omega) \cap \mathscr{H}^{0,-m}_A(\Omega)]. \end{aligned}$$

Recall also that, by (3.33)

$$(3.48)\qquad q(\phi,\phi) \leqq 0, \quad \forall \phi \in \gamma[\mathscr{H}^{0,-m}_A(\Omega) \cap \mathscr{H}^{0,-m}_{A'}(\Omega)]$$

Definition 3.5. When A is strongly elliptic satisfying (3.24), we denote by Q the ps.d.o. in Γ of type $(-k, -2m+j+1)_{j,k\in M_0}$ defined by

$$(3.49)\qquad Q = -P^r_{\gamma,\chi^r} + \operatorname{Re} P_{\gamma,\chi} \qquad [= -P^r_{\gamma,\chi^r} + \tfrac{1}{2}(P_{\gamma,\chi} + P'_{\gamma,\chi'})].$$

(Cf. Corollary 2.7.)

Proposition 3.6. *Let A be strongly elliptic satisfying* (3.24). *When $\phi \in \gamma[D(A_1) \cap D(A'_1)]$, then $Q\phi \in \prod_{j\in M_0} H^{j+\frac{1}{2}}(\Gamma)$ and*

$$(3.50)\qquad q(\phi,\phi) = \langle \underset{\{j+\frac{1}{2}\}}{Q\phi}, \underset{\{-j-\frac{1}{2}\}}{\phi} \rangle.$$

PROOF. Let $\phi \in \gamma[D(A_1) \cap D(A_1')] = \gamma[D(A_1^r) \cap D(A_1)] = \gamma[Z_{A^r}^0(\Omega) \cap D(A_1)]$. Let $y = (\gamma_Z^r)^{-1}\phi$, it belongs to $Z_{A^r}^0(\Omega)$ and satisfies $Ay = -A'y \in L^2(\Omega)$, so $pr_\gamma y$ and $pr_\gamma' y$ are well defined elements of $H_0^m(\Omega) \cap H^{2m}(\Omega)$. Then

$$\begin{aligned} q(\phi,\phi) &= \operatorname{Re}\langle Ay, pr_\gamma' y\rangle = \operatorname{Re}\,(Ay, pr_\gamma' y) \\ &= \tfrac{1}{2}[(A\,pr_\gamma y, pr_\gamma' y) + (pr_\gamma' y, A\,pr_\gamma y)] \\ &= \tfrac{1}{2}[(A\,pr_\gamma y, pr_\gamma' y) + (A'\,pr_\gamma' y, pr_\gamma y)] \\ &= \tfrac{1}{2}[(Ay, y - pr_\zeta' y) + (A'y, y - pr_\zeta y)] \\ &= \tfrac{1}{2}((A + A')y, y) - \tfrac{1}{2}[(Ay, pr_\zeta' y) + (A'y, pr_\zeta y)]. \end{aligned}$$

Here $(A + A')y = 0$, moreover, we find by Proposition 2.7, that

$$q(\phi,\phi) = -\tfrac{1}{2}[\langle \mu y, \phi\rangle + \langle \mu' y, \phi\rangle] = \langle -\tfrac{1}{2}(\mu + \mu')y, \phi\rangle,$$

where $(\mu + \mu')y \in \prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma)$. Now, (cf. (2.44) and (2.34))

$$\mu y = \chi y - P_{\gamma,\chi}\gamma y = [\mathscr{A}_{M_0M_1}P_{\gamma,\nu}^r + \tfrac{1}{2}\mathscr{A}_{M_0M_0} - P_{\gamma,\chi}]\phi,$$

since $y \in Z_{A^r}^0(\Omega)$ with $\gamma y = \phi$. Similarly,

$$\mu' y = [\mathscr{A}_{M_0M_1}' P_{\gamma,\nu}^r + \tfrac{1}{2}\mathscr{A}_{M_0M_0}' - P_{\gamma,\chi'}']\phi.$$

Altogether,

$$\begin{aligned} -\tfrac{1}{2}(\mu + \mu')y = [&-\tfrac{1}{2}(\mathscr{A}_{M_0M_1} + \mathscr{A}_{M_0M_1}')P_{\gamma,\nu}^r - \tfrac{1}{4}(\mathscr{A}_{M_0M_0} + \mathscr{A}_{M_0M_0}') \\ &+ \tfrac{1}{2}(P_{\gamma,\chi} + P_{\gamma,\chi'}')]\phi \end{aligned}$$

$$= [-\mathscr{A}_{M_0M_1}^r P_{\gamma,\nu}^r - \tfrac{1}{2}\mathscr{A}_{M_0M_0}^r + \tfrac{1}{2}(P_{\gamma,\chi} + P_{\gamma,\chi'}')]\phi = [-P_{\gamma,\chi^r}^r + \operatorname{Re} P_{\gamma,\chi}]\phi,$$

by Corollary 2.7; so that

$$q(\phi,\phi) = \langle(-P_{\gamma,\chi^r}^r + \operatorname{Re} P_{\gamma,\chi})\phi, \phi\rangle,$$

which shows the proposition.

Using these considerations, we finally obtain from Theorem 3.3

THEOREM 3.6. *Assume that A is strongly elliptic satisfying* (3.24). *With $s \in [0, m]$ and U denoting a linear space satisfying $H^{2m}(\Omega) \subset U \subset \mathscr{H}_A^{0,-m}(\Omega)$, we consider the two statements*

(3.51) $\quad \exists c > 0,\ \lambda \in \mathbb{R}$ *s.th.* $\operatorname{Re}(Au, u) \geqq c\| u \|_s^2 - \lambda \| u \|_0^2$, *all* $u \in D(\tilde{A}) \cap U$;

$$(3.52)\quad \begin{cases} \text{(i)}\ \ D(L) \cap \gamma[U \cap D(A_1)] \subset Y, \\ \text{(ii)}\ \ \exists c' > 0, \lambda' \in \mathbb{R} \textit{ such that for all } \phi \in D(L) \cap \gamma[U \cap D(A_1)], \\ \operatorname{Re}\langle L\phi, \phi\rangle + q(\phi,\phi) \geqq c' \| \phi \|_{\{s-j-\frac{1}{2}\}}^2 - \lambda' \| \phi \|_{\{-j-\frac{1}{2}\}}^2 \end{cases}$$

(where $q(\phi,\phi)$ takes the form $\langle Q\phi,\phi\rangle$ when $U \subset D(A_1'))$. Here (3.51) *implies* (3.52) *for all* $s \in [0,m]$, *and* (3.52) *implies* (3.51) *when* $s \in]m-\frac{1}{2},m]$. *When* $s \in [0, m-\frac{1}{2}]$, (3.52) *implies* (3.51) *if furthermore* $\lambda'\alpha' < m(A_0)$, *where*

$$\alpha' = \sup\{\|\phi\|^2_{\{-j-\frac{1}{2}\}} \mid \phi \in D(L) \cap \gamma[U \cap D(A_1)] \text{ with } \|(\gamma_Z^{\Gamma})^{-1}\phi\|_0 = 1\}.$$

(Here $-\lambda > 0$ *or* $-\lambda \geqq 0$ *implies* $-\lambda' > 0$ *resp.* $-\lambda' \geqq 0$, *and vice versa.)*

4. Application to normal boundary problems

4.1. *Reduction of the boundary condition to a special form.* We assume Assumption 3.1, and now furthermore

ASSUMPTION 4.1. J and K denote two complementing subsets, each consisting of m elements, of the set $M = \{0,1,\cdots,2m-1\}$. For each pair $\{j,k\} \in J \times K$, F_{jk} denotes a ps.d.o. in Γ of order $j-k$, such that $F_{jk} = 0$ when $k > j$.

With J, K and $(F_{jk})_{j\in J, k\in K}$ given in this way, we consider the system of boundary conditions

$$\gamma_j u - \sum_{k\in K, k<j} F_{jk}\gamma_k u = 0, \quad j \in J, \tag{4.1}$$

and shall study the realization $\tilde{A}$ of A defined by

$$D(\tilde{A}) = \{u \in D(A_1) \mid u \text{ satisfies } (4.1)\}. \tag{4.2}$$

(4.1) is a reduced version of the usual homogeneous normal boundary condition, generalized to include ps.d.o.'s in the boundary.

Recall the definitions (2.1), (2.2) of the sets of integers M_0, M_1, J_0, J_1, J_1' etc. ...; note that they form the disjoint unions

$$M_0 = J_0 \cup K_0 = J_1' \cup K_1', \; M_1 = J_1 \cup K_1 = J_0' \cup K_0'.$$

We set

$$F_0 = (F_{jk})_{j\in J_0, k\in K_0}, \; F_1 = (F_{jk})_{j\in J_1, k\in K_0}, \text{ and } F_2 = (F_{jk})_{j\in J_1, k\in K_1}; \tag{4.3}$$

here $(F_{jk})_{j\in J_\alpha, k\in K_\beta}$ is a ps.d.o. in Γ of type $(-k,-j)_{j\in J_\alpha, k\in K_\beta}$ for $\alpha,\beta = 0,1$ (cf. Definition 2.1). (Note that $(F_{jk})_{j\in J_0, k\in K_1}$ is zero.) Then, with the notations of Section 2.2, (4.1) may be written

$$\gamma_{J_0} u = F_0 \gamma_{K_0} u; \quad \nu_{J_1} u = F_1 \gamma_{K_0} u + F_2 \nu_{K_1} u. \tag{4.4}$$

(We use the convention that empty index sets give zero terms). Recall the boundary operator defined in Section 2.5

$$\chi = \mathscr{A}_{M_0M_1}\nu + \tfrac{1}{2}\mathscr{A}_{M_0M_0}\gamma. \tag{4.5}$$

Our first step will be to reduce (4.4) to the form

(4.6) $$\gamma_{J_0}u = F_0\gamma_{K_0}u;\ \chi_{J_1'}u = G_1\gamma_{K_0}u + G_2\chi_{K_1'}u,$$

where G_1 and G_2 are suitable ps.d.o.'s in Γ. It is fairly evident that this may be done. To do it in a precise and explicit form, we introduce some notation:

DEFINITION 4.1. With F_0 as in (4.3) we define Φ as the $M_0 \times K_0$-matrix of ps.d.o.'s in Γ for which $\Phi_{K_0K_0} = I_{K_0K_0}$ and $\Phi_{J_0K_0} = F_0$. Briefly written,

$$\Phi = \begin{pmatrix} I_{K_0K_0} \\ F_0 \end{pmatrix};$$

it sends $\phi \in \prod_{k\in K_0} H^{s-k}(\Gamma)$ into $\psi \in \prod_{j\in M_0} H^{s-j}(\Gamma)$, where $\psi_{K_0} = \phi$, $\psi_{J_0} = F_0\phi$.

In a similar way, we define Θ and Ψ as the $M_1 \times J_1$—resp. $M_0 \times J_1'$-matrices of ps.d.o.'s satisfying

$$\Theta_{J_1J_1} = I_{J_1J_1},\ \Theta_{K_1J_1} = -F_2;\ \text{in short } \Theta = \begin{pmatrix} I_{J_1J_1} \\ -F_2^* \end{pmatrix};$$

resp.

$$\Psi_{J_1'J_1'} = I_{J_1'J_1'},\ \Psi_{K_1'J_1'} = -G_2^*;\ \text{in short } \Psi = \begin{pmatrix} I_{J_1'J_1'} \\ -G_2^* \end{pmatrix};$$

F_2 and G_2 being $J_1 \times K_1$- resp. $J_1' \times K_1'$-matrices of ps.d.o.'s.

For the inverse of

(4.7) $$\mathscr{A} = \begin{pmatrix} \mathscr{A}_{M_0M_0} & \mathscr{A}_{M_0M_1} \\ \mathscr{A}_{M_1M_0} & 0 \end{pmatrix}$$

(cf. (2.32)) we introduce the notation $\mathscr{A}^{-1} = \mathscr{B}$, so

(4.8) $$\mathscr{B} = \begin{pmatrix} 0 & \mathscr{A}^{-1}_{M_1M_0} \\ \mathscr{A}^{-1}_{M_0M_1} & -\mathscr{A}^{-1}_{M_0M_1}\mathscr{A}_{M_0M_0}\mathscr{A}^{-1}_{M_1M_0} \end{pmatrix};$$

in particular, $\mathscr{A}^{-1}_{M_0M_1} = \mathscr{B}_{M_1M_0}$. Then, by (4.5),

(4.9) $$v = \mathscr{B}_{M_1M_0}\chi - \tfrac{1}{2}\mathscr{B}_{M_1M_0}\mathscr{A}_{M_0M_0}\gamma.$$

PROPOSITION 4.1. *Let (F_0, F_1, F_2) denote a triple of ps.d.o.'s. in Γ of types $(-k, -j)_{j\in J_0, k\in K_0}$, $(-k, -j)_{j\in J_1, k\in K_0}$ resp. $(-k, -j)_{j\in J_1, k\in K_1}$, with $(F_i)_{jk} = 0$ when $k > j$.*[3] *Let (F_0, G_1, G_2) denote a triple of ps.d.o.'s. in Γ with F_0 as before, and G_1 and G_2 of types $(k, -2m + j + 1)_{j\in J_1, k\in K_0}$ resp. $(k, j)_{j\in J_1', k\in K_1'}$, with $(G_2)_{jk} = 0$ when $j > k$.*

[3] In [13] the operators were said to be subtriangular when having this property.

There is a 1-1 correspondence between the class of triples $\{(F_0, F_1, F_2)\}$ *and the class of triples* $\{(F_0, G_1, G_2)\}$, *in which* (F_0, F_1, F_2) *corresponds to* (F_0, G_1, G_2) *if and only if* (4.10)$\Leftrightarrow$(4.11) *for all* $u \in \mathscr{D}(\bar{\Omega})$, *where*

$$\gamma_{J_0}u = F_0\gamma_{K_0}u, \quad \nu_{J_1}u = F_1\gamma_{K_0}u + F_2\nu_{K_1}u, \tag{4.10}$$

$$\gamma_{J_0}u = F_0\gamma_{K_0}u, \quad \chi_{J_1'}u = G_1\gamma_{K_0}u + G_2\chi_{K_1'}u. \tag{4.11}$$

In this correspondence, G_1 *and* G_2 *are expressed by* (F_0, F_1, F_2) *by*

$$G_1 = (\Theta^*\mathscr{B}_{M_1J_1'})^{-1}(F_1 + \tfrac{1}{2}\Theta^*\mathscr{B}_{M_1M_0}\mathscr{A}_{M_0M_0}\Phi) \tag{4.12}$$

$$G_2 = -(\Theta^*\mathscr{B}_{M_1J_1'})^{-1}\ \Theta^*\mathscr{B}_{M_1K_1'}; \tag{4.13}$$

and F_1 *and* F_2 *are expressed by* (F_0, G_1, G_2) *by*

$$F_1 = (\Psi^*\mathscr{A}_{M_0J_1})^{-1}(G_1 - \tfrac{1}{2}\Psi^*\mathscr{A}_{M_0M_0}\Phi) \tag{4.14}$$

$$F_2 = -(\Psi^*\mathscr{A}_{M_0J_1})^{-1}\Psi^*\mathscr{A}_{M_0K_1}. \tag{4.15}$$

PROOF. We note first that a triple (F_0, F_1, F_2) is uniquely determined by the boundary condition (4.10) on $\mathscr{D}(\bar{\Omega})$ that it gives rise to, since $\rho: \mathscr{D}(\bar{\Omega}) \to \prod_{j \in M}\mathscr{D}(\Gamma)$ is surjective, and since a ps.d.o. is determined by its action on smooth functions. Also, a triple (F_0, G_1, G_2) is uniquely determined by the boundary condition (4.11) that it gives rise to.

Let (F_0, F_1, F_2) be given. Then (4.10) may be written in the form

$$\gamma u = \Phi\gamma_{K_0}u, \qquad \Theta^*\nu u = F_1\gamma_{K_0}u, \tag{4.16}$$

(cf. Definition 4.1). Let u belong to $\mathscr{D}(\bar{\Omega})$ and satisfy (4.16). By use of (4.9) we find

$$\begin{aligned} 0 &= \Theta^*\nu u - F_1\gamma_{K_0}u = \Theta^*\mathscr{B}_{M_1M_0}\chi u - \tfrac{1}{2}\Theta^*\mathscr{B}_{M_1M_0}\mathscr{A}_{M_0M_0}\gamma u - F_1\gamma_{K_0}u \\ &= \Theta^*\mathscr{B}_{M_1J_1'}\chi_{J_1'}u + \Theta^*\mathscr{B}_{M_1K_1'}\chi_{K_1'}u - (\tfrac{1}{2}\Theta^*\mathscr{B}_{M_1M_0}\mathscr{A}_{M_0M_0}\Phi + F_1)\gamma_{K_0}u, \end{aligned}$$

whence

$$\Theta^*\mathscr{B}_{M_1J_1'}\chi_{J_1'}u = (F_1 + \tfrac{1}{2}\Theta^*\mathscr{B}_{M_1M_0}\mathscr{A}_{M_0M_0}\Phi)\gamma_{K_0}u - \Theta^*\mathscr{B}_{M_1K_1'}\chi_{K_1'}u. \tag{4.17}$$

Here, $\Theta^*B_{M_1J_1'} = \mathscr{B}_{J_1J_1'} - F_2\mathscr{B}_{K_1J_1'}$ where $B_{J_1J_1'}$ is skew-triangular *invertible* with zeroes above the second diagonal, and $F_2\mathscr{B}_{K_1J_1'}$ is skew-triangular with zeroes *in and above* the second diagonal (since $(F_2)_{jk} = 0$ for $k \geqq j$ and $\mathscr{B}_{kl} = 0$ for $k + l < 2m - 1$, such that $(F_2\mathscr{B}_{K_1J_1'})_{jl} = \sum_{k \in K_1}F_{jk}\mathscr{B}_{kl} = 0$ when $\{j, l\} \in J_1 \times J_1'$ with $j + l \leqq 2m - 1$). Thus

$$\Theta^*\mathscr{B}_{M_1J_1'} \text{ is invertible,} \tag{4.18}$$

so that we get from (4.17)

$$\begin{aligned}\chi_{J_1'}u &= (\Theta^*\mathscr{B}_{M_1J_1'})^{-1}(F_1 + \tfrac{1}{2}\Theta^*\mathscr{B}_{M_1M_0}\mathscr{A}_{M_0M_0}\Phi)\gamma_{K_0}u \\ &\quad - (\Theta^*\mathscr{B}_{M_1J_1'})^{-1}\Theta^*\mathscr{B}_{M_1K_1'}\chi_{K_1'}u.\end{aligned} \tag{4.19}$$

Thus when u satisfies (4.10), it satisfies (4.11) with G_1 and G_2 defined by (4.12)–(4.13). Conversely when u satisfies (4.11) with (4.12)–(4.13), backwards calculations give that it satisfies (4.10).

Now let (F_0, G_1, G_2) be given, and write (4.11) in the form

$$\gamma u = \Phi\gamma_{K_0}u, \qquad \Psi^*\chi u = G_1\gamma_{K_0}u. \tag{4.20}$$

Using (4.5) we now find that when u satisfies (4.20),

$$\begin{aligned} 0 &= \Psi^*\mathscr{A}_{M_0M_1}\nu u + \tfrac{1}{2}\Psi^*\mathscr{A}_{M_0M_0}\gamma u - G_1\gamma_{K_0}u \\ &= \Psi^*\mathscr{A}_{M_0J_1}\nu_{J_1}u + \Psi^*\mathscr{A}_{M_0K_1}\nu_{K_1}u - (G_1 - \tfrac{1}{2}\Psi^*\mathscr{A}_{M_0M_0}\Phi)\gamma_{K_0}u.\end{aligned}$$

Here $\Psi^*\mathscr{A}_{M_0J_1} = \mathscr{A}_{J_1J_1'} - G_2\mathscr{A}_{K_1'J_1}$ is invertible by arguments analogous to those establishing (4.18). So we find that

$$\nu_{J_1}u = (\Psi^*\mathscr{A}_{M_0J_1})^{-1}(G_1 - \tfrac{1}{2}\Psi^*\mathscr{A}_{M_0M_0}\Phi)\gamma_{K_0}u - (\Psi^*\mathscr{A}_{M_0J_1})^{-1}\Psi^*\mathscr{A}_{M_0K_1}\nu_{K_1}u, \tag{4.21}$$

and obtain that (4.20) is equivalent with (4.10), when F_1 and F_2 are determined by (4.14)–(4.15).

There is an alternative description of G_1 and G_2 in terms of (F_0, F_1, F_2) that will be useful later.

LEMMA 4.1. *When (F_0, F_1, F_2) corresponds to (F_0, G_1, G_2) as in Proposition 4.1, then*

$$G_2 = (\mathscr{A}_{J_1'K_1} + \mathscr{A}_{J_1'J_1}F_2)(\mathscr{A}_{K_1'K_1} + \mathscr{A}_{K_1'J_1}F_2)^{-1} \tag{4.22}$$

and, with Ψ^* defined from this,

$$G_1 = \Psi^*\mathscr{A}_{M_0J_1}F_1 + \tfrac{1}{2}\Psi^*\mathscr{A}_{M_0M_0}\Phi. \tag{4.23}$$

PROOF. Multiplication with $\Psi^*\mathscr{A}_{M_0J_1}$ in (4.15) gives

$$\Psi^*\mathscr{A}_{M_0J_1}F_2 = -\Psi^*\mathscr{A}_{M_0K_1}$$

or

$$\mathscr{A}_{J_1'J_1}F_2 - G_2\mathscr{A}_{K_1'J_1}F_2 = -\mathscr{A}_{J_1'K_1} + G_2\mathscr{A}_{K_1'K_1}$$

Then

$$G_2(\mathscr{A}_{K_1'K_1} + \mathscr{A}_{K_1'J_1}F_2) = \mathscr{A}_{J_1'K_1} + \mathscr{A}_{J_1'J_1}F_2,$$

and, since $\mathscr{A}_{K_1'K_1} + \mathscr{A}_{K_1'J_1}F_2$ is invertible by the usual argument, we get (4.22). (4.23) is then obtained straightforwardly from (4.14).

We shall also need

COROLLARY 4.1. *Let (F_0, F_1, F_2) correspond to (F_0, G_1, G_2) as in Proposition 4.1. Then*

$$\Phi = \Psi$$

or, equivalently,

$$J_1' = K_0 \text{ and } -G_2^* = F_0, \tag{4.24}$$

if and only if

$$J_1' = K_0 \text{ and } F_2 = -(\Phi^* \mathscr{A}_{M_0 J_1})^{-1} \Phi^* \mathscr{A}_{M_0 K_1}. \tag{4.25}$$

PROOF. When $\Phi = \Psi$, formula (4.15) takes the form (4.25). Conversely, when (4.25) holds, it reduces as in the proof of Lemma 4.1 to

$$-F_0^* = (\mathscr{A}_{J_1' K_1} + \mathscr{A}_{J_1' J_1} F_2)(\mathscr{A}_{K_1' K_1} + \mathscr{A}_{K_1' J_1} F_2)^{-1}$$

whence, by comparison with (4.22), $-F_0^* = G_2$.

4.2. *The description of X, Y and L.* In view of Proposition 4.1, we may now restrict our attention to boundary conditions in the form (4.11), or, with Definition 4.1, of the equivalent form (4.20)

$$\text{(i) } \gamma_{J_0} u = F_0 \gamma_{K_0} u, \qquad \text{(ii) } \chi_{J_1'} u = G_1 \gamma_{K_0} u + G_2 \chi_{K_1'} u, \tag{4.11}$$

$$\text{(i) } \gamma u = \Phi \gamma_{K_0} u, \qquad \text{(ii) } \Psi^* \chi u = G_1 \gamma_{K_0} u. \tag{4.20}$$

So now $D(\tilde{A}) = \{u \in D(A_1) \,|\, (4.11) \text{ holds}\}$. Since $\gamma_{K_0} u$ varies freely, at least among smooth functions, when $u \in D(\tilde{A})$, we see from (4.20) (i)

$$X = \overline{\gamma D(\tilde{A})} = \Phi\Big(\prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Gamma)\Big). \tag{4.26}$$

To determine Y, we need some information on $\tilde{A}^*$.
Recall that

$$D(\tilde{A}^*) = \{u \in D(A_1') \,|\, (Au, v) - (u, A'v) = 0, \ \forall u \in D(\tilde{A})\}. \tag{4.27}$$

For each $s \in [0, 2m]$ one has for $u \in D(\tilde{A}) \cap H^s(\Omega)$, $v \in D(\tilde{A}^*) \cap H^{2m-s}(\Omega)$ (cf. section 2.5)

$$\begin{aligned} 0 &= (Au, v) - (u, A'v) = \langle \chi u, \gamma v \rangle - \langle \gamma u, \chi' v \rangle \\ &= \langle \chi_{K_1'} u, \gamma_{K_1'} v \rangle + \langle G_1 \gamma_{K_0} u + G_2 \chi_{K_1'} u, \gamma_{J_1'} v \rangle - \langle \Phi \gamma_{K_0} u, \chi' v \rangle; \end{aligned}$$

$$0 = \langle \chi_{K_1'} u, \gamma_{K_1'} v + G_2^* \gamma_{J_1'} v \rangle + \langle \gamma_{K_0} u, G_1^* \gamma_{J_1'} v - \Phi^* \chi' v \rangle. \tag{4.28}$$

Introduce the "adjoint" boundary condition

$$\gamma_{K_1'}v = -G_2^*\gamma_{J_1'}v, \qquad \Phi^*\chi' v = G_1^*\gamma_{J_1'}v, \tag{4.29}$$

i.e., using Definition 4.1,

$$\gamma v = \Psi\gamma_{J_1'}v, \qquad \Phi^*\chi' v = G_1^*\gamma_{J_1'}v, \tag{4.30}$$

and define the realization $\tilde{A}'$ of A' by

$$D(\tilde{A}') = \{v \in D(A_1') \,|\, v \text{ satisfies } (4.30)\}. \tag{4.31}$$

Then (4.28) gives, by varying s in $[0, 2m]$, that

$$D(\tilde{A}') \;\cap\; H^{2m}(\Omega) \;\subset\; D(\tilde{A}^*) \;\subset\; D(\tilde{A}'). \tag{4.32}$$

This suffices to conclude (like for X)

$$Y = \overline{\gamma D(\tilde{A}^*)} = \Psi\left(\sum_{j \in J_1'} H^{-j-\frac{1}{2}}(\Gamma))\right). \tag{4.33}$$

($\tilde{A}^*$ will be precisely characterized later).

We proceed to determine L.

Consider the three spaces $\prod_{j \in K_0} H^{-j-\frac{1}{2}}(\Gamma)$, $X = \Phi(\prod_{j \in K_0} H^{-j-\frac{1}{2}}(\Gamma))$, and $\prod_{j \in M_0} H^{-j-\frac{1}{2}}(\Gamma)$. We denote by Φ_1 the restriction[4] of the ps.d.o. Φ with domain space $\prod_{j \in K_0} H^{-j-\frac{1}{2}}(\Gamma)$ and range space X; then, with i_X denoting the injection $i_X : X \hookrightarrow \prod_{j \in M_0} H^{-j-\frac{1}{2}}(\Gamma)$,

$$\Phi = i_X \Phi_1 \qquad \left(\text{on} \prod_{j \in K_0} H^{-j-\frac{1}{2}}(\Gamma)\right). \tag{4.34}$$

By taking adjoints, we obtain the formula

$$\Phi^* = \Phi_1^* \, pr_{X'} \qquad \left(\text{on} \prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma)\right), \tag{4.35}$$

here $i_X^* = pr_{X'} : \prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma) \to X'$, as defined in Definition 3.3, and Φ_1^* sends X' into $\prod_{j \in K_0} H^{j+\frac{1}{2}}(\Gamma)$. Now, since X is a graph, Φ_1 is evidently a bijection with $I_{K_0 M_0}$ as inverse, or, more precisely

$$\Phi_1^{-1} = I_{K_0 M_0} i_X. \tag{4.36}$$

This gives us the formula for $(\Phi_1^*)^{-1} : \prod_{j \in K_0} H^{j+\frac{1}{2}}(\Gamma) \to X'$.

$$(\Phi_1^*)^{-1} = (\Phi_1^{-1})^* = i_X^* \, I_{K_0 M_0}{}^* = pr_{X'} I_{M_0 K_0}. \tag{4.37}$$

In a similar way, defining Ψ_1 as the restriction of the p.s.o. Ψ with domain $\prod_{j \in J_1'} H^{-j-\frac{1}{2}}(\Gamma)$ and range $Y = \Psi(\prod_{j \in J_1'} H^{-j-\frac{1}{2}}(\Gamma))$, we have the formulae

[4] We use the word "restriction" in a general sense; the important part of the above restriction takes place in the *range space.*

$$\Psi = i_Y \Psi_1 \quad \text{on } \prod_{j \in J_1'} H^{-j-\frac{1}{2}}(\Gamma); \tag{4.38}$$

$$\Psi^* = \Psi_1^* \, pr_{Y'} \quad \text{on } \prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma); \tag{4.39}$$

$$\Psi_1^{-1} = I_{J_1' M_0} i_Y; \tag{4.40}$$

$$(\Psi_1^*)^{-1} = pr_{Y'} I_{M_0 J_1'} \quad \text{on } \prod_{j \in J_1'} H^{j+\frac{1}{2}}(\Gamma). \tag{4.41}$$

REMARK 4.1. Since X and Y are usually understood to be subspaces of $\prod_{j \in M_0} H^{-j-\frac{1}{2}}(\Gamma)$, we shall often omit i_X and i_Y in formulae; on the contrary, $pr_{X'}$ and $pr_{Y'}$ of course cannot be omitted. In continuation of Remark 3.4, let us mention that when $\prod_{j \in M_0} H^{-j-\frac{1}{2}}(\Gamma)$ is provided with a norm $\langle E\phi, \phi \rangle^{\frac{1}{2}}$, for which the associated isometry E of $\prod_{j \in M_0} H^{-j-\frac{1}{2}}(\Gamma)$ onto $\prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma)$ is a ps.d.o. (e.g. R, R' or R^r of Example 6.3, or $\Lambda^{(-2j-1)}$ of [12, Appendix]), then $pr_{Y'}$ (say) may be considered as the restriction of a ps.d.o.: Identify Y' with EY, and consider the commutative diagrams representing (4.38) and (4.39), connected by $E\colon Y \to Y'$ and $E\colon \prod_{j \in M_0} H^{-j-\frac{1}{2}}(\Gamma) \to \prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma)$. It is seen that $\Psi^* E \Psi = \Psi_1^* E \Psi_1$, which is composed of isomorphisms, so that $\Psi^* E \Psi$ maps $\prod_{j \in J_1'} H^{-j-\frac{1}{2}}(\Gamma)$ isomorphically onto $\prod_{j \in J_1'} H^{j+\frac{1}{2}}(\Gamma)$. Next, one finds that

$$pr_{Y'} = E\Psi_1(\Psi^* E \Psi)^{-1} \Psi^* \quad \text{on } \prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma), \tag{4.42}$$

so that $pr_{Y'}$ may be considered a restriction (cf.[4]) of the ps.d.o. $E\Psi(\Psi^* E \Psi)^{-1}\Psi^*$. (All this is seen easily from the indicated diagram, which we omit drawing, since the remark is of minor importance.)

LEMMA 4.2. *When $u \in D(A_1)$, u satisfies* (4.20) *if and only if*

(i) $\gamma u = \Phi \gamma_{K_0} u$

(ii) $(G_1 - \Psi^* P_{\gamma,\chi} \Phi)\gamma_{K_0} u$ *belongs to* $\prod_{j \in J_1'} H^{j+\frac{1}{2}}(\Gamma)$

and equals $\Psi^* \mu u$.

PROOF. The lemma is evident from the fact that, when $u \in D(A_1)$, $\mu u = \chi u - P_{\gamma,\chi} \gamma u \in \prod_{j \in M_0} H^{j+\frac{1}{2}}(\Gamma)$ (Proposition 2.7).

DEFINITION 4.2. Define the ps.d.o. $\mathscr{L}_1$ in Γ by

$$\mathscr{L}_1 = G_1 - \Psi^* P_{\gamma,\chi} \Phi, \tag{4.43}$$

it is of type $(-k, -2m + j + 1)_{j \in J_1', k \in K_0}$.

We note that $\mathscr{L}_1$ in general sends $\phi \in \prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Gamma)$ into

$$\mathscr{L}_1 \phi \in \prod_{j \in J_1'} H^{-2m+j+\frac{1}{2}}(\Gamma),$$

so that (ii) in Lemma 4.2 puts a nontrivial requirement on $\mathscr{L}_1 \gamma_{K_0} u$. This is essential in the description of L.

PROPOSITION 4.2. *The operator $L: X \to Y'$ associated with $\tilde{A}$ by Proposition 3.5 is defined by*

$$D(L) = \left\{ \phi \in X \,\middle|\, \mathscr{L}_1 \phi_{K_0} \in \prod_{j \in J_1'} H^{j+\frac{1}{2}}(\Gamma) \right\} \tag{4.44}$$

$$L\phi = pr_{Y'} I_{M_0 J_1'} \mathscr{L}_1 I_{K_0 M_0} \phi, \text{ when } \phi \in D(L). \tag{4.45}$$

PROOF. Let us compare the statement of Proposition 4.2 with the equation defining L in Proposition 3.5:

$$D(\tilde{A}) = \{u \in D(A_1) \,|\, \gamma u \in D(L), L\gamma u = pr_{Y'} \mu u\}. \tag{3.39}$$

When $u \in D(\tilde{A})$, we have by Lemma 4.2

$$\Psi^* \mu u = \mathscr{L}_1 \gamma_{K_0} u \in \prod_{j \in J_1'} H^{j+\frac{1}{2}}(\Gamma). \tag{4.46}$$

Inserting (4.39) in this, we find

$$\Psi_1^* pr_{Y'} \mu u = \mathscr{L}_1 \gamma_{K_0} u,$$

which, by application of (4.41), is equivalent with

$$pr_{Y'} \mu u = (\Psi_1^*)^{-1} \mathscr{L}_1 \gamma_{K_0} u = pr_{Y'} I_{M_0 J_1'} \mathscr{L}_1 I_{K_0 M_0} \gamma u. \tag{4.47}$$

This shows (4.45); and then another application of Lemma 4.2 shows (4.44).

REMARK 4.2. When E is chosen as in Remark 4.1, L is a restriction of the ps.d.o. $E\Psi(\Psi^* E \Psi)^{-1} \Psi^* I_{M_0 J_1'} \mathscr{L}_1 I_{K_0 M_0}$.

The following alternative definition of L is convenient in some questions.

LEMMA 4.3. *Let L_1 be the operator from $\prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Gamma)$ to $\prod_{j \in J_1'} H^{j+\frac{1}{2}}(\Gamma)$ obtained by restricting $\mathscr{L}_1$ to*

$$D(L_1) = \{\phi \in \prod_{k \in K_0} H^{-k-\frac{1}{2}}(\Gamma) \,|\, \mathscr{L}_1 \phi \in \prod_{j \in J_1'} H^{j+\frac{1}{2}}(\Gamma)\} \tag{4.48}$$

Then

$$L = pr_{Y'} I_{M_0 J_1'} L_1 I_{K_0 M_0} i_X. \tag{4.49}$$

PROOF. Evident, in view of Proposition 4.2.

This gives, concerning L^* (which defines $\tilde{A}^*$ by (3.40))

COROLLARY 4.2. $L^* = pr_X I_{M_0K_0} L_1^* I_{J_1'M_0} i_Y$.

From the definition of L_1 as a *maximal* operator for $\mathscr{L}_1$ it is now seen by standard arguments that L_1^* is a *minimal* operator for $\mathscr{L}_1^*$:

LEMMA 4.4. L_1^*: $\prod_{j\in J_1'} H^{-j-\frac{1}{2}}(\Gamma) \to \prod_{k\in K_0} H^{k+\frac{1}{2}}(\Gamma)$ *is the restricion of the ps.d.o.*

$$\mathscr{L}_1^* = G_1^* - \Phi^* P_{\gamma,\chi}^* \Psi \tag{4.50}$$

with domain equal to the closure of $\prod_{j\in J_1'}\mathscr{D}\Gamma)$ *in the graphnorm.*

Hereby $\tilde{A}^*$ is determined; moreover we find

PROPOSITION 4.3. *With* $\tilde{A}'$ *defined by* (4.30)–(4.31), *one has* $\tilde{A}^* = \tilde{A}'$ *exactly when* $\mathscr{L}_1 = G_1 - \Psi^* P_{\gamma,\chi}\Phi$ *has the property:*

$$\text{(4.51)}\quad \begin{cases} \prod_{k\in K_0}\mathscr{D}(\Gamma) \text{ is dense in } D(L_1) \text{ (defined by (4.48)) with respect} \\ \text{to the graph-norm } (\|\phi\|^2_{\{-k-\frac{1}{2}\},k\in K_0} + \|\mathscr{L}_1\phi\|^2_{\{j+\frac{1}{2}\},j\in J_1'})^{\frac{1}{2}}. \end{cases}$$

PROOF. $\tilde{A}^* = \tilde{A}'$ if and only if $\tilde{A} = (\tilde{A}')^*$, so the proposition may be proved by applying the whole set-up to $\tilde{A}'$. One may also observe that (4.51) and its analogue hold or do not hold for $\mathscr{L}_1$ and $\mathscr{L}_1^*$ simultaneously.

In particular, we obtain a characterization of the selfadjoint realizations of normal boundary problems, without any à priori regularity assumptions.

COROLLARY 4.3. $\tilde{A} = \tilde{A}^*$ *if and only if*: $A = A'$, $\Phi = \Psi$ *(i.e.* $J_1' = K_0$ *and* $F_2 = -(\Phi^*\mathscr{A}_{M_0J_1})^{-1}\Phi^*\mathscr{A}_{M_0K_1}$), *and* $\mathscr{L}_1$ *is selfadjoint as a ps.d.o. and satisfies* (4.51).

Density properties like (4.51) are often in the literature labeled "weak = strong" properties. They have been widely discussed, often in connection with regularity estimates, but they can also take place in cases without regularity.

4.3. *Application of the general theory.* We continue in the terminology of Section 4.2, so $\tilde{A}$ is the realization determined by (4.1) transformed into (4.11) or (4.20); X and Y are then given by (4.26) and (4.33), and $L: X \to Y'$ is determined by Proposition 4.2 (or Lemma 4.3).

In order to apply Theorems 3.4–3.6 we shall identify some of the things appearing there.

PROPOSITION 4.4. *Let* $H^{2m}(\Omega) \subset U \subset \mathscr{H}^{0,-m}_{A'}(\Omega)$.

The condition

$$D(L) \cap \gamma[U \cap D(A_1)] \subset Y \tag{4.52}$$

is equivalent with

$$J_1' = K_0 \text{ and } F_2 = -(\Phi^*\mathscr{A}_{M_0J_1})^{-1}\Phi^*\mathscr{A}_{M_0K_1}, \tag{4.53}$$

and then also with

$$X = Y. \tag{4.54}$$

PROOF. Recall $X = \Phi(\prod_{j\in K_0} H^{-j-\frac{1}{2}}(\Gamma))$, $Y = \Psi(\prod_{j\in J_1'} H^{-j-\frac{1}{2}}(\Gamma))$. Evidently,

$$X \cap \prod_{j\in M_0} H^{2m-j-\frac{1}{2}}(\Gamma) \subset D(L) \subset X,$$

then, since $\gamma[U \cap D(A_1)] \subset \gamma H^{2m}(\Omega) = \prod_{j\in M_0} H^{2m-j-\frac{1}{2}}(\Gamma)$, $D(L) \cap \gamma[U \cap D(A_1)]$ is dense in X, so that (4.52) is equivalent with

$$\Phi\left(\prod_{j\in K_0} H^{-j-\frac{1}{2}}(\Gamma)\right) \subset \Psi\left(\prod_{j\in J_1'} H^{-j-\frac{1}{2}}(\Gamma)\right). \tag{4.55}$$

Recall from Corollary 4.1 that (4.53) is equivalent with

$$\Phi = \Psi \quad (\text{i.e., } K_0 = J_1',\ F_0 = -G_2^*), \tag{4.56}$$

and thus with (4.54); we shall therefore simply show that (4.55) implies (4.56).

Both spaces in (4.55) are *graphs* (of F_0 resp. $-G_2^*$) and furthemore, because of the subtriangular property: $(F_0)_{jk} = 0$ for $j < k$, $\{j,k\} \in J_0 \times K_0$, and $(G_2^*)_{jk} = 0$ for $j < k$, $\{j,k\} \in K_1' \times J_1'$; the elements $\phi = \{\phi_0, \cdots, \phi_{m-1}\}$ in each space have the property that for each $l \in M_0$, ϕ_l depends at most on all the *preceding* entries $\{\phi_0, \cdots, \phi_{l-1}\}$. Then (4.55) implies successively, for each $l \in M_0$

$$K_0 \cap \{0, \cdots, l\} \subset J_1' \cap \{0, \cdots, l\},$$

so that $K_0 \subset J_1'$, whence, since $|K_0| = |J_1'|$,

$$K_0 = J_1'.$$

Then also $J_0 = K_1'$; and the two spaces are now graphs of operators F_0 resp. $-G_2^*$ with same domain $\prod_{j\in K_0} H^{-j-\frac{1}{2}}(\Gamma)$ and same range space $\prod_{j\in J_0} H^{-j-\frac{1}{2}}(\Gamma)$, thus (4.55) implies

$$F_0 = -G_2^*.$$

REMARK 4.3. Note that the condition (4.53) is concerned not only with principal symbols or symbols but with the complete structure of the operators F_0 and F_2. In this sense we call it a *global* condition.

LEMMA 4.5. *When $\Phi = \Psi$, and ϕ and $\psi \in D(L)$, then*

$$\langle L\phi, \psi\rangle = \langle \mathscr{L}_1\phi_{K_0}, \psi_{K_0}\rangle, \tag{4.57}$$

the left bracket denoting the duality between X' and X, and the right between $\prod_{j\in K_0} H^{j+\frac{1}{2}}(\Gamma)$ and $\prod_{j\in K_0} H^{-j-\frac{1}{2}}(\Gamma)$.

PROOF. We find by use of Lemma 4.3 and that $X = Y$, $K_0 = J_1'$,

$$\langle L\phi,\psi\rangle = \langle pr_{X'} I_{M_0K_0} L_1 I_{K_0M_0} i_X\phi,\psi\rangle = \langle L_1 I_{K_0M_0} i_X\phi, I_{K_0M_0} i_X\psi\rangle,$$

which we may write, omitting i_X, as

$$= \langle \mathscr{L}_1 \phi_{K_0}, \psi_{K_0}\rangle.$$

LEMMA 4.6. *When $\Phi = \Psi$,*

$$G_1 = \Phi^*\mathscr{A}_{M_0J_1}F_1 + \tfrac{1}{2}\Phi^*\mathscr{A}_{M_0M_0}\Phi \tag{4.58}$$

and

$$\mathscr{L}_1 = G_1 - \Phi^* P_{\gamma,\chi}\Phi = \Phi^*\mathscr{A}_{M_0J_1}F_1 - \Phi^*\mathscr{A}_{M_0M_1}P_{\gamma,\nu}\Phi; \tag{4.59}$$

here

$$\sigma^0(\mathscr{A}_{M_0M_1}P_{\gamma,\nu}) = -\,i A_{2m} I^\times S_m^- S_0^+, \tag{4.60}$$

in the notation of the Appendix (for $\mathscr{A}_{M_0J_1}$ cf. (2.32) *or* (6.28)).

PROOF. (4.58) and (4.59) follow from (4.23), (4.43) and (2.43); (4.60) follows from (6.28), (6.25).

With Q denoting the operator introduced in Definition 3.5, we have

LEMMA 4.7. *Assume A strongly elliptic. When $\Phi = \Psi$, and*

$$\phi \in D(L) \cap \prod_{j\in M_0} H^{m-j-\frac{1}{2}}(\Gamma), \textit{ then}$$

$$\mathrm{Re}\langle L\phi,\phi\rangle + \langle Q\phi,\phi\rangle = \langle(\mathrm{Re}\,\mathscr{L}_1 + \Phi^*Q\Phi)\phi_{K_0},\phi_{K_0}\rangle \tag{4.61}$$

(*the brackets denoting dualities between X' and X, $\prod_{j\in M_0} H^{-m+j+\frac{1}{2}}(\Gamma)$ and $\prod_{j\in M_0} H^{m-j-\frac{1}{2}}(\Gamma)$, resp. $\prod_{j\in K_0} H^{-m+j+\frac{1}{2}}(\Gamma)$ and $\prod_{j\in K_0} H^{m-j-\frac{1}{2}}(\Gamma)$*)*.*

PROOF. All expressions are well defined, in view of the types of $\mathscr{L}_1$, Φ and Q. Using Lemma 4.5 and that $\phi = \Phi\phi_{K_0}$ we have

$$\begin{aligned}\mathrm{Re}\langle L\phi,\phi\rangle + \langle Q\phi,\phi\rangle &= \mathrm{Re}\langle \mathscr{L}_1\phi_{K_0},\phi_{K_0}\rangle + \langle Q\Phi\phi_{K_0},\Phi\phi_{K_0}\rangle \\ &= \langle(\tfrac{1}{2}(\mathscr{L}_1 + \mathscr{L}_1^*) + \Phi^*Q\Phi)\phi_{K_0},\phi_{K_0}\rangle = \langle(\mathrm{Re}\,\mathscr{L}_1 + \Phi^*Q\Phi)\phi_{K_0},\phi_{K_0}\rangle.\end{aligned}$$

REMARK 4.4. When $\mathscr{L}_1$ has the property (4.51), the validity of (4.61) extends to $D(L) \cap \gamma[D(A_1') \cap D(A_1)]$, cf. Proposition 3.6 and Lemma 4.4 (the brackets suitably interpreted).

DEFINITION 4.3. When $\Phi = \Psi$, we define the ps.d.o. $\mathscr{K}$ in Γ by

$$\mathscr{K} = \operatorname{Re}\mathscr{L}_1 + \Phi^* Q \Phi. \tag{4.62}$$

It is of type $(-k, -2m+j+1)_{j,k \in K_0}$.

REMARK 4.5. $\mathscr{K}$ is the real part of the ps.d.o. named $\mathscr{K}$ in [14].

LEMMA 4.8.

$$\mathscr{K} = \operatorname{Re}[\Phi^* \mathscr{A}_{M_0 J_1} F_1] - \Phi^*(\mathscr{A}^r_{M_0 M_1} P^r_{\gamma,\nu} + \tfrac{1}{2}\mathscr{A}'_{M_0 M_0})\Phi; \tag{4.63}$$

here, with the notation of the Appendix,

$$\begin{aligned}\sigma^0(\mathscr{A}^r_{M_0M_1}P^r_{\gamma,\nu} + \tfrac{1}{2}\mathscr{A}'_{M_0M_0}) = i\, I^\times[&-(\operatorname{Re} A_{2m})\overline{S^r_m}S^r_0 \\ &+ \tfrac{1}{2}\overline{A_{2m}}(\overline{S_0^+}\,\overline{S_m^-} + \overline{S_m^+}\,\overline{S_0^-})],\end{aligned} \tag{4.64}$$

(*for* $\mathscr{A}_{M_0J_1}$ *cf.* (2.32) *or* (6.28)).

PROOF. Applying (4.59), (3.49) and (2.43) (noting (2.33)) to (4.62), we find

$$\begin{aligned}\mathscr{K} &= \operatorname{Re}[\Phi^*\mathscr{A}_{M_0J_1}F_1 + \tfrac{1}{2}\Phi^*\mathscr{A}_{M_0M_0}\Phi - \Phi^*P_{\gamma,\chi}\Phi] + \Phi^*[-P^r_{\gamma,\chi^r} + \operatorname{Re} P_{\gamma,\chi}]\Phi \\ &= \operatorname{Re}[\Phi^*\mathscr{A}_{M_0J_1}F_1] - \Phi^*[P^r_{\gamma,\chi^r} - \tfrac{1}{2}\operatorname{Re}\mathscr{A}_{M_0M_0}]\Phi \\ &= \operatorname{Re}[\Phi^*\mathscr{A}_{M_0J_1}F_1] - \Phi^*[\mathscr{A}^r_{M_0M_1}P^r_{\gamma,\nu} + \tfrac{1}{4}(\mathscr{A}_{M_0M_0} + \mathscr{A}'_{M_0M_0}) \\ &\qquad\qquad - \tfrac{1}{4}(\mathscr{A}_{M_0M_0} - \mathscr{A}'_{M_0M_0})]\Phi \\ &= \operatorname{Re}[\Phi^*\mathscr{A}_{M_0J_1}F_1] - \Phi^*[\mathscr{A}^r_{M_0M_1}P^r_{\gamma,\nu} + \tfrac{1}{2}\mathscr{A}'_{M_0M_0}]\Phi.\end{aligned}$$

Next, an application of (6.25), (6.27) and (6.28) to A^r resp. A' (noting (6.21) and (6.22)) gives (4.64).

Finally, note the following immediate consequence of the definition of Φ:

LEMMA 4.9. *For each* $s \in \mathbb{R}$, *there exist constants* $c_s > 0$, $c_s' > 0$ *such that for all* $\phi \in \prod_{j \in K_0} H^{s-j}(\Gamma)$,

$$\|\phi\|_{\{s-j\},j\in K_0} \leqq c_s \|\Phi\phi\|_{\{s-j\},j\in M_0} \leqq c_s'\|\phi\|_{\{s-j\},j\in K_0}.$$

We are now ready to apply Theorems 3.4–3.6.

THEOREM 4.1. *Let A be properly elliptic satisfying Definition* 2.2, *and let $\tilde{A}$ be the realization of A determined by* (4.2). *Let* $H^{2m}(\Omega) \subset U \subset H^m(\Omega)$. *There exists* $\lambda \in \mathbb{R}$ *such that*

$$\operatorname{Re}(Au, u) \geqq -\lambda \|u\|_m^2 \ \textit{for all}\ u \in D(\tilde{A}) \cap U \tag{4.65}$$

if and only if, in the terminology of Section 4.1,

$$J_1' = K_0,\ F_2 = -(\Phi^*\mathscr{A}_{M_0J_1})^{-1}\Phi^*\mathscr{A}_{M_0K_1}; \tag{4.66}$$

and then $\tilde{A}$ represents the boundary condition

$$(4.67)\qquad \gamma u = \Phi\gamma_{K_0}u, \quad \Phi^*\chi u = G_1\gamma_{K_0}u,$$

where $G_1 = \Phi^*\mathscr{A}_{M_0J_1}F_1 + \frac{1}{2}\Phi^*\mathscr{A}_{M_0M_0}\Phi$.

PROOF. Since $U \subset H^m(\Omega)$, $\gamma[U \cap D(A_1)] \subset \prod_{j\in M_0} H^{m-j-\frac{1}{2}}(\Gamma)$. Then, when (4.66) holds, we have for $\phi \in D(L) \cap \gamma[U \cap D(A_1)]$

$$(4.68)\qquad \begin{aligned} |\langle L\phi,\phi\rangle| &= |\langle \mathscr{L}_1\phi_{K_0}, \phi_{K_0}\rangle_{\{j+\frac{1}{2}\}\,\{-j-\frac{1}{2}\}}| = |\langle \mathscr{L}_1\phi_{K_0}, \phi_{K_0}\rangle_{\{-m+j+\frac{1}{2}\}\,\{m-j-\frac{1}{2}\}}| \\ &\leqq c\|\phi_{K_0}\|^2_{\{m-j-\frac{1}{2}\}} \leqq c'\|\phi\|^2_{\{m-j-\frac{1}{2}\}}, \end{aligned}$$

where we used Lemmas 4.5 and 4.9, and the type of $\mathscr{L}_1$. Then Theorem 3.4 implies that (4.65) is equivalent with (4.66), in view of Proposition 4.4. The last statement is now seen from Corollary 4.1 and Lemma 4.1.

Since (4.66) is independent of λ, we find

COROLLARY 4.4. *With the assumptions of Theorem* 4.1, $\tilde{A}$ *satisfies*

$$(4.69)\qquad |(Au,v)| \leqq c\|u\|_m\|v\|_m \quad \text{for } u,v\in D(\tilde{A})\cap U$$

if and only if (4.66) *holds.*

Here, Corollary 4.4 could also have been derived from Corollary 3.4, by use of the notations in Remarks 4.1 and 4 2.

EXAMPLE 4.1. When $\Omega = \mathbb{R}^n_+$ and $A = \Delta^2$, one finds that boundary conditions of the form

$$\gamma_1 u = F_{10}\gamma_0 u, \qquad \gamma_3 u = F_{30}\gamma_0 u + F_{32}\gamma_2 u,$$

satisfy $J_1' = K_0 (= \{0\})$; and that here the second part of (4.66) means exactly that $F_{32} = -F_{10}^*$.

REMARK 4.6. In continuation of Remark 3.6, we note that Theorem 4.1 and its corollary may be proved without the assumption that A has uniquely solvable Dirichlet problem, by use of techniques from Chapter 5; this will be done in Theorem 5.2.

Application of Theorem 3.5 leads to

THEOREM 4.2. *Let A and $\tilde{A}$ be as in Theorem* 4.1. *Let* $H^{2m}(\Omega) \subset U \subset \mathscr{H}^{0,-m}_{A'}(\Omega)$ *If there exists* $\lambda \in \mathbb{R}$ *such that*

$$(4.70)\qquad \operatorname{Re}(Au,u) \geqq -\lambda(\|u\|_0^2 + \|A'u\|^2_{-m}), \quad \forall u \in D(\tilde{A}) \cap U;$$

then, with the terminology of Section 4.1 *and Lemma* 4.6,

$$(4.71)\begin{cases}\text{(i)} & J_1' = K_0, \quad F_2 = -(\Phi^*\mathscr{A}_{M_0J_1})^{-1}\Phi^*\mathscr{A}_{M_0K_1}, \\ \text{(ii)} & \exists\lambda' \in \mathbb{R}\ s.th.\ \operatorname{Re}\langle L\phi,\phi\rangle \geqq -\lambda'\|\phi\|^2_{\{-j-\frac12\}}, \forall\phi\in D(L)\cap\gamma[U\cap D(A_1)] \\ \text{(iii)} & \operatorname{Re}\rho^0(\mathscr{L}_1)(y,\eta)\geqq 0,\ all\ (y,\eta)\in S(\Gamma).\end{cases}$$

Proof. That (4.70) implies (4.71) (i)–(ii) is immediate from Theorem 3.5 and Proposition 4.4. An application of Proposition 2.2 (i) then yields (iii): Choose a pseudo-differential isomorphism E of type $(-k-\frac12,\ j+\frac12)_{j,k\in M_0}$ as in Remark 4.1, then (4.71) (ii) implies that for $\phi\in D(L)\cap\prod_{j\in M_0}\mathscr{D}(\Gamma)$,

$$0\leqq \operatorname{Re}\langle L\phi,\phi\rangle + \lambda'\langle E\phi,\phi\rangle = \langle(\operatorname{Re}\mathscr{L}_1+\lambda'\Phi^*E\Phi)\phi_{K_0},\phi_{K_0}\rangle,$$

whence, since $\mathscr{L}_1$ is of type $(m-k-\frac12,\ -m+j+\frac12)_{j,k\in M_0}$,

$$\sigma^0(\operatorname{Re}\mathscr{L}_1+\lambda'\Phi^*E\Phi)=\sigma^0(\operatorname{Re}\mathscr{L}_1)=\operatorname{Re}\sigma^0(\mathscr{L}_1)\geqq 0 \text{ on } S(\Gamma).$$

Finally, we apply Theorem 3.6.

Theorem 4.3. *Assume that A is strongly elliptic, and that $\tilde A$ is the realization of A defined by* (4.2). *Then there exists $c>0$, $\lambda\in\mathbb{R}$ such that*

$$\operatorname{Re}\,(Au,u)\geqq c\|u\|_m^2-\lambda\|u\|_0^2\ for\ all\ u\in D(\tilde A) \tag{4.72}$$

if and only if, with the terminology of Section 4.1 *and Lemma* 4.8,

$$(4.73)\quad\begin{cases}\text{(i)} & J_1'=K_0\ and\ F_2=-(\Phi^*\mathscr{A}_{M_0J_1})^{-1}\Phi^*\mathscr{A}_{M_0K_1}\\ \text{(ii)} & \sigma^0(\mathscr{K})(y,\eta)>0\ for\ all\ (y,\eta)\in S(\Gamma).\end{cases}$$

In the affirmative case $D(\tilde A)\subset H^{2m}(\Omega)$; and $\tilde A^*$ is the realization of A' determined by the adjoint boundary condition

$$\gamma v=\Phi\gamma_{K_0}v,\quad \Phi^*\chi'v=G_1^*\gamma_{K_0}v, \tag{4.74}$$

and has the analogous properties.

Proof. When (4.72) holds, then (4.73) (i) holds by Theorem 4.1; and Theorem 3.6 assures that

$$(4.75)\quad \begin{aligned}&\langle L\phi,\phi\rangle+\langle Q\phi,\phi\rangle\geqq c'\|\phi\|^2_{\{m-j-\frac12\}}-\lambda'\|\phi\|^2_{\{-j-\frac12\}}\\ &\qquad\text{for } \phi\in D(L)\cap\prod_{j\in M_0}H^{2m-j-\frac12}(\Gamma).\end{aligned}$$

By Definition 4.3 and Lemmas 4.7–4.9, this implies

$$(4.76)\quad \begin{aligned}&\langle\mathscr{K}\phi_{K_0},\phi_{K_0}\rangle\geqq c''\|\phi_{K_0}\|^2_{\{m-j-\frac12\}}-\lambda''\|\phi_{K_0}\|^2_{\{-j-\frac12\}},\\ &\qquad\text{all } \phi_{K_0}\in\prod_{j\in K_0}H^{2m-j-\frac12}(\Gamma).\end{aligned}$$

Then, since $\mathscr{K}$ is of type $(m-k-\frac{1}{2},\ -m+j+\frac{1}{2})_{j,k\in M_0}$, Proposition 2.2 (ii) gives that $\sigma^0(\mathscr{K})>0$ on $S(\Gamma)$.

Conversely, assume (4.73). Then Proposition 2.2 (ii) (with the inequality extended by continuity to $\prod_{k\in K_0}H^{2m-j-\frac{1}{2}}(\Gamma)$) together with Theorem 3.6 gives that (4.73)(i) and (ii) imply

$$\text{(4.77)} \qquad \operatorname{Re}(Au,u)\geqq c\|u\|_m^2-\lambda\|u\|_0^2 \text{ on } D(\tilde{A})\cap H^{2m}(\Omega).$$

Now, since $\langle Q\phi,\phi\rangle\leqq 0$ (cf. (3.48)), we have

$$\text{(4.78)} \qquad \sigma^0(Q)\leqq 0 \text{ on } S(\Gamma),$$

so that (4.73) in particular implies

$$\operatorname{Re}\sigma^0(\mathscr{L}_1)=\sigma^0(\mathscr{K})-\sigma^0(\Phi)^*\sigma^0(Q)\sigma^0(\Phi)>0 \text{ on } S(\Gamma).$$

This shows that $\mathscr{L}_1$ is *elliptic*, and then (cf. Proposition 4.2, where $\phi=\Phi\phi_{K_0}$)

$$D(L)=\Phi\left(\prod_{j\in K_0}H^{2m-j-\frac{1}{2}}(\Gamma))\right)\subset\prod_{j\in M_0}H^{2m-j-\frac{1}{2}}(\Gamma),$$

which implies $D(T)\subset Z_A^{2m}(\Omega)$ and thus (c.f. (3.5))

$$D(\tilde{A})\subset H^{2m}(\Omega).$$

Then (4.77) implies (4.72). The statements concerning $\tilde{A}^*$ follow by use of Proposition 4.3, where (4.51) is valid by the ellipticity of $\mathscr{L}_1$.

REMARK 4.7. When (4.73) holds, the ellipticity of $\mathscr{L}_1$ assures in fact that for for all $t\geqq 0$

$$\text{(4.79)} \qquad u\in D(\tilde{A}),\ Au\in H^t(\Omega) \Rightarrow u\in H^{t+2m}(\Omega),$$

with the analogous property of $\tilde{A}^*$. This is so, because ellipticity of $\mathscr{L}_1$ is equivalent with the well known "complementing condition" (generalized to the ps.d.o.-case). We refrain from further discussion, since this aspect is so well covered in the literature (cf. e.g. [5], [25]).

EXAMPLE 4.2. For the boundary problem in Example 4.1 one finds, when $F_{32}=-F_{10}^*$: $\sigma^0(\mathscr{K}_1)=\operatorname{Re}\sigma^0(\mathscr{L}_1)=-\operatorname{Im}f_{30}(y,\eta)-4|\eta|^2\operatorname{Im}f_{10}(y,\eta)+2|\eta|\,|f_{10}(y,\eta)|^2+2|\eta|^3$ on $T.^*(\Gamma)$ (denoting $\sigma^0(F_{jk})$ by f_{jk}); positivity of this function is necessary and sufficient for 2-coerciveness.

Theorem 4.3 gives a concrete solution of the problem of characterizing the m-coercive realizations of normal boundary problems. For $s<m$ we shall let do with Theorems 3.6 and 4.1, together with the explicit description of $L\colon X\to Y'$ given above; except for the following remarks:

When $s \in]m - \frac{1}{2}, m]$, s-coerciveness is equivalent with m-coerciveness. Fujiwara treated in [9], [10] the case $s = m - \frac{1}{2}$ for a special class of boundary problems; his results should be extendable to the general case by the techniques of the present paper (the relevant condition on the ps.d.o. in the boundary is related to Hörmander's subellipticity [17]). For $s < m - \frac{1}{2}$, some results on the required inequalities may be found in Calderón [8]. Concerning "0-coerciveness", we note the consequence of Theorem 3.6 and Proposition 2.2 (i):

THEOREM 4.4. *Assumptions of Theorem* 4.3. *If, for some* $\lambda \in \mathbb{R}$,

$$\operatorname{Re}(Au, u) \geqq -\lambda \|u\|_0^2, \quad \forall u \in D(\tilde{A}) \cap H^{2m}(\Omega), \tag{4.90}$$

then

$$\begin{cases} \text{(i)} \quad J_1' = K_0, \quad F_2 = -(\Phi^* \mathscr{A}_{M_0 J_1})^{-1} \Phi^* \mathscr{A}_{M_0 K_1}; \\ \text{(ii)} \quad \sigma^0(\mathscr{K}) \geqq 0 \ \text{on}\ S(\Gamma). \end{cases} \tag{4.91}$$

As a corollary we find Agmon's result [2]:

COROLLARY 4.5. *In addition to the assumptions of Theorem* 4.3, *assume that* $D(\tilde{A}) \subset H^{2m}(\Omega)$ *and that* $\tilde{A}$ *is selfadjoint. Then* (4.90), (*or just* (4.91) (ii)) *implies that* $\tilde{A}$ *is m-coercive.*

PROOF. $D(\tilde{A}) \subset H^{2m}(\Omega)$ is equivalent with ellipticity of $\mathscr{L}_1$; then in view of Corollary 4.3, $\tilde{A}$ is selfadjoint if and only if $A = A'$, (4.91)(i) holds, and $\mathscr{L}_1 = \mathscr{L}_1^*$. Now $A = A'$ implies $Q = 0$, thus $\mathscr{K} = \operatorname{Re}\mathscr{L}_1 = \mathscr{L}_1$. By Theorem 4.4, (4.90) implies $\sigma^0(\mathscr{L}_1) \geqq 0$ on $S(\Gamma)$; this together with the ellipticity (i.e. $\sigma^0(\mathscr{L}_1) \neq 0$ on $S(\Gamma)$) gives $\sigma^0(\mathscr{L}_1) > 0$, whence $\tilde{A}$ is m-coercive by Theorem 4.3.

REMARK 4.8. Note however, that (4.90) together with regularity do not *in general* imply m-coerciveness. As a counter example, let $\mathscr{L}_1 = S - \Phi^* Q \Phi$, where S is skew-selfadjoint and elliptic with a large enough ellipticity constant such that $\mathscr{L}_1$ is also elliptic. Then $D(\tilde{A}) \subset H^{2m}(\Omega)$, but $\mathscr{K} = \operatorname{Re} S - \Phi^* Q \Phi + \Phi^* Q \Phi = 0$; and then, by Theorem 3.6, $\operatorname{Re}(Au, u) \geqq 0$ on $D(\tilde{A})$ but $\tilde{A}$ is not m-coercive. (The presence of Q plays no important rôle in this argument.)

Finally, we have a curious observation:

COROLLARY 4.6. *Assumptions of Theorem* 4.3. *When Q is elliptic, then* (4.90) *implies* $D(\tilde{A}) \subset H^{2m}(\Omega)$.

This follows by observing that (4.90) here implies $\operatorname{Re}\sigma^0(\mathscr{L}_1) = \sigma^0(\mathscr{K}) - \sigma^0(\Phi^* Q \Phi) \geqq -\sigma^0(\Phi^* Q \Phi) > 0$ on $S(\Gamma)$. An example, where Q *is* elliptic: Let

$n=2$, and let $a(y,\eta,\tau)=\tau^2+3i\eta\tau+4\eta^2$ on a component Γ_0 of Γ. One finds on this component: $a^+(y,\eta,\tau)=\tau-i\eta$ when $\eta>0$, and $\tau+4i\eta$ when $\eta<0$; $a^-(y,\eta,\tau)=\tau+4i\eta$ when $\eta>0$, and $\tau-i\eta$ when $\eta<0$; and $a^r(y,\eta,\tau)=\tau^2+4\eta^2$ $=(\tau-2i\eta)\ (\tau+2i\eta)$. This gives, by formulae (6.20)–(6.22) and (6.29) that $\sigma^0(P_{\gamma,\chi})=-\frac{5}{2}|\eta|$ and $\sigma^0(P^r_{\gamma,\chi^r})=-2|\eta|$; thus $\sigma^0(Q)=-\frac{1}{2}|\eta|$. If we take $\Omega\simeq\Gamma_0\times\,]0,1[$, we may obtain a similar result on the other component of Γ.

5. Comparison with prior results

The most fundamental previous result on m-coerciveness is due to Agmon [1] 1958, who characterized those normal boundary problems, associated in a certain way with sesquilinear forms, that determine m-coercive realizations.

We assume in this chapter, that $\bar{\Omega}\subset\mathbb{R}^n$ and is provided with the metric of $\mathbb{R}^n$. We use the standard notation for differential operators in terms of coordinates: With $x=(x_1,\cdots,x_n)$, let $D_j=i^{-1}\partial/\partial x_j$, and, for any multi-index $p=(p_1,\cdots,p_n)$, where each p_j denotes a non-negative integer,

$$D^p=D_1^{p_1}D_2^{p_2}\cdots D_n^{p_n}\quad(=D_x^p),\tag{5.1}$$

here $|p|=p_1+\cdots+p_n$ is the order of D^p.

Let us remark that we as usual assume for simplicity that all our differential operators have C^∞ coefficients, which is much more smoothness than Agmon required. To present his result, let us first recall

LEMMA 5.1. *Let there be given an integro-differential sesquilinear form*

$$c(u,v)=\int_\Omega\sum_{|p|,|q|\leq m}c_{pq}(x)\,D^qu\,\overline{D^pv}\,dx,\tag{5.2}$$

where each $c_{pq}(x)\in\mathscr{D}(\bar{\Omega})$, so that $c(u,v)$ is defined and continuous for $\{u,v\}\in H^m(\Omega)\times H^m(\Omega)$. Furthermore, let there be given a normal system $\beta=\{\beta_j\}_{j\in M_0}$ of boundary differential operators of orders j, $j\in M_0$. Denote by C the differential operator defined on $\mathscr{D}(\bar{\Omega})$ by

$$Cu=\sum_{|p|,|q|\leq m}D^p(c_{pq}D^ru).\tag{5.3}$$

Then there exists a unique system of boundary differential operators $\kappa=\{\kappa_j\}_{j\in M_0}$, with κ_j of order $2m-j-1$, such that

$$c(u,v)=(Cu,v)-\langle\kappa u,\beta v\rangle,\quad\textit{all}\quad u,v\in\mathscr{D}(\bar{\Omega}).\tag{5.4}$$

When C is elliptic, κ is furthermore a normal system.

HINT OF PROOF. Note first that $\beta = \mathscr{B}\gamma$ for some invertible differential operator $\mathscr{B}$ in Γ, so, by replacing κ by $\tilde{\kappa} = \mathscr{B}^*\kappa$ we may assume $\beta = \gamma$. Now integration by parts gives $(Cu, v) - c(u, v)$ as an integral over Γ involving derivatives of u up to order $2m - 1$ and of v up to order $m - 1$, this may be arranged in a form $\langle \kappa u, \gamma v \rangle$.–For the elliptic case, cf. e.g. [20, Section 2.2.4].

In the following we denote by $a(u, v)$ a sesquilinear form

$$a(u,v) = \int_\Omega \sum_{|p|,|q| \leqq m} a_{pq} D^q u\, \overline{D^p v}\, dx \tag{5.5}$$

with $a_{pq} \in \mathscr{D}(\bar{\Omega})$, for which the associated differential operator

$$A = \sum_{|p|,|q| \leqq m} D\; a_{pq} D^q \tag{5.6}$$

(understood like (5.3) and then extended to $\mathscr{D}'(\Omega)$) is properly elliptic. Here

$$\sigma^0(A)(x, \xi) = \sum_{|p| = |q| = m} a_{pq}(x) \xi^{p+q}. \tag{5.7}$$

We recall that A is determined by a, but not vice versa; in fact this is central for the discussion to follow.

THEOREM 5.1. (Agmon [1]). *Let a and A be given as above, with A strongly elliptic. Let $J_0 \subset M_0$, and let $\beta = \{\beta_j\}_{j \in J_0}$ be a normal system of differential boundary operators of orders j, $j \in J_0$, β_j of the form*

$$\beta_j = \gamma_j - \sum_{k<j} B_{jk} \gamma_k. \tag{5.8}$$

Then there exists $c > 0$, $\lambda \in \mathbb{R}$ such that

$$\operatorname{Re} a(u, u) \geqq c \|u\|_m^2 - \lambda \|u\|_0^2 \text{ for all } u \in H^m(\Omega) \text{ with } \beta u = 0, \tag{5.9}$$

if and only if the following condition is satisfied at each $y \in \Gamma$:

Introduce a local coordinate system (with generic point $z \in \mathbb{R}^n$) where $z_n = 0$ on Γ, $z = 0$ at y, and the normal derivative D_t at y goes into D_n at 0. With $\frac{1}{2}(A + A')$ denoted A^r, and $(\xi_1, \cdots, \xi_{n-1})$ denoted ξ', write in this coordinate system

$$\sigma^0(A^r)(0, \xi) = \sum_{l=0}^{2m} a_l^r(0, \xi') \xi_n^l,$$

$$\sigma_{|p|}(D_x^p)(0, \xi) = \sum_{l=0}^{|p|} d_{pl}(0, \xi') \xi_n^l,$$

and

$$\sigma_{j-k}(B_{jk})(0, \xi') = b_{jk}(\xi').$$

Then, for each $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, *the non-zero, bounded solutions of the ordinary differential boundary problem (where we denote* $(i^{-1}d/dt)^l v(t)$ *by* $v^{(l)}(t)$*)*

$$\begin{cases} \sum_{l=0} a_l^r(0,\xi')v^{(l)}(t) = 0 \text{ for } t>0, \\ \left[v^{(j)}(t) - \sum_{k<j} b_{jk}(\xi')v^{(k)}(t)\right]_{t=0} = 0, \text{ each } j \in J; \end{cases}$$

satisfy

$$\operatorname{Re}\int_0^\infty \sum_{|p|=|q|=m} a_{pq}(0)\left(\sum_{l=0}^m d_{ql}(0,\xi')v^{(l)}(t)\right)\overline{\left(\sum_{l=0}^m d_{pl}(0,\xi')v^{(l)}(t)\right)}\,dt > 0.$$

In view of Lemma 5.1, Theorem 5.1 has the corollary

COROLLARY 5.1. *Assumptions of Theorem 5.1. Let* $K_0 = M_0\setminus J_0$, *and choose normal boundary differential operators* β_j *of orders* j *for* $j \in K_0$. *Denoting* $\{\beta_j\}_{j\in M_0}$ *by* β, *let* κ *be the system of boundary operators with which*

$$a(u,v) = (Au,v) - \langle \kappa u, \beta v\rangle,\ u,v \in \mathscr{D}(\bar{\Omega}). \tag{5.9}$$

Then the realization $\tilde{A}$ *of* A *with domain*

$$D(\tilde{A}) = \{u \in D(A_1) \mid \beta_j u = 0 \text{ for } j \in J_0;\ \kappa_j u = 0 \text{ for } j \in K_0\} \tag{5.10}$$

is m-coercive if and only if the condition of Theorem 5.1 *holds.*

PROOF. Under the given circumstances one finds from (5.9) (after an extension by continuity to $H^{2m}(\Omega)$)

$$a(u,v) = (Au,v) \text{ for } u \in D(\tilde{A}) \cap H^{2m}(\Omega). \tag{5.11}$$

Then the statement follows from Theorem 5.1 by using (as Agmon pointed out) that the condition in Theorem 5.1 in particular implies that the system

$$[\{\beta_j\}_{j\in J_0}, \{\kappa_j\}_{j\in K_0}]$$

satisfies the complementing condition (cf. [5]), whence $D(\tilde{A}) \subset H^{2m}(\Omega)$.

This corollary indicates the applicability of Agmon's theorem. Although one may think of more general consequences of Theorem 5.1, there always remains the problem of finding a sesquilinear form a associated with A and with the particular boundary condition, such that (5.11) or a suitable generalization holds for functions satisfying the boundary condition. This puts a restriction on the class of realizations that may be tested by Agmon's theorem; a restriction that up until now does not seem to have been systematically characterized. (That there is really a restriction may be seen by noting that the condition in Theorem

5.1 concerns principal symbols, whereas we know from Theorem 4.3 that a global condition (like Theorem 4.3(i), cf. Remark 4.3) is necessary.) We shall go deeper into this question further below.

The subsequent treatments of the coerciveness problem avoided involving a sesquilinear form. In [2], [3] (1960 and 1962) Agmon gave a simple necessary and suffient condition for m-coerciveness of selfadjoint realizations of normal boundary problems, the so-called "strong complementing condition". (It concerns only principal symbols, which is well in accordance with Agmon's remark in [2], that such selfadjoint realizations may, at least locally, be brought into the framework of the sesquilinear froms. See also Corollary 4.5. The "strong complementing condition" may be defined generally for nonselfadjoint normal problems [3], for these however, the corresponding realizations are usually not semibounded, simply because our global condition need not be satisfied.) It has been known for a long time that the *orders* of the boundary operators must comply with the condition $J_1' = K_0$ in order for m-coerciveness to hold. Recently, Shimakura [26] aborded the problem again, giving a sufficient condition applicable to the boundary problems where

$$K_0 = J_1' = \{m-p, \cdots, m-1\}, \text{ for some } 0 \leqq p \leqq m. \tag{5.12}$$

This was soon after improved to a necessary and sufficient condition by Shimakura and Fujiwara [27] (see also [9]), and (independently, for formally selfadjoint A) by Grubb [12]. (The condition again concerns only principal symbols, but then again Theorem 4.3 (i) is trivially satisfied: F_0, F_2 and $\Phi^*\mathscr{A}_{M_0K_1}$ are all zero because of (5.12) and the "subtriangular" property, cf.[3].) Finally, Theorem 4.3 of the present paper, announced for formally selfadjoint A in [13], and for general A in [14], characterizes m-coerciveness of normal problems completely.

We shall now show how the problem of associating a sesquilinear form with $\tilde{A}$ fits into the general result. The fundamental step is expressed in Corollary 5.2, which is in a sense analogous to Lemma 2.2, but much more involved.

LEMMA 5.2. *Let a be associated with A by* (5.5)–(5.6). *Then*

$$a(u,v) = (Au,v) - \langle \chi u + S\gamma u, \gamma v\rangle, \textit{ all } u,v \in \mathscr{D}(\bar{\Omega}) \tag{5.13}$$

where S is a differential operator in Γ of type $(-k, -2m+j+1)_{j,k\in M_0}$.

PROOF. Define

$$a^*(v,u) = \overline{a(u,v)} \quad \text{for } u,v \in H^m(\Omega). \tag{5.14}$$

Then it is easily seen that a^* is associated analogously with A' (for $A' = \sum_{|p|,|q|\leq m} D^q \overline{a_{pq}} D^p$). Now, by Lemma 5.1,

$$a(u,v) = (Au,v) - \langle \kappa u, \gamma v\rangle, \quad a^*(v,u) = (A'v,u) - \langle \kappa' v, \gamma u\rangle, \tag{5.15}$$

for certain normal systems κ, κ', and then by (5.14)

$$(Au,v) - (u,A'v) = \langle \kappa u, \gamma v\rangle - \langle \gamma u, \kappa' v\rangle,$$

for $u,v \in \mathscr{D}(\bar{\Omega})$. Then Lemma 2.2 gives that $\kappa = \chi + S\gamma$ in the desired fashion.

LEMMA 5.3. *Let c and C be as in Lemma 5.1. If for some differential operators $\mathscr{B}_1$ and $\mathscr{B}_2$ in Γ of types $(-k, -2m+j+1)_{j\in M_0, k\in M_1}$ resp. $(-k, -2m+j+1)_{j,k\in M_0}$,*

$$c(u,v) = \langle \mathscr{B}_1 \nu u + \mathscr{B}_2 \gamma u, \gamma v\rangle, \quad \text{all } u,v \in \mathscr{D}(\bar{\Omega}), \tag{5.16}$$

then $C = 0$ and $\mathscr{B}_1 = 0$.

PROOF. Use Lemma 5.1, then a comparison of (5.4) with (5.16) gives

$$(Cu,v) = \langle \kappa u + \mathscr{B}_1 \nu u + \mathscr{B}_2 \gamma u, \gamma v\rangle, \quad \text{all } u,v \in \mathscr{D}(\bar{\Omega}). \tag{5.17}$$

In particular, for $u,v \in \mathscr{D}(\Omega)$, $(Cu,v) = 0$, whence

$$C = 0. \tag{5.18}$$

For one thing, this implies $\kappa = -\mathscr{B}_1 \nu - \mathscr{B}_2 \gamma$, by the uniqueness of κ. But also, the formal adjoint $C' = 0$, whence, by applying Lemma 5.1 to $c^*(v,u) = \overline{c(u,v)}$,

$$c^*(v,u) = -\langle \kappa' v, \gamma u\rangle, \quad u,v \in \mathscr{D}(\bar{\Omega}) \tag{5.19}$$

for some κ'. Together with (5.16) this gives the identity

$$\langle \mathscr{B}_1 \nu u + \mathscr{B}_2 \gamma u, \gamma v\rangle = -\langle \gamma u, \kappa' v\rangle, \quad u,v \in \mathscr{D}(\bar{\Omega}),$$

from which it follows, by letting u run through $\mathscr{D}(\bar{\Omega}) \cap H_0^m(\Omega)$, that $\mathscr{B}_1 = 0$.

In contrast with this, $\mathscr{B}_2$ may take any value, as will be shown now.

PROPOSITION 5.1. *When S is a differential operator in Γ of type $(-k, -2m+j+1)_{j,k\in M_0}$, there exist $c_{pq} \in \mathscr{D}(\bar{\Omega})$ such that, with $c(u,v)$ defined by* (5.2),

$$c(u,v) = \langle S\gamma u, \gamma v\rangle, \quad \text{all } u, v \in H^m(\Omega) \tag{5.20}$$

(then in particular C, defined by (5.3), *is zero).*

Proof. Of course it suffices to verify (5.20) for $u,v \in \mathscr{D}(\bar{\Omega})$.

1°. Consider the case where $\Omega = \mathbb{B}^n_+ = \mathbb{R}^n_+ \cap \mathbb{B}^n$ and $S = \sum_{k=1}^{n-1} a_k(x')D_k + a_0(x')$, each $a_k(x') \in \mathscr{D}(\mathbb{B}^{n-1})$[5].

Let $\zeta(x_n)$ denote a function in $\mathscr{D}(\mathbb{R})$, which is 1 near 0, and for which $\bigcup_{k=0}^{n-1}$ (supp $a_k \times$ supp $\zeta) \subset \mathbb{B}^n$. Denote $a_k(x')\zeta(x_n) = \tilde{a}_k(x)$. Now let $u, v \in \mathscr{D}(\Omega)$. Then

$$\int_{\mathbb{B}^{n-1}} a_0(x')\gamma_0 u \overline{\gamma_0 v}\, dx' = \int_{\mathbb{B}^{n-1}} \gamma_0(\tilde{a}_0 u\bar{v})\, dx'$$
$$= -i \int_\Omega D_n(\tilde{a}_0 u\bar{v}) dx = \int_\Omega (-i\tilde{a}_0(D_n u)\bar{v} + i\tilde{a}_0 u\overline{D_n v} - i(D_n\tilde{a}_0)u\bar{v})\, dx.$$

To handle the first order terms we note that, when $b_{kl} \in \mathscr{D}(\bar{\Omega}) \cap \mathscr{D}(\mathbb{B}^n)$, $k, l = 1, \cdots, n$, then

$$\sum_{k,l=1}^{n} \int_\Omega b_{kl}(x)\, D_k u \overline{D_l v}\, dx = \sum_{k,l=1}^{n} \int_\Omega \left[i\frac{\partial}{\partial x_l}(b_{kl} D_k u\ \bar{v}) + D_l(b_{kl} D_k u)\bar{v} \right] dx$$
$$= \int_{\mathbb{B}^{n-1}} \left(-i \sum_{k=1}^{n-1} b_{kn}(x')\, D_k\gamma_0 u - i\, b_{nn}\gamma_1 u \right) \overline{\gamma_0 v}\, dx' + \sum_{k,l=1}^{n} \int_\Omega D_l(b_{kl} D_k u)\,\bar{v}\, dx.$$

In particular, when $b_{kl} = 0$ for $k, l = 1, \cdots, n-1$, and $b_{kn} = -b_{nk}$ for $k = 1, \cdots, n$ then

$$\sum_{k,l=1}^{n} D_l(b_{kl} D_k u) = \sum_{k=1}^{n-1} (D_n b_{kn}) D_k u - \sum_{k=1}^{n-1} (D_k b_{kn})\, D_n u,$$

so that

$$\int_{\mathbb{B}^{n-1}} \left(\sum_{k=1}^{n-1} b_{kn}(x')\, D_k\gamma_0 u \right) \overline{\gamma_0 v}\, dx'$$
$$= \int_\Omega \left[i \sum_{k,l=1}^{n} b_{kl} D_k u\, \overline{D_l v} - i \sum_{k=1}^{n-1} (D_n b_{kn}) D_k u\ \bar{v} + i \sum_{l=1}^{n-1} (D_l b_{ln}) D_n u\ \bar{v} \right] dx$$

This leads to the following choice of the c_{pq}, $|p|$, $|q| \leqq 1$ (we denote the kth unit vector by e_k):

$$c_{e_k,e_l} = 0 \quad \text{for } k, l = 1, \cdots, n-1;\ c_{e_n,e_n} = 0;$$
$$c_{e_k,e_n} = -\, c_{e_n,e_k} = i\,\tilde{a}_k \quad \text{for } k = 1, \cdots, n-1;$$
$$c_{e_k,0} = -\, i D_n\,\tilde{a}_k \quad \text{for } k = 1, \cdots, n-1;\ c_{e_n,0} = i \sum_{l=1}^{n-1} D_l\,\tilde{a}_l - i\,\tilde{a}_0;$$
$$c_{0,e_k} = 0 \quad \text{for } k = 1, \cdots, n-1;\ c_{0,e_n} = i\,\tilde{a}_0;\ c_{0,0} = -\, i\, D_n\,\tilde{a}_0;$$

[5] $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$; $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$; and $\{x \in \mathbb{R}^n \mid x_n = 0, |x| < 1\}$ is identified with $\mathbb{B}^{n-1}$.

with which one has

$$\int_{\mathbb{B}^{n-1}} \left(\sum_{n=1}^{k-1} a_k D_k + a_0 \right) \gamma_0 u \overline{\gamma_0 v}\, dx' = \int_\Omega \sum_{|p|,|q| \leqq 1} c_{pq} D^q u\, \overline{D^p v}\, dx.$$

The idea in this proof stems from Agmon [2], [4].

2°. Let again $\Omega = \mathbb{B}^n_+$, and let now S_{jk} denote a differential operator in $\mathbb{R}^{n-1}$ with coefficients in $\mathscr{D}(\mathbb{B}^{n-1})$, of order $l = 2m - j - k$, where j and k are two nonnegative integers $\leqq m - 1$. We consider for $u, v \in \mathscr{D}(\bar{\Omega})$ the integral

$$\int_{\mathbb{B}^{n-1}} S_{jk} \gamma_k u\, \overline{\gamma_j v}\, dx'.$$

To treat this, write S_{jk} as a finite sum

$$S_{jk} = \sum_i Q_i R_i T_i,$$

where the Q_i, R_i and T_i are differential operators in $\mathbb{R}^{n-1}$, with coefficients in $\mathscr{D}\,(\mathbb{B}^{n-1})$, and of orders respectively $m - 1 - j$, 1 and $m - 1 - k$. Then

$$\int_{\mathbb{B}^{n-1}} S_{jk} \gamma_k u\, \overline{\gamma_j v}\, dx' = \sum_i \int_{\mathbb{B}^{n-1}} R_i (T_i \gamma_k u) \overline{(Q_i^* \gamma_j v)}\, dx'.$$

Here $T_i \gamma_k u = \gamma_0 \tilde{T}_i u$, $Q_i^* \gamma_j v = \gamma_0 \tilde{Q}_i v$, for suitable smooth differential operators $\tilde{T}_i$, $\tilde{Q}_i$ of orders $m - 1$ in Ω. This reduces the problem to 1°, whereby one altogether finds functions $c_{pq}(x) \in \mathscr{D}(\bar{\Omega}) \cap \mathscr{D}(\mathbb{B}^n)$ for which

$$\int_{\mathbb{B}^{n-1}} S_{jk} \gamma_k u\, \overline{\gamma_j v}\, dx' = \int_\Omega \sum_{|p|,|q| \leqq m} c_{pq} D^q u\, \overline{D^p v}\, dx,$$

all $u, v \in \mathscr{D}(\bar{\Omega})$. When $S = (S_{jk})_{j,k \in M_0}$ is a differential operator in $\mathbb{R}^{n-1}$ of type $(-k, -2m + j + 1)_{j,k \in M_0}$ with coefficients in $\mathscr{D}(\mathbb{B}^{n-1})$,

$$\int_{\mathbb{B}^{n-1}} S\gamma u \cdot \overline{\gamma v}\, dx' = \sum_{j,k \in M_0} \int_{\mathbb{B}^{n-1}} S_{jk} \gamma_k u\, \overline{\gamma_j v}\, dx',$$

where each summand is handled by the above argument.

3°. The general case, where Ω is a smooth bounded domain in $\mathbb{R}^n$ and S is a system of differential operators in the boundary Γ, is now reduced to 2° by use of a finite system of local coordinates. (The arguments involved are standard, and will not be reproduced here.)

Altogether, Lemmas 5.2 and 5.3 and Proposition 5.1 imply (after an extension by continuity, cf. Proposition 2.4):

COROLLARY 5.2. *When $a(u,v)$ is a sesquilinear form associated with A, then*

(5.21) $$a(u,v)=(Au,v)-\langle \chi u+S\gamma u,\gamma v\rangle,\ \textit{all}\ u\in \mathscr{H}_A^{m,0}(\Omega), v\in H^m(\Omega),$$

where S is a differential operator in Γ of type $(-k,\ -2m+j+1)_{j,k\in M_0}$. Conversely, when S is such an operator, there exists a sesquilinear form $a(u,v)$ associated with A, satisfying (5.21).

We can now show that the differential boundary problems to which Corollary 5.1 may be applied are exactly those which satisfy the "global" condition (4.66) (i.e., Theorem 4.3(i)). At the same time, we shall show how Theorem 4.1 may be proved (after reduction the case $\bar{\Omega}\subset\mathbb{R}^n$) without the assumption that A has uniquely solvable Dirichlet problem, cf. Remark 4.6.

THEOREM 5.2. *Assume merely proper ellipticity of A. Let $\tilde{A}$ be the realization of A determined by a boundary condition* (4.1). *Then there exist $\lambda\in\mathbb{R}$ such that*

(5.22) $$\operatorname{Re}(Au,u)\geqq-\lambda\|u\|_m^2,\ \textit{all}\ u\in D(\tilde{A})\cap H^m(\Omega)$$

if and only if (4.66) *holds.*

If furthermore the operators F_{jk} are differential operators, (4.66) *is equivalent with the existence of a sesquilinear form $a(u,v)$ (as* (5.5)*) such that*

(5.23) $$(Au,v)=a(u,v)\ \textit{for all}\ u,v\in D(\tilde{A})\cap H^m(\Omega).$$

PROOF. Pick an arbitrary sesquilinear form $a_0(u,v)$ associated with A, it satisfies an equation (5.21). Inserting this in (5.22), we find that (5.22) is equivalent with the validity of an inequality

(5.24) $$\operatorname{Re}\langle\chi u,\gamma u\rangle\geqq-\lambda_1\|u\|_m^2,\ \text{all}\ u\in D(\tilde{A})\cap H^m(\Omega),$$

since $a_0(u,u)$ and $\langle S\gamma u,\gamma u\rangle$ are continuous on $H^m(\Omega)$.

To prove the first statement, let us begin with assuming that (5.24) holds. Let $u_0\in D(\tilde{A})\cap H^m(\Omega)$, and let w run through $\mathscr{D}(\Omega)$, then insertion of $u=u_0+w$ in (5.24) gives

$$\operatorname{Re}\langle\chi u_0,\gamma u_0\rangle\geqq-\lambda_1\|u_0+w\|_m^2,\quad \text{all}\ w\in\mathscr{D}(\Omega),\ \text{whence (assuming}\ \lambda_1\geqq 0)$$

$$\operatorname{Re}\langle\chi u_0,\gamma u_0\rangle\geqq-\lambda_1\inf_{w\in\mathscr{D}(\Omega)}\|u_0+w\|_m^2\geqq-\lambda_2\|\gamma u_0\|_{\{m-j-\frac12\}}^2$$

(cf. Proposition 2.1). Now, using that u_0 satisfies (4.1), written in the form (4.11), we have

$$\operatorname{Re}\left[\langle\chi_{K_1'}u_0,\gamma_{K_1'}u_0\rangle+\langle G_1\gamma K_0u_0+G_2\chi_{K_1'}u_0,\gamma_{J_1'}u_0\rangle\right]$$
$$\geqq-\lambda_2\|\gamma u_0\|_{\{m-j-\frac12\}}^2,$$

whence, using the type of G_1, and the fact that $\gamma u_0=\Phi\gamma_{K_0}u_0$,

$$\text{(5.25)} \qquad \operatorname{Re}\langle \chi_{K_1'}u_0, \gamma_{K_1'}u_0 + G_2^*\gamma_{J_1'}u_0\rangle \geqq -\lambda_3 \| \gamma_{K_0}u_0 \|^2_{\{m-j-\frac{1}{2}\}}.$$

Here, $\chi_{K_1'}u_0$ and $\gamma_{K_0}u_0$ independently run through all smooth values, when u_0 runs through $D(\tilde{A}) \cap H^{2m}(\Omega)$, and therefore (5.25) implies

$$\text{(5.26)} \qquad \gamma_{K_1'}u_0 + G_2^*\gamma_{J'_1}u = 0 \text{ for all } u_0 \in D(\tilde{A}) \cap H^{2m}(\Omega).$$

This means in particular that, with the notations of Definition 4.1,

$$\text{(5.27)} \qquad \Phi\left(\prod_{j\in K_0} H^{2m-j-\frac{1}{2}}(\Gamma)\right) \subset \Psi\left(\prod_{j\in J_1'} H^{2m-j-\frac{1}{2}}(\Gamma)\right),$$

to which the proof of Proposition 4.4 may be applied to conclude that $\Phi = \Psi$, i.e., (4.66) holds (cf. Corollary 4.1).

Conversely, assume (4.66). This means that the boundary condition has the form

$$\text{(5.28)} \qquad \gamma u = \Phi\gamma_{K_0}u, \qquad \Phi^*\chi u = G_1\gamma_{K_0}u.$$

Then, for $u, v \in D(\tilde{A}) \cap H^m(\Omega)$,

$$\text{(5.29)} \qquad \begin{aligned} (Au, v) &= a_0(u,v) + \langle \chi u + S\gamma u, \gamma v\rangle \\ &= a_0(u,v) + \langle \Phi^*\chi u + \Phi^* S\gamma u, \gamma_{K_0}v\rangle \\ &= a_0(u,v) + \langle (G_1 + \Phi^*S\Phi)\gamma_{K_0}u, \gamma_{K_0}v\rangle. \end{aligned}$$

Thus, in view of the type of $G_1 + \Phi^*S\Phi$,

$$\text{(5.30)} \qquad |(Au,v)| \leqq c\|u\|_m \|v\|_m \text{ for all } u, v \in D(\tilde{A}) \cap H^m(\Omega),$$

which in particular shows (5.22).

For the second statement we proceed as follows:

When (5.23) holds, (5.22) is an immediate consequence, so (4.66) holds by the first part. Conversely, when (4.66) holds and the F_{jk} are differential operators, the boundary condition is of the form (5.28), so that one has (5.29) for $u, v \in D(\tilde{A}) \cap H^m(\Omega)$, with $G_1 + \Phi^*S\Phi$ a differential operator of type

$$(-k, -2m + j + 1)_{j,k\in K_0}.$$

By Proposition 5.1 there exists a sesquilinear form $c(u,v)$ with $C = 0$ such that

$$c(u,v) = \langle (G_1 + \Phi^*S\Phi)\gamma_{K_0}u, \gamma_{K_0}v\rangle, \text{ all } u, v \in H^m(\Omega),$$

then

$$a(u,v) = a_0(u,v) + c(u,v)$$

fits together with $\tilde{A}$ in (5.23).

REMARK 5.1. This theorem actually does not use the ellipticity of A, but rather that Γ is nowhere characteristic for A (cf. (2.26)).

In the course of the proof we also found

COROLLARY 5.3. *Under the assumptions of Theorem* 5.2, (4.66) *is also equivalent with* (5.30).

Finally, we have

COROLLARY 5.4. *The normal differential boundary problems for A that may be brought into the framework of Corollary* 5.1 *are exactly those which satisfy* (4.66).

When $\tilde{A}$ *is a realization determined by such a boundary condition, and* $a(u,v)$ *has been chosen to satisfy* (5.23), *and A is strongly elliptic, then Agmon's condition in Theorem* 5.1 (*on principal symbols*) *is equivalent with our condition* (ii) *in Theorem* 4.3.

6. Appendix. Further details on P.

In this appendix, we consider Ω as an open subset of a compact manifold Σ without boundary, as described in Section 2.1. We shall then also assume, as we may, that A is defined and properly elliptic throughout Σ. Now $\gamma_j (j \in M)$ is defined on smooth functions in Σ as usual as $(D_t^j u)\big|_\Gamma$, but for the extended definitions as in Proposition 2.4, there is a distinction between $\gamma_j^+ = \mathscr{H}_A^{s,t}(\Omega) \to H^{s-j-\frac{1}{2}}(\Gamma)$ and $\gamma_j^- : \mathscr{H}_A^{s,t}(\Sigma \setminus \bar{\Omega}) \to H^{s-j-\frac{1}{2}}(\Gamma)$, when $s < 2m$. We set $\rho^\pm = \{\gamma_j^\pm\}_{j \in M}$, $\gamma^\pm = \rho_{M_0}^\pm$ and $\nu^\pm = \rho_{M_1}^\pm$.

We assume as usual that A in Ω has uniquely solvable Dirichlet problem (Definition 2.2) (but remark that all considerations may be carried through without it, with evident modifications); then in particular, $\{u \in \mathscr{D}(\Sigma) \mid Au = 0$ in $\Sigma, \rho u = 0\} = \{0\}$. With this assumption, the result of Seeley [24] (cf. also Caldéron [7], Hörmander [17]) takes the form

PROPOSITION 6.1. *For each* $s \in \mathbb{R}$

$$(6.1) \qquad \prod_{j \in M} H^{s-j-\frac{1}{2}}(\Gamma) = \rho^+ Z_A^s(\Omega) \dotplus \rho^- Z_A^s(\Sigma \setminus \bar{\Omega}),$$

topological direct sum; and the projections Q^+ *and* Q^- *in* $\prod_{j \in M} H^{s-j-\frac{1}{2}}(\Gamma)$ *defined by* (6.1) *with ranges* $\rho^+ Z_A^s(\Omega)$ *resp.* $\rho^- Z_A^s(\Sigma \setminus \bar{\Omega})$ *are ps. d. o.'s in* Γ *of types* $(-k, -j)_{j,k \in M}$. *Moreover, their principal symbols are determined by the analogous construction for the ordinary differential operator*

$$(6.2)\qquad a\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right)=\sum_{l=0}^{2m}a_l(y,\eta)\left(\frac{1}{i}\frac{d}{dt}\right)^l,\quad t\in\mathbb{R},$$

at each fixed $(y,\eta)\in T^*.(\Gamma)$: here Ω,Γ and Σ are replaced by $\mathbb{R}_+$, $\{0\}$ and $\mathbb{R}$, respectively, and ρ^+ and ρ^- map

$$Z_a(\mathbb{R}_+)=\left\{z(t)\in\mathscr{S}(\mathbb{R}_+)\,\middle|\,a\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right)z(t)=0\text{ on }\mathbb{R}_+\right\}^6$$

resp.

$$Z_a(\mathbb{R}_-)=\left\{z(t)\in\mathscr{S}(\mathbb{R}_-)\,\middle|\,a\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right)z(t)=0\text{ on }\mathbb{R}_-\right\}$$

into $\prod_{j\in M}\mathbb{C}\quad(=\mathbb{C}^{2m})$, which decomposes

$$(6.3)\qquad \prod_{j\in M}\mathbb{C}=\rho^+Z_a(\mathbb{R}_+)\dotplus\rho^-Z_a(\mathbb{R}_-),$$

such that the projection $M\times M$-matrices $q^+(y,\eta)$ and $q^-(y,\eta)$ determined by (6.3) *are the principal symbols of Q^+ resp. Q^- at (y,η).*

Seeley proves this by showing that, when A is an invertible operator on $\mathscr{D}(\Sigma)$, $Q^+=-\rho^+A^{-1}\rho^*\mathscr{A}^{-1}$ and $Q^-=\rho^-A^{-1}\rho^*\mathscr{A}^{-1}$ (cf. (2.31)–(2.32)), which he modifies to the general case where A has an index in the appendices of [24] and [25].

We shall now rapidly indicate how Proposition 2.6 is proved on the basis of this.

Recalling (2.29)–(2.30), we define for each $(y,\eta)\in T.^*(\Gamma)$ the differential operators in $\mathbb{R}$

$$(6.4)\qquad \begin{cases} a^+\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right)=\sum_{l=0}^{m}s_l^+(y,\eta)\left(\frac{1}{i}\frac{d}{dt}\right)^l\\ a^-\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right)=\sum_{l=0}^{m}s_l^-(y,\eta)\left(\frac{1}{i}\frac{d}{dt}\right)^l. \end{cases}$$

Then

$$(6.5)\qquad a\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right)=A_{2m}(y)a^+\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right)a^-\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right)$$
$$=A_{2m}(y)a^-\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right)a^+\left(y,\eta,\frac{1}{i}\frac{d}{dt}\right).$$

---

6 $\mathscr{S}(\mathbb{R}\pm)$ denotes the space of functions $u\in C^\infty(\overline{\mathbb{R}\pm})$ for which $t^kd^l/dt^lu(t)$ is bounded on $\mathbb{R}_+$ resp. $\mathbb{R}_-$, for all $k\geqq 0$, $l\geqq 0$.

(We shall omit the dependence on (y,η) whenever convenient). The following lemma is well known.

LEMMA 6.1. *A function* $u \in C^\infty(\overline{\mathbb{R}_\pm})$ *belongs to* $Z_a(\mathbb{R}_\pm)$ *if and only if*

$$(6.6)^\pm \qquad a^\pm\left(\frac{1}{i}\frac{d}{dt}\right)u(t) = 0 \text{ on } \mathbb{R}_\pm.$$

(The "if" part follows from the fact that a solution of $(6.6)^\pm$ is a linear combination of exponentials $\exp(i\tau_k^\pm t)$, where the $\tau_k^\pm$ are the roots of $a^\pm(\tau)$, and thus belongs to $\mathscr{S}(\mathbb{R}_+)$. The "only if" part uses the Paley Wiener theorem.)

Now, a solution u of $(6.6)^\pm$ is uniquely determined by the value of γu, therefore we may introduce

DEFINITION 6.1. $p^\pm$ is the $M_1 \times M_0$-matrix sending γu into νu for $u \in Z_a(\mathbb{R}_+)$.

Furthermore, Lemma 6.1 reduces the computation of $p^\pm$ to an algebraic manipulation:

LEMMA 6.2. $p^\pm$ *is the matrix* $(p_{jk})_{j\in M_1, k\in M_0}$ *whose elements are the coefficients in the rest polynomials of the division equations*

$$(6.7)^\pm \qquad \begin{gathered} \tau^j = \left(\sum_{k\in M_0} c_{jk}\tau^k\right)a^\pm(\tau) - \sum_{k\in M_0} p_{jk}^\pm\tau^k, \qquad j\in M_1 \\ (i.e., \tau^j \equiv \sum_{k\in M_0} p_{jk}^\pm \tau^k \ (\operatorname{mod} a^\pm(\tau)), \textit{ for each } j \in M_1). \end{gathered}$$

By use of the identities $(6.7)^\pm$ one easily shows

LEMMA 6.3. p^+, p^- *and* $p^+ - p^-$ *are invertible matrices.*

This leads to the construction of $q^\pm$ from $p^\pm$: Let $\phi \in \prod_{j\in M}\mathbb{C}$, and denote elements of $\prod_{j\in M_0}\mathbb{C}$ by x, y, then the equation

$$(6.8) \qquad \{\phi_{M_0}, \phi_{M_1}\} = \{x, p^+x\} + \{y, p^-y\}$$

(where evidently $\{x, p^+x\} \in \rho Z_a(\mathbb{R}_+)$ and $\{y, p^-y\} \in \rho Z_a(\mathbb{R}_-)$) has the unique solution

$$(6.9) \qquad \begin{cases} x = (p^+ - p^-)^{-1}(-p^-\phi_{M_0} + \phi_{M_1}) \\ y = (p^+ - p^-)^{-1}(p^+\phi_{M_0} - \phi_{M_1}). \end{cases}$$

This gives that

$$(6.10) \qquad q^+ = \begin{pmatrix} -(p^+ - p^-)^{-1}p^- & (p^+ - p^-)^{-1} \\ -p^+(p^+ - p^-)^{-1}p^- & p^+(p^+ - p^-)^{-1} \end{pmatrix}$$

and

$$(6.11) \qquad q^- = \begin{pmatrix} (p^+ - p^-)^{-1}p^+ & -(p^+ - p^-)^{-1} \\ p^-(p^+ - p^-)^{-1}p^+ & -p^-(p^+ - p^-)^{-1} \end{pmatrix} = I_{MM} - q^+.$$

The observation that we shall use is that, by application of Lemma 6.3 again, each of the four $m \times m$-blocks in q^+ and q^- is *invertible*, and that p^+ and p^- may in fact be derived from q^+ or q^-. More precisely, we shall use

LEMMA 6.4. *In the $M \times M$-matrix q^+, each of the minors $q_{M_iM_j}$ $(i,j = 0,1)$ is invertible, and*

$$p^+ = (q^+_{M_0M_1})^{-1}(I_{M_0M_0} - q^+_{M_0M_0}). \tag{6.12}$$

This lemma leads immediately to an analogue in the non-symbolic set-up:

PROPOSITION 6.2. *In the $M \times M$-matrix of ps.d.o.'s Q^+, each of the minors $Q^+_{M_iM_j}$ $(i,j = 0,1)$ is* e l l i p t i c, *and, with T denoting a parametrix of $Q^+_{M_0M_1}$,*

$$P^A_{\gamma,\nu} = T(I_{M_0M_0} - Q^+_{M_0M_0}) + S, \tag{6.13}$$

where S is of order $-\infty$.

PROOF. The invertibility of $q^+_{M_iM_j}(y,\eta)$ for each $(y,\eta) \in T.^*(\Gamma)$ means exactly that $Q^+_{M_iM_j}$ is elliptic. Now, the elements $\phi \in \rho^+ Z_A(\Omega)$ satisfy

$$\phi_{M_1} = P^A_{\gamma,\nu}\phi_{M_0}, \text{ where } \phi_{M_0} \text{ runs through} \prod_{j \in M_0} H^{s-j-\frac{1}{2}}(\Gamma), \tag{6.14}$$

and on the other hand $Q^+\phi = \phi$, which may be written

$$Q^+_{M_0M_0}\phi_{M_0} + Q^+_{M_0M_1}\phi_{M_1} = \phi_{M_0}, \tag{6.15}$$

$$Q^+_{M_1M_0}\phi_{M_0} + Q^+_{M_1M_1}\phi_{M_1} = \phi_{M_1}; \tag{6.16}$$

then an insertion of (6.14) in (6.15) gives

$$Q^+_{M_0M_0} + Q^+_{M_0M_1}P^A_{\gamma,\nu} = I_{M_0M_0},$$

from which (6.13) follows by composition with T.

In view of (6.12) and Lemma 6.2, this proves Proposition 2.6: that $P^A_{\gamma,\nu}$ is a ps.d.o. of type $(-k,-j)_{j \in M_1, k \in M_0}$ with principal symbol $p^+(y,\eta)$. (Computation of the complete symbol of $P^A_{\gamma,\nu}$ may also be based on (6.13).)

REMARK 6.1 The convenient aspect of this proof of Proposition 2.6 is that it never moves outside of ps.d.o.'s and standard boundary operators (and their adjoints), as would be required if one tried to generalize the considerations behind Lemma 6.2 directly; it seems hard to associate with $a^+(y,\eta,\tau)$ a workable global operator in $\bar{\Omega}$.

Definition 2.5 of $P_{\gamma,\beta} = \beta\gamma_Z^{-1}$ may be extended to arbitrary β, of the form $\beta = \{\beta_j\}_{j \in J}$ where J is any finite subset of $N \cup \{0\}$, $\{m_j\}_{j \in J}$ is any real J-vector, and

(6.17)
$$\beta_j = \sum_{l=0}^{j} B_{jl}\gamma_l, \quad j \in J,$$

with each B_{jl} denoting a (scalar) ps.d.o. in Γ of order $m_j - l$; by use of the identity $D_t^{2m}u = -A_{2m}^{-1}(\sum_{l=0}^{2m-1} A_l D_t^l u)$ near Γ when $u \in Z_A^s(\Omega)$ (cf. (2.26)). Then one finds easily

PROPOSITION 6.3. *With β as above, $P_{\gamma,\beta}^A$ is a ps.d.o. in Γ of type $(-k, -m_j)_{j\in J, k\in M_0}$ with principal symbol $(p_{jk})_{j\in J, k\in M_0}$ consisting, at each $(y,\eta) \in T^*.(\Gamma)$, of the coefficients p_{jk} in the polynomials determined by*

(6.18)
$$\sum_{l=0}^{j} \sigma^0(B_{jl})\tau^l \equiv \sum_{k\in M_0} p_{jk}\tau^k (\operatorname{mod} a^+(\tau)), \; j \in J.$$

REMARK 6.2. When β is in particular a normal system of m differential boundary operators of orders $j \in J \subset M$, ellipticity of $P_{\gamma,\beta}^A$ means exactly that the "complementing condition" is satisfied (cf. [5]); this is equivalent with well-posedness of the boundary problem $Au = 0$ in Ω, $\beta u = \phi$ on Γ; which is here also seen using $P_{\gamma,\beta}^A = \beta\gamma_Z^{-1}$.

When $P_{\gamma,\beta}^A$ is an isomorphism, and κ is another system of boundary operators, we may of course define

(6.19)
$$P_{\beta,\kappa}^A = P_{\gamma,\kappa}^A (P_{\gamma,\beta}^A)^{-1}$$

(and corresponding modifications when we admit a finite index).

We shall now describe some concrete examples. To do this, we introduce the matrices $S_m^{\pm}(y,\eta) = (s_{m+k-j}^{\pm}(y,\eta))_{j,k\in M_0}$ and $S_0^{\pm}(y,\eta) = (s_{k-j}^{\pm}(y,\eta))_{j,k\in M_0}$; here the $s_l^{\pm}$ are the coefficients in $a^+(\tau)$ resp. $a^-(\tau)$ (cf. (2.30)) and we put $s_l^{\pm} = 0$ for $l \notin [0,m]$. So

(6.20)
$$S_m^{\pm} = \begin{bmatrix} s_m^{\pm} & 0 \;\cdots\; 0 \\ s_{m-1}^{\pm} & s_m^{\pm} \cdots 0 \\ \vdots & \vdots \qquad \vdots \\ s_1^{\pm} & s_2^{\pm} \cdots s_m^{\pm} \end{bmatrix}; \quad S_0^{\pm} = \begin{bmatrix} s_0^{\pm} & s_1^{\pm} \cdots s_{m-1}^{\pm} \\ 0 & s_0^{\pm} \cdots s_{m-2}^{\pm} \\ \vdots & \vdots \qquad \vdots \\ 0 & 0 \;\cdots\; s_0^{\pm} \end{bmatrix}.$$

Noting that $\overline{a(\tau)} = \overline{A_{2m}}\;\overline{a^-(\tau)}\;\overline{a^+(\tau)}$, we have

(6.21)
$$S_p^{\pm}(A') = \overline{S_p^{\mp}(A)}, \qquad p = 0, m;$$

furthermore, since $\operatorname{Re} a(\tau) = a^r(\tau) = (\operatorname{Re} A_{2m}) a^{r,+}(\tau)\, a^{r,-}(\tau)$, where $a^{r,-}(\tau) = \overline{a^{r,+}(\tau)}$, we denote:

(6.22) $$S_p^+(A^r) = S_p^r, \quad S_p^-(A^r) = \overline{S_p^r} \quad p = 0, m.$$

EXAMPLE 6.1: $\sigma^0(P^A_{\gamma,\nu})$.

Solving the equation $(6.7)^+$ is equivalent with solving the matrix equation

(6.23) $$(0 \quad I) = C(S_0^+ \quad S_m^+) + (T, \quad 0),$$

where 0 and I temporarily denote the $(m \times m)$ zero resp. identity matrix, and C and T are the unknown $(m \times m)$-matrices. (6.23) splits up in

(6.24) $$0 = CS_0^+ + T, \qquad I = CS_m^+,$$

whence, since S_m^+ is invertible (in fact $s_m^+ = 1$),

$$T = -(S_m^+)^{-1}S_0.$$

This shows that

(6.25) $$\sigma^0(P^A_{\gamma,\nu})(y,\eta) = -(S_m^+(y,\eta))^{-1}S_0^+(y,\eta).$$

EXAMPLE 6.2. $\sigma^0(P^A_{\gamma,\chi})$.

Comparison of (2.27) and (2.29) gives

$$\sum_{l=0}^{2m} \sigma^0(A_l)\tau^l = A_{2m}\left(\sum_{p=0}^{m} s_p^-\tau^p\right)\left(\sum_{q=0}^{m} s_q^+\tau^q\right) = A_{2m}\sum_{l=0}^{2m}\left(\sum_{p+q=l} s_p^- s_q^+\right)\tau^l,$$

whence (cf. (2.32))

$$\sigma^0(\mathscr{A}_{jk}) = iA_{2m}\sum_{p+q=j+k+1} s_p^- s_q^+.$$

Denoting the $(m \times m)$ skew-unit matrix by $I^\times$

(6.26) $$I^\times = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix},$$

we thus find that

$$\sigma^0(\mathscr{A}) = iA_{2m}\begin{pmatrix} I^\times S_0^- & I^\times S_m^- \\ I^\times S_m^- & 0 \end{pmatrix}\begin{pmatrix} S_m^+ & 0 \\ S_0^+ & S_m^+ \end{pmatrix} = iA_{2m}\begin{pmatrix} I^\times(S_0^- S_m^- + S_m^- S_0^+) & I^\times S_m^- S_m^+ \\ I^\times S_m^- S_m^+ & 0 \end{pmatrix}$$

In particular,

(6.27) $$\sigma^0(\mathscr{A}_{M_0M_0}) = iA_{2m}I^\times(S_0^- S_m^+ + S_m^- S_0^+),$$

$$\sigma^0(\mathscr{A}_{M_0M_1}) = iA_{2m}I^\times S_m^- S_m^+, \tag{6.28}$$

whence, by (6.25), and (2.43)

$$\begin{aligned}\sigma^0(P^A_{\gamma,\chi}) &= \sigma^0(\mathscr{A}_{M_0M_1})\sigma^0(P^A_{\gamma,\nu}) + \tfrac{1}{2}\sigma^0(\mathscr{A}_{M_0M_0})\\ &= iA_{2m}I^\times[S_m^- S_m^+(-S_m^+)^{-1}S_0^+ + \tfrac{1}{2}(S_0^- S_m^+ + S_m^- S_0^+)]\\ &= \tfrac{1}{2}iA_{2m}I^\times(S_0^- S_m^+ - S_m^- S_0^+).\end{aligned}$$

We have proved

$$\sigma^0(P^A_{\gamma,\chi}) = \tfrac{1}{2}i\, A_{2m}I^\times(S_0^- S_m^+ - S_m^- S_0^+). \tag{6.29}$$

EXAMPLE 6.3: The isomorphism R.

By the isomorphism $\gamma: Z(A_1) \to \prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$, the $L^2(\Omega)$-inner product in $Z(A_1)$ induces an inner product in $\prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma)$; thereby giving rise to an isomorphism $R: \prod_{j\in M_0} H^{-j-\frac{1}{2}}(\Gamma) \to \prod_{j\in M_0} H^{j+\frac{1}{2}}(\Gamma)$, with which

$$(z_1, z_2) = \langle R\gamma z_1, \gamma z_2\rangle, \text{ all } z_1, z_2 \in Z(A_1).$$

It may be shown that R is a selfadjoint positive definite elliptic ps.d.o. in Γ of type $(-k-\frac{1}{2}, j+\frac{1}{2})_{j,k\in M_0}$; that it is the operator

$$R = P^{AA'}_{\{\gamma,\gamma A'\},\chi'} I_{MM_1} \tag{6.30}$$

(in the notation of Remark 6.2, cf. (6.19)), and that

$$\sigma^0(R) = i\, I^\times[\overline{S_0^+}(\overline{S_m^+})^{-1} - \overline{S_m^+}(S_m^+)^{-1}S_0^+\overline{S_m^+}]^{-1}. \tag{6.31}$$

Since R is not essential for the present paper, we omit proofs and further details. Note that, as should be expected, $\sigma^0(R)$ depends only on a^+.

R' and R^r are defined analogously relative to A' and A^r.

A final remark. In the article [12] we used some results, for which the proofs were deferred to a later paper with the provisional title "On the regularity of a general class of boundary problems". These results were in part concerned with with the operators $P_{\gamma,\beta}$, for which the present Appendix provides the proofs; in part they were concerned with the connection between the regularity of $\tilde{A}$ and of L. For the latter part, Remark 4.7 of the present paper covers what is used in [12]; a more systematic study is easy to set up, and may be included in a future paper.

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