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Pseudo-differential boundary problems in L_p spaces

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1. Introduction.

The theory of pseudo-differential boundary problems has been developed, partly as an interesting subject in its own right, partly because it comes up naturally when boundary problems for differential operators are subjected to reductions like for example composition with solution operators for auxiliary problems, and other "algebraic" manipulations. The theory originally evolved through works of Vishik, Eskin and Boutet de Monvel (see [V-E], [BM1,2], [E]), and is further treated in the books of Rempel and Schulze [R-S1], Grubb [G2] and a number of journal articles. These works are almost exclusively concerned with the L_2 theory, whereas little has been done in an L_p framework. The purpose of this paper is to extend the calculus to L_p generalizations of Sobolev spaces, mainly the Bessel-potential spaces H_p^s and the Besov spaces B_p^s (and $B_{p,q}^s$) for $1 < p < \infty$.

In Section 2 we recall some facts on these spaces, defined over $\mathbb{R}^n$ as well as over smooth subsets $\overline{\Omega}$ (with reference primarily to Triebel's book [T1]). In Section 3 we prove the continuity of the pseudo-differential boundary operators in these spaces. For example, we have the continuity of

$$(1.1)\qquad \mathcal{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} : \begin{matrix} H_p^s(\overline{\Omega}, E) \\ \times \\ B_p^{s-1/p}(\partial\Omega, F) \end{matrix} \to \begin{matrix} H_p^{s-d}(\overline{\Omega}, E') \\ \times \\ B_p^{s-d-1/p}(\partial\Omega, F') \end{matrix},$$

for all $s > r + 1/p - 1$, when $\mathcal{A}$ is of order $d \in \mathbb{R}$ and class $r \in \mathbb{Z}$; systems with $P_\Omega + G$ and T of negative class are included here. (1.1) also holds when H_p^s spaces are replaced by B_p^s or $B_{p,q}^s$ spaces; moreover, G maps $H_p^s \cup B_p^s$ into $H_p^s \cap B_p^s$. (A simple application is mentioned: the projection onto divergence free vector fields in $H^s_p(\overline{\Omega})^n$.) Section 4 gives the first conclusions on elliptic systems. Here we begin a thorough analysis of a tool in the theory that has played an important role since the beginning: the "order-reducing" operators. A new variant of the operators is introduced for $\mathbf{R}^n_+$ in Section 4, and in Section 5 we use the ellipticity results of Section 4 and the parameter-dependent calculus of [G2] to set them up for manifolds in a very convenient way, where bijectiveness properties are obtained, not by a delicate elimination of kernels and cokernels, but by choosing a certain parameter large enough; then the construction is stable under small perturbations.

The order-reducing operators are used in Section 5 to obtain a full treatment of elliptic problems (and overdetermined and underdetermined elliptic problems), their parametrices and solvability properties. In particular, we establish the Fredholm property of (1.1) in the elliptic case for all $s > r+1/p-1$, showing that there is a parametrix of class $r-d$ (here $d \in \mathbf{Z}$), regardless of the sign of d and r, and with a precise description of the kernel and range. We include Douglis-Nirenberg systems at the end. — Besides extending the classical results on differential operator boundary problems, this extends results of e.g. Solonnikov [So1,2] (overdetermined differential problems in L_p-type spaces), Schulze [S] (one-sided elliptic problems in L_2-type spaces), [T1,2] (normal differential problems in $F_{p,q}$ and $B_{p,q}$ spaces) and of [R-S1].

A treatment of related questions for parabolic pseudo-differential problems (where the L_2 theory has been developed in [G2] and in Grubb and Solonnikov [G-S]) is presently being worked out.

Relation to other work: This paper was written when we found out that the extension to L_p Sobolev spaces in [R-S1] was not fully achieved (cf. Remark 3.3 below), so that a fresh treatment would be required. After submitting the first version for publication, we were informed that there is a dissertation of Franke [F2] treating pseudo-differential boundary problems in the framework of Besov-Triebel-Lizorkin spaces. The results are announced in [F1], which indicates an article to appear in Math. Nachr., that has never materialized; we have recently been told that details will be published in a volume of "Surveys in Analysis, Geometry and Mathematical Physics", Teubner Texte, Leipzig. The particular interest of this is that [F1,2] shows continuity of the pseudo-differential boundary operators in a larger scale of L_p-related spaces, namely the spaces $F^s_{p,q}$ ($p \in]0,\infty[$, $q \in]0,\infty]$) and $B^s_{p,q}$ (p, $q \in]0,\infty]$) under some hypotheses; his arguments seem quite different from ours. In comparison, we have a slightly better statement on singular Green operators (that can undoubtedly be generalized to his setting). Moreover, drawing on material from [G3] that was written after our first version, we include operators $P_\Omega + G$ (and T) of negative class. This leads to the best

possible result on the relation between the class of $\mathcal{A}$ and the class of its parametrix (not covered in [F1,2], where negative class is defined for operators G and T, but the elliptic systems $\mathcal{A}$ are taken of class ≥ 0); and it allows the lowest possible values of s. In our analysis of elliptic problems, we establish the more stable order-reducing operators; and we give a precise characterization of the range in the surjectively elliptic case by a finite set of linear conditions defined by smooth functions, where [F2], like its main reference [R-S1], just shows the existence of a smooth finite dimensional complement.

2. Review of the relevant spaces.

Let $1 < p < \infty$. We recall that a constant-coefficient pseudo-differential operator Q with symbol $q(\xi)$,

$$Qu = \mathrm{OP}(q(\xi))u = \mathcal{F}^{-1}(q\mathcal{F}u)$$

(where $\mathcal{F}$ denotes the Fourier transformation $f(x) \mapsto \hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$) is continuous in L_p, if one of the following criteria holds:

$$\text{(2.1)}\qquad \begin{aligned} &\sup_{|\alpha|\leq[n/2]+1} \sup_{\xi} |\xi|^{|\alpha|} |D^\alpha_\xi q(\xi)| < \infty \quad \text{(Mihlin [M])},\\ &\sup_{\alpha_i=0,1} \sup_{\xi} |(\xi_1 D_{\xi_1})^{\alpha_1} \cdots (\xi_n D_{\xi_n})^{\alpha_n} q(\xi)| < \infty \quad \text{(Lizorkin [L])}; \end{aligned}$$

(We denote by $[s]$ the largest integer $\leq s$.) Lizorkin's criterion covers a different type of symbols than Mihlin's, requiring estimates of a higher total order but with less restrictions on mixed derivatives (we shall not here go into further generalizations). These criteria are satisfied by "classical" ps.d.o.s with x-independent symbols in $S^0_{1,0}$, the operator norm depending of course on a finite number of estimates only.

The Sobolev spaces $W^s_p(\mathbf{R}^n)$ are defined for $s \in \mathbf{N} = \{0,1,2,\dots\}$ by

$$\text{(2.2)}\qquad W^s_p(\mathbf{R}^n) = \{\, f \in \mathcal{S}'(\mathbf{R}^n) \mid D^\alpha f \in L_p \text{ for } |\alpha| \leq s \,\}.$$

(Here and in the following, the space is a Banach space with the appropriate norm inferred from the definition.) For $s \in \mathbf{R} \setminus \mathbf{N}$, the concept has been generalized in several ways. There is a nice description of the results in the monograph of Triebel [T1], with full details and ample references; for convenience, we use this as a main reference.

On one hand, there are the H^s_p spaces (where we write $(1+|\xi|^2)^{1/2} = \langle\xi\rangle$):

$$\text{(2.3)}\qquad H^s_p(\mathbf{R}^n) = \{\, f \in \mathcal{S}'(\mathbf{R}^n) \mid \Xi^s f \in L_p \}, \text{ where } \Xi^s = \mathrm{OP}(\langle\xi\rangle^s),\ s \in \mathbf{R},$$

that coincide with the $W^s_p(\mathbf{R}^n)$ for $s \in \mathbf{N}$ and satisfy $H^s_p \subset H^{s'}_p$ for $s > s'$; they are called Lebesgue spaces by some authors, and Bessel-potential spaces by others (since $\mathcal{F}^{-1}\langle\xi\rangle^s$ can be expressed by Bessel functions and the Ξ^s therefore have been called Bessel potential operators). For a constant-coefficient ps.d.o. P with symbol in $S^d_{1,0}$, $\Xi^{s-d}P\Xi^{-s}$ has symbol in $S^0_{1,0}$, so, by the criteria in (2.1), P is continuous from $H^s_p(\mathbb{R}^n)$ to $H^{s-d}_p(\mathbb{R}^n)$ for any $s \in \mathbb{R}$. On the other hand, there are the Sobolev-Slobodetskiĭ spaces $W^s_p(\mathbb{R}^n)$, defined for $s \in \mathbb{R}_+ \setminus \mathbb{N}$ by:

(2.4)
$$f \in W^s_p(\mathbb{R}^n) \iff \|f\|^p_{L_p} + \sum_{|\alpha|\le[s]} \int_{\mathbb{R}^{2n}} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+p(s-[s])}}\,dxdy < \infty,$$

that also fill out the gaps between the positive integers, and satisfy $W^s_p \subset W^{s'}_p$ when $s > s'$. These spaces coincide with the *Besov spaces* $B^s_{p,p}(\mathbb{R}^n)$ for $s \in \mathbb{R}_+ \setminus \mathbb{N}$; see the definition of the $B^s_{p,q}$ spaces e.g. in Triebel [T1] or Bergh and Löfström [B-L] ($s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$). Since we shall not so often mention the $B^s_{p,q}$ spaces with $q \neq p$, we use from now on the simplifying notation

$$B^s_{p,p} = B^s_p. \tag{2.5}$$

When $p \neq 2$, the spaces B^s_p are different from the H^s_p *for all* $s \in \mathbb{R}$, in particular for integer values; more precisely,

$$B^s_p(\mathbb{R}^n) \subset H^s_p(\mathbb{R}^n) \text{ when } p \le 2, \quad B^s_p(\mathbb{R}^n) \supset H^s_p(\mathbb{R}^n) \text{ when } p \ge 2; \tag{2.6}$$

with strict inclusions if and only if $p \neq 2$. Stein [St, p. 161] gives an example for $n = 1$, $s = 1$: The function f_σ defined by a lacunary series

$$f_\sigma(x) = e^{-\pi x^2} \sum_{k=1}^{\infty} 2^{-k} k^{-\sigma} e^{2\pi i 2^k x}, \quad x \in \mathbb{R}, \tag{2.7}$$

satisfies $f_\sigma \in H^1_p \setminus B^1_p$ when $1/2 < \sigma < 1/p$, and $f_\sigma \in B^1_p \setminus H^1_p$ when $1/p < \sigma < 1/2$. In any case, we can define $H^s_p(\mathbb{R}^n) \cap B^s_p(\mathbb{R}^n)$ resp. $H^s_p(\mathbb{R}^n) \cup B^s_p(\mathbb{R}^n)$ as the smallest, resp. largest, of the two spaces, provided with the topology of that space. One has that $B^s_p(\mathbb{R}^n) \subset H^{s-\varepsilon}_p(\mathbb{R}^n) \subset B^{s-2\varepsilon}_p(\mathbb{R}^n)$ for $\varepsilon > 0$, all s.

All these spaces are interrelated by interpolation formulas, of which we shall mention a few. For p, p_0 and $p_1 \in]1,\infty[$, s and $t \in \mathbb{R}$, $\theta \in]0,1[$, $q \in [1,\infty]$, one has:

(2.8)
$$(H^s_p, H^t_p)_{\theta,p} = (H^s_p, B^t_p)_{\theta,p} = (B^s_p, B^t_p)_{\theta,p} = B^{(1-\theta)s+\theta t}_p \quad \text{for } s \neq t;$$
$$(H^s_{p_0}, H^s_{p_1})_{\theta,p_2} = H^s_{p_2},\ (B^s_{p_0}, B^s_{p_1})_{\theta,p_2} = B^s_{p_2},\ 1/p_2 = (1-\theta)/p_0 + \theta/p_1;$$
$$(B^s_p, B^t_p)_{\theta,q} = B^{(1-\theta)s+\theta t}_{p,q} \quad \text{for } s \neq t;$$

cf. [T1], [B-L]. One consequence is that ps.d.o.s $\mathrm{OP}(q(\xi))$ of order d map B^s_p and $B^s_{p,q}$ continuously into B^{s-d}_p resp. $B^{s-d}_{p,q}$.

To illustrate the integer cases further, we mention that for example

$$f \in B^s_p(\mathbb{R}^n) \iff \|f\|^p_{L_p} + \int_{\mathbb{R}^{2n}} \frac{|f(x) + f(y) - 2f((x+y)/2)|^p}{|x-y|^{n+ps}}\,dxdy < \infty$$

holds for $0 < s < 2$, in particular for $s = 1$; and $B^{s-t}_p(\mathbb{R}^n) = \Xi^t B^s_p(\mathbb{R}^n)$ in general.

Summing up, the Sobolev-Slobodetskiĭ spaces W^s_p, defined for $s \ge 0$, are related to the Bessel-potential spaces and Besov spaces by:

$$W^s_p = H^s_p \quad \text{for } s \in \mathbb{N}, \quad W^s_p = B^s_p \quad \text{for } s \in \mathbb{R}_+ \setminus \mathbb{N}; \tag{2.9}$$

we shall rarely mention the W^s_p spaces explicitly in the rest of this paper.

The Besov spaces are inevitable when one wants a complete calculus of boundary value problems in L_p, because of the fundamental trace theorem:

THEOREM 2.1. *Let $1 < p < \infty$ and let $s > 1/p$. The mapping $\tilde\gamma_0 \colon u(x', x_n) \mapsto u(x', 0)$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{n-1})$ extends by continuity to continuous mappings:*

$$\tilde\gamma_0 \colon H^s_p(\mathbb{R}^n) \to B^{s-1/p}_p(\mathbb{R}^{n-1}), \quad \tilde\gamma_0 \colon B^s_p(\mathbb{R}^n) \to B^{s-1/p}_p(\mathbb{R}^{n-1}); \tag{2.10}$$

that are surjective.

We use the notation $\tilde\gamma_0$ here for the "two-sided" trace operator, to save the notation γ_0 for the one-sided trace operator from $\mathbb{R}^n_+$ to $\mathbb{R}^{n-1}$. For precision, we observe:

LEMMA 2.2. *The continuity in (2.10) cannot be extended to $s \le 1/p$. More precisely, for any $s \le 1/p$ there exists a sequence of functions $u_m \in \mathcal{S}(\mathbb{R}^n)$ converging to 0 in H^s_p or B^s_p such that $\tilde\gamma_0 u_m$ is constant $\neq 0$.*

PROOF: For $s < 1/p$, this is easy to see; for example, $w_m(x_n) = e^{-m|x_n|}$ has $w_m(0) = 1$ for all $m \in \mathbb{N}$ and goes to zero in $H^s_p(\mathbb{R})$ and $B^s_p(\mathbb{R})$ for $m \to \infty$ (for $B^s_p(\mathbb{R})$, (2.4) shows that $\|w_m\|_{B^s_p}$ is $O(m^{s-1/p})$, and for $H^s_p(\mathbb{R})$, one can use that the convergence in $B^{s'}_p(\mathbb{R})$ for $s < s' < 1/p$ implies convergence in $H^s_p(\mathbb{R})$). This gives a counterexample to (2.10) in the case $n = 1$, and by multiplication of $w_m(x_n)$ by $v(x') \in \mathcal{S}(\mathbb{R}^{n-1})\setminus\{0\}$, we get a counterexample for $n > 1$.

For $s = 1/p$, the functions $w_m(x_n) = -\varphi(x_n)\log(|x_n| + m^{-1})/\log m$, $m \geq 2$, with $\varphi \in C_0^\infty(\mathbf{R})$, $\varphi = 1$ on a neighborhood of 0, have the properties that $w_m(0) = 1$ and $\|w_m\|_{B_p^{1/p}} \leq C[\log m]^{1/p-1} \to 0$ for $m \to \infty$ (recall that $p > 1$). As a counterexample for $H_p^{1/p}(\mathbf{R})$, one can take $w_m(x_n) = u_m(x_n)/u_m(0)$, where $\hat{u}_m(\xi_n) = \psi(\xi_n/m)\langle\xi_n\rangle^{-1}$, with $\psi \in \mathcal{S}(\mathbf{R})$ having $\tilde{\psi} = \mathcal{F}^{-1}\psi$ supported in $]-1,1[$ and $\psi(0) = 1$. Here one can show that $u_m(0) \sim 2\log m$, whereas $\|u_m\|_{H_p^{1/p}} = \|\mathcal{F}^{-1}[\psi(\xi_n/m)\langle\xi_n\rangle^{1/p-1}]\|_{L_p} = \|m\tilde{\psi}(mx_n) * \mathcal{F}^{-1}\langle\xi_n\rangle^{1/p-1}\|_{L_p}$ is $O([\log m]^{1/p})$; thus $w_m(0) = 1$ with $\|w_m\|_{H_p^{1/p}} = O([\log m]^{1/p-1}) \to 0$ for $m \to \infty$. Again, these counterexamples for $n = 1$ give counterexamples for $n > 1$ by multiplication by functions in $\mathcal{S}(\mathbf{R}^{n-1})$. ∎

The non-extendability of γ_0 for $s \leq 1/p$ is well known to the specialists on these spaces, but we have not been able to find specific counterexamples as above for $s = 1/p$ in the literature; they were kindly constructed by L. Hörmander when we told him about the question.

In the following, Ω denotes an open subset of $\mathbf{R}^n$, either equal to $\mathbf{R}^n_+$ or equal to a bounded set with C^∞ boundary $\partial\Omega = \Gamma$ (i.e., there is a finite cover of Γ by open sets $U_i \subset \mathbf{R}^n$ with associated diffeomorphisms (local coordinates) $\kappa_i \colon U_i \to \mathbf{B} = \{\, y \in \mathbf{R}^n \mid |y| < 1 \,\}$ such that $U_i \cap \Omega$ is mapped onto $\mathbf{B} \cap \mathbf{R}^n_+$ and $U_i \cap \Gamma$ is mapped onto $\mathbf{B} \cap \{\, y \in \mathbf{R}^n \mid |y| < 1,\ y_n = 0 \,\}$). The operator of *restriction to* Ω (going from $\mathcal{D}'(\mathbf{R}^n)$ to $\mathcal{D}'(\Omega)$) is denoted r_Ω, or $r^\pm$ if $\Omega = \mathbf{R}^n_\pm$; and the operator of *extension by zero on* $\mathbf{R}^n \setminus \Omega$ (sending functions on Ω into functions on $\mathbf{R}^n$) is denoted e_Ω, or $e^\pm$ if $\Omega = \mathbf{R}^n_\pm$. Then we define

$$(2.11)\qquad \begin{aligned} H_p^s(\overline{\Omega}) &= r_\Omega H_p^s(\mathbf{R}^n), & H_{p;0}^s(\overline{\Omega}) &= \{\, u \in H_p^s(\mathbf{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega} \,\}; \\ B_p^s(\overline{\Omega}) &= r_\Omega B_p^s(\mathbf{R}^n), & B_{p;0}^s(\overline{\Omega}) &= \{\, u \in B_p^s(\mathbf{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega} \,\} \end{aligned}$$

(and extend the notation also to define the spaces $W_p^s(\overline{\Omega})$, $W_{p;0}^s(\overline{\Omega})$, $B_{p,q}^s(\overline{\Omega})$ and $B_{p,q;0}^s(\overline{\Omega})$). Here $C_{(0)}^\infty(\overline{\Omega}) = r_\Omega C_0^\infty(\mathbf{R}^n)$ is dense in $H_p^s(\overline{\Omega})$ and in $B_p^s(\overline{\Omega})$, and $e_\Omega C_0^\infty(\Omega)$ (usually written as $C_0^\infty(\Omega)$) is dense in $H_{p;0}^s(\overline{\Omega})$ and in $B_{p;0}^s(\overline{\Omega})$. Note that for negative s, the space obtained by use of the restriction r_Ω is generally not a function space, so e_Ω does not apply. On the other hand, when $s \geq 0$ (resp. > 0), the space $H_{p;0}^s(\overline{\Omega})$ (resp. $B_{p;0}^s(\overline{\Omega})$) is e_Ω of a function space over Ω, that we usually identify it with.

(2.6) and (2.8) generalize to these spaces; and for any $N \geq 0$, there are extension operators $l_{\Omega,N}$ and $l'_{\Omega,N}$ that are continuous:

$$(2.12)\qquad \begin{aligned} l_{\Omega,N} &\colon H_p^s(\overline{\Omega}) \to H_p^s(\mathbf{R}^n), \\ l'_{\Omega,N} &\colon B_p^s(\overline{\Omega}) \to B_p^s(\mathbf{R}^n), \quad \text{for } s \in [-N, N]; \end{aligned}$$

with $r_\Omega l_{\Omega,N} = I$ on $H_p^s(\overline{\Omega})$ and $r_\Omega l'_{\Omega,N} = I$ on $B_p^s(\overline{\Omega})$.

Spaces over Γ are defined by use of local coordinates.

One has the following trace theorem for Ω, where $\gamma_j \colon u \mapsto (\partial_\nu^j u)|_\Gamma$, ∂_ν denoting the derivative along the normal to Γ:

THEOREM 2.3. *Let $1 < p < \infty$. For any $s > 1/p$, the mapping γ_0 from $C_{(0)}^\infty(\overline{\Omega})$ to $C_0^\infty(\Gamma)$ extends by continuity to continuous mappings:*

$$(2.13)\qquad \gamma_0 \colon H_p^s(\overline{\Omega}) \to B_p^{s-1/p}(\Gamma), \quad \gamma_0 \colon B_p^s(\overline{\Omega}) \to B_p^{s-1/p}(\Gamma);$$

that are surjective.

For $s > m + 1/p - 1$, $m \in \mathbf{N} \setminus \{0\}$, the operators

$$(2.14)\qquad \varrho^{(m)} = \{\gamma_0, \dots, \gamma_{m-1}\}$$

define continuous mappings from $H_p^s(\overline{\Omega})$ resp. $B_p^s(\overline{\Omega})$ to $\prod_{j=0}^{m-1} B_p^{s-j-1/p}(\Gamma)$ that are surjective; the kernels equal $H_{p;0}^s(\overline{\Omega})$ resp. $B_{p;0}^s(\overline{\Omega})$ when $s < m+1/p$.

One proves the theorem first for the case $\Omega = \mathbf{R}^n_+$, and then for a smooth bounded set by use of a partition of unity and local coordinates. The counterexamples given in Lemma 2.2 work also for the trace operators from $\mathbf{R}^n_+$.

The spaces (2.11) are discussed in detail in [T1, Ch. 2, 4] (and most of the proofs of the assertions can be found there), although with a slightly different notation. Our $H_p^s(\overline{\Omega})$ and $B_p^s(\overline{\Omega})$ equal the spaces $H_p^s(\Omega)$ resp. $B_{p,p}^s(\Omega)$ there, and our $H_{p;0}^s(\overline{\Omega})$ and $B_{p;0}^s(\overline{\Omega})$ equal the spaces $\tilde{H}_p^s(\Omega)$ resp. $\tilde{B}_{p,p}^s(\Omega)$ there. Triebel moreover defines the spaces $\mathring{H}_p^s(\Omega)$ and $\mathring{B}_{p,p}^s(\Omega)$, as the closures of $C_0^\infty(\Omega)$ in $H_p^s(\overline{\Omega})$ resp. $B_p^s(\overline{\Omega})$; note that this need not be the same as taking the closure of $C_0^\infty(\Omega)$ in $H_p^s(\mathbf{R}^n)$ resp. $B_p^s(\mathbf{R}^n)$, which gives $H_{p;0}^s(\overline{\Omega})$ resp. $B_{p;0}^s(\overline{\Omega})$. In fact (cf. [T1, 4.3.2]), for $s > 1/p$, $B_{p;0}^s(\overline{\Omega})$ coincides with $\mathring{B}_{p,p}^s(\Omega)$ *if and only if $s - 1/p$ is not integer*; and in the exceptional cases $s - 1/p = k \in \mathbf{N}$, our $B_{p;0}^s(\overline{\Omega})$ is the set of $u \in B_p^s(\overline{\Omega})$ such that $\gamma_j u = 0$ for $j < k$ and the following strictly stronger norm is finite:

$$(2.15)\qquad \|u\|^p_{B_p^s(\overline{\Omega})} + \textstyle\sum_{|\alpha|=k} \|\operatorname{dist}(x,\Gamma)^{-1/p} D^\alpha u(x)\|^p_{L_p(\Omega)}.$$

In contrast with this, $\overset{\circ}{B}{}^{1/p}_{p,p}(\Omega) = B^{1/p}_p(\overline{\Omega})$; and more generally, $\overset{\circ}{B}{}^{k+1/p}_{p,p}(\Omega)$ equals for $k \in \mathbb{N}$ the closed subspace of $B^{k+1/p}_p(\overline{\Omega})$ on which γ_j is 0 for $j = 0, \dots, k-1$. See [T1] for related statements concerning the H^s_p spaces.

We emphasize the spaces $H^s_{p;0}(\overline{\Omega})$ and $B^s_{p;0}(\overline{\Omega})$ rather than the spaces $\overset{\circ}{H}{}^s_p(\Omega)$ and $\overset{\circ}{B}{}^s_{p,p}(\Omega)$, because they have the most convenient duality and interpolation properties. (The different preferences are also seen in earlier presentations of the L_2 theory, where e.g. the book of Lions and Magenes [L-M] primarily studies the $H^s_2(\overline{\Omega})$-closure of $C_0^\infty(\Omega)$, whereas the book of Hörmander [H1] (with reference to the thesis of Peetre [P]) emphasizes the $H^s_2(\mathbb{R}^n)$-closure. The latter is likewise preferred in the extensive Russian literature, with norms formulated like (2.4), (2.15).)

Throughout the paper, we shall denote $1 - 1/p = 1/p'$. One has the identifications of dual spaces (with respect to extensions of the pairing $(u,v)_\Omega = \int_\Omega u(x) \cdot \bar{v}(x)\,dx$, $\Omega = \mathbb{R}^n$ or a smooth bounded subset):

$$\text{(2.16)} \quad \begin{aligned} &H^s_p(\mathbb{R}^n)^* \simeq H^{-s}_{p'}(\mathbb{R}^n),\ B^s_p(\mathbb{R}^n)^* \simeq B^{-s}_{p'}(\mathbb{R}^n);\\ &H^s_p(\overline{\Omega})^* \simeq H^{-s}_{p';0}(\overline{\Omega}),\quad B^s_p(\overline{\Omega})^* \simeq B^{-s}_{p';0}(\overline{\Omega});\quad \text{for } s \in \mathbb{R}. \end{aligned}$$

For $s = 0$, $H^s_p(\overline{\Omega})$ and $H^s_{p;0}(\overline{\Omega})$ both identify with $L_p(\Omega)$ (cf. however (2.6) for B^s_p). The identification of H^s_p with $H^s_{p;0}$ extends to a small interval around 0, and a similar statement holds for the Besov spaces:

$$\text{(2.17)} \quad H^s_p(\overline{\Omega}) \simeq H^s_{p;0}(\overline{\Omega}),\quad B^s_p(\overline{\Omega}) \simeq B^s_{p;0}(\overline{\Omega}),\quad \text{for } 1/p - 1 < s < 1/p.$$

It is convenient for the consideration of the spaces defined relative to $\mathbb{R}^n_+$ to replace Ξ^r by other ps.d.o.s. One possibility is to use the operators

$$\text{(2.18)} \quad \Xi^r_\pm = \mathrm{OP}((\langle \xi' \rangle \pm i\xi_n)^r) \quad \text{for } r \in \mathbb{R},$$

as in [P] and in works of Vishik and Eskin (see e.g. [E]). Their symbols are *not in* $S^r_{1,0}(\mathbb{R}^n)$, since $\langle \xi' \rangle$ does not satisfy all the estimates in terms of powers of $\langle \xi \rangle$ required for that (they are "of regularity 1" with respect to the parameter ξ_n, in the terminology of [G2]). But they do have the property that their compositions with constant coefficient $S^{-r}_{1,0}$-ps.d.o.s satisfy Lizorkin's criterion (2.1), and clearly $(\Xi^r_\pm)^{-1} = \Xi^{-r}_\pm$, so they do define homeomorphisms from $H^s_p(\mathbb{R}^n)$ to $H^{s-r}_p(\mathbb{R}^n)$ and from $B^s_p(\mathbb{R}^n)$ to $B^{s-r}_p(\mathbb{R}^n)$ for all s. By use of the Paley-Wiener theorem one finds that the Ξ^r_+ preserve the property of being supported in $\overline{\mathbb{R}}^n_+$, so that they moreover define homeomorphisms between the $H^s_{p;0}(\overline{\mathbb{R}}^n_+)$ spaces. Then, identifying $H^s_p(\overline{\mathbb{R}}^n_+)$ with $H^s_p(\mathbb{R}^n)/H^s_{p;0}(\overline{\mathbb{R}}^n_-)$, one finds in view of (2.16), and the fact that $(\Xi^r_\pm)^* = \Xi^r_\mp$, that the "truncated" operators $\Xi^r_{-,\Omega} = r^+ \Xi^r_- e^+$ define homeomorphisms between the $H^s_p(\overline{\mathbb{R}}^n_+)$ spaces. (When $s < 0$ and e^+ does not apply, these operators are defined by extension by continuity from the dense subspace $\mathcal{S}(\overline{\mathbb{R}}^n_+) = r^+\mathcal{S}(\mathbb{R}^n)$.) Altogether, one has for $s \in \mathbb{R}$,

$$\text{(2.19)} \quad \begin{aligned} &\Xi^r_+ : H^s_{p;0}(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} H^{s-r}_{p;0}(\overline{\mathbb{R}}^n_+), &&\Xi^r_+ : B^s_{p;0}(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} B^{s-r}_{p;0}(\overline{\mathbb{R}}^n_+),\\ &\Xi^r_{-,\mathbb{R}^n_+} : H^s_p(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} H^{s-r}_p(\overline{\mathbb{R}}^n_+), &&\Xi^r_{-,\mathbb{R}^n_+} : B^s_p(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} B^{s-r}_p(\overline{\mathbb{R}}^n_+). \end{aligned}$$

Since the operators $\Xi^r_\pm$ are not $S^r_{1,0}$ pseudo-differential operators, they are in the Boutet de Monvel calculus usually replaced by other operators with almost as convenient properties; we return to this later.

All the above Banach spaces have the property that multiplication of the elements with a $C_0^\infty(\mathbb{R}^n)$ function φ maps the space into itself. Then we can define the "local" versions, for $s \in \mathbb{R}$, by

$$\begin{aligned} H^s_{p,\mathrm{loc}}(\mathbb{R}^n) &= \{\, u \in \mathcal{D}'(\mathbb{R}^n) \mid \varphi u \in H^s_p(\mathbb{R}^n) \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^n) \,\},\\ H^s_{p,\mathrm{comp}}(\mathbb{R}^n) &= \{\, u \in H^s_p(\mathbb{R}^n) \mid \operatorname{supp} u \text{ is compact} \,\}; \end{aligned}$$

topologized as usual as a Fréchet space resp. an inductive limit of Banach spaces; here $H^s_{p,\mathrm{loc}}$ and $H^{-s}_{p',\mathrm{comp}}$ are duals of one another. Similar definitions are introduced for the B^s_p and $B^s_{p,q}$ spaces and the derived spaces over $\overline{\Omega}$.

We shall not here say much about the important imbedding relations between the various spaces, but only list some simple facts, for the case where $\overline{\Omega}$ is compact:

(2.20)

$$\begin{aligned} &H^s_p(\overline{\Omega}) \subset B^{s-\varepsilon}_p(\overline{\Omega}) \subset B^{s-2\varepsilon}_{p,q}(\overline{\Omega}) \subset H^{s-3\varepsilon}_p(\overline{\Omega}) \text{ compactly, when } \varepsilon > 0;\\ &H^s_p(\overline{\Omega}) \subset H^s_{p_1}(\overline{\Omega}),\ B^s_p(\overline{\Omega}) \subset B^s_{p_1}(\overline{\Omega}),\ B^s_{p,q}(\overline{\Omega}) \subset B^s_{p_1,q}(\overline{\Omega}), \text{ when } p_1 \le p;\\ &H^s_p(\overline{\Omega}),\ B^s_p(\overline{\Omega}),\ B^s_{p,q}(\overline{\Omega}) \subset C^m(\overline{\Omega}), \text{ when } s - n/p > m, \text{ and hence}\\ &\textstyle\bigcap_{s\in\mathbb{R}} H^s_p(\overline{\Omega}) = \bigcap_{s\in\mathbb{R}} B^s_p(\overline{\Omega}) = \bigcap_{s\in\mathbb{R}} B^s_{p,q}(\overline{\Omega}) = C^\infty(\overline{\Omega}). \end{aligned}$$

3. Continuity of pseudo-differential boundary operators in Bessel-potential spaces and Besov spaces.

The theory of pseudo-differential boundary operators satisfying the transmission condition was introduced in Boutet de Monvel [BM1,2], and is described with further details in the books of Rempel and Schulze [R-S1] and Grubb [G2], in the paper [G1], and now most recently in the lectures [G3]. To save space, we build on [G1-3], recalling only information of special importance for the proofs in the following.

For an open set $\Omega \subset \mathbf{R}^n$, the symbol space $S^d_{1,0}(\Omega, \mathbf{R}^n)$ is defined as the set of C^∞ functions $p(x,\xi)$ on $\Omega \times \mathbf{R}^n$ satisfying a system of estimates

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le c_{\alpha,\beta}(x)\langle\xi\rangle^{d-|\alpha|} \quad \text{for all } \alpha \in \mathbf{N}^n,\ \beta \in \mathbf{N}^n;$$

here $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, and the $c_{\alpha,\beta}(x)$ are continuous functions on Ω. $p(x,\xi)$ is said to be *polyhomogeneous* when it moreover has an asymptotic expansion $p \sim \sum_{l\in\mathbf{N}} p_{d-l}$ with $p - \sum_{l<\mathbf{N}} p_{d-l} \in S^{d-\mathbf{N}}_{1,0}$ for all $N \in \mathbf{N}$ and each $p_{d-l}(x,\xi)$ homogeneous of degree $d-l$ in ξ for $|\xi| \ge 1$; the space of polyhomogeneous symbols of order d is denoted $S^d(\Omega, \mathbf{R}^n)$. If the estimates hold for $\Omega = \mathbf{R}^n$ with constants independent of x, we speak of *uniform symbol estimates*, and denote the symbol spaces $S^d_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ resp. $S^d(\mathbf{R}^n \times \mathbf{R}^n)$. (More details on these spaces in Hörmander [H3, 18.1].)

Pseudo-differential operators on $\mathbf{R}^n$ of order d are defined as operators of the form, with $p(x,\xi) \in S^d_{1,0}(\mathbf{R}^n, \mathbf{R}^n)$,

$$Pu(x) = \mathrm{OP}\,(p(x,\xi))u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\,d\xi, \tag{3.1}$$

plus a smoothing term $\mathcal{P}\colon \mathcal{E}'(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n)$ (of order $-\infty$). We use the notation OP' resp. OP_n, when the definition is applied with respect to the variables (x',ξ') resp. (x_n,ξ_n). It is well known that the formula (3.1), suitably interpreted, defines continuous operators

$$P\colon H^s_p(\mathbf{R}^n) \to H^{s-d}_p(\mathbf{R}^n), \quad P\colon B^s_p(\mathbf{R}^n) \to B^{s-d}_p(\mathbf{R}^n), \tag{3.2}$$

when $p(x,\xi) \in S^d_{1,0}(\mathbf{R}^n, \mathbf{R}^n)$ and satisfies suitable additional hypotheses. E.g., when $p(x,\xi)$ is constant in x, it follows directly from Mihlin's criterion (2.1), and when $p(x,\xi)$ vanishes for x outside a compact set one gets it by a superposition argument; then it follows when $p(x,\xi)$ satisfies uniform symbol estimates on $\mathbf{R}^n$ by [H2, proof of Th. 3.5]. The continuity was shown for polyhomogeneous symbols with uniform estimates in Seeley [Se2]. For more general P of order d, one finds, by use of partitions of unity,

$$P\colon H^s_{p,\mathrm{comp}}(\mathbf{R}^n) \to H^{s-d}_{p,\mathrm{loc}}(\mathbf{R}^n), \quad P\colon B^s_{p,\mathrm{comp}}(\mathbf{R}^n) \to B^{s-d}_{p,\mathrm{loc}}(\mathbf{R}^n). \tag{3.3}$$

The restriction P_Ω to the open set $\Omega \subset \mathbf{R}^n$ is defined as

$$\begin{gathered} P_\Omega = r_\Omega P e_\Omega; \quad \text{in particular} \\ P_\Omega = r_{\mathbf{R}^n_+} P e_{\mathbf{R}^n_+} = r^+ P e^+, \text{ when } \Omega = \mathbf{R}^n_+. \end{gathered} \tag{3.4}$$

We shall first consider the case $\Omega = \mathbf{R}^n_+$ in detail. When $u \in C^\infty_{(0)}(\overline{\mathbf{R}}^n_+)$, e^+u has a jump at $x_n = 0$, which gives a singularity for Pe^+u. Boutet de Monvel introduced a condition assuring that $r^+Pe^+u \in C^\infty(\overline{\mathbf{R}}^n_+)$, called the transmission condition (or the $\mathcal{H}$-condition in [G3], or the two-sided transmission property in [G-H]); we shall assume that P satisfies this condition at $x_n = 0$, and refer to [G3, Sect. 1] for details. We consider $S^d_{1,0}$-symbols of any order $d \in \mathbf{R}$. When p is polyhomogeneous of order $d \notin \mathbf{Z}$, this transmission condition is quite restrictive, for it then implies that p and its derivatives at $x_n = 0$ are of order $-\infty$ in ξ_n. However, symbols with such a behavior (generally not polyhomogeneous) play a role in the proof of Theorem 3.1 below.

Before studying the mapping properties of P_Ω, we shall consider Poisson operators. In order to get good global estimates over $\mathbf{R}^n_+$, we take operators with uniformly estimated symbols; local estimates for more general symbols follow from this. Recall that a Poisson operator of order d is an operator of the form (up to addition of a smoothing term $r^+\mathcal{K}$, where $\mathcal{K}\colon \mathcal{E}'(\mathbf{R}^{n-1}) \to C^\infty(\mathbf{R}^n)$, and with suitable interpretation of the integrals):

$$\begin{aligned} (Kv)(x) &= \mathrm{OPK}\,(k(x',\xi))v = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{ix\cdot\xi} k(x',\xi)\hat{v}(\xi')\,d\xi \\ &= \mathrm{OPK}\,(\tilde{k}(x,\xi'))v = (2\pi)^{1-n}\int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'}\tilde{k}(x,\xi')\hat{v}(\xi')\,d\xi', \end{aligned} \tag{3.5}$$

where $k(x',\xi)$ is the symbol and $\tilde{k}(x',x_n,\xi') = \mathcal{F}^{-1}_{\xi_n\to x_n} k$ (vanishing for $x_n < 0$) is the so-called symbol-kernel. We take k and $\tilde{k}$ in the spaces $S^{d-1}_{1,0}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \mathcal{H}^+)$ resp. $S^{d-1}_{1,0}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \mathcal{S}(\overline{\mathbf{R}}_+))$, which means that for all indices α' and $\beta' \in \mathbf{N}^{n-1}$, m and $m' \in \mathbf{N}$, all x' and ξ',

$$\|h^+ D^{\beta'}_{x'} D^{\alpha'}_{\xi'} D^m_{\xi_n} \xi_n^{m'} k(x',\xi)\|_{L_{2,\xi_n}} \dot{\le} \langle\xi'\rangle^{d-1/2-|\alpha'|-m+m'}, \tag{3.6}$$

$$\|D^{\beta'}_{x'} x_n^m D^{m'}_{x_n} D^{\alpha'}_{\xi'} \tilde{k}(x,\xi')\|_{L_{2,x_n}(\mathbf{R}_+)} \dot{\le} \langle\xi'\rangle^{d-1/2-|\alpha'|-m+m'}, \tag{3.7}$$

respectively. (The sign $\dot{\le}$ stands for: "less than or equal to a constant depending on the indices times".)

In these formulas, $\mathcal{H}^+ = \mathcal{F}(e^+\mathcal{S}(\overline{\mathbf{R}}_+))$, and h^+ is the Fourier transformed version of the projection e^+r^+; $k \in \mathcal{H}^+$ in ξ_n and h^+ can be omitted in (3.6) when $m \ge m'$. ($\mathcal{H}^+$ is a space of boundary values of bounded C^∞ functions holomorphic in $\{\xi_n \in \mathbf{C} \mid \mathrm{Im}\,\xi_n < 0\}$, described in detail in the quoted works.) Other equivalent formulations of the system of estimates (3.7) are:

(3.8) $$\sup_{x_n\geq 0} |D^{\beta'}_{x'} x_n^m D^{m'}_{x_n} D^{\alpha'}_{\xi'} \tilde{k}(x,\xi')| \mathrel{\dot{\le}} \langle\xi'\rangle^{d-|\alpha'|-m+m'},$$

(3.9) $$\|D^{\beta'}_{x'} x_n^m D^{m'}_{x_n} D^{\alpha'}_{\xi'} \tilde{k}(x,\xi')\|_{L_{1,x_n}(\mathbb{R}_+)} \mathrel{\dot{\le}} \langle\xi'\rangle^{d-1-|\alpha'|-m+m'};$$

since we can pass back and forth between the estimates in (3.7)–(3.9) by use of elementary inequalities for functions $f(t)\in \mathcal{S}(\overline{\mathbb{R}}_+)$, such as

(3.10) $$\begin{aligned} &\sup_t |f(t)| \le \|f\|_{L_2}^{1/2}\|D_t f\|_{L_2}^{1/2}, \quad \|f\|_{L_2} \le \frac{c}{\sqrt{\sigma}}\sup_t |(1+\sigma t)f(t)|,\\ &\|f\|_{L_1} \le \frac{c}{\sqrt{\sigma}}\|(1+\sigma t)f\|_{L_2}, \quad \sup_t |f(t)| \le \|D_t f\|_{L_1}, \end{aligned}$$

applied with $\sigma = \langle\xi'\rangle$. The expressions

(3.11) $$\begin{aligned} &|||\tilde{k}|||_{\alpha',\beta',m,m',L_2}\\ &= \sup_{\xi'\in\mathbb{R}^{n-1}} \langle\xi'\rangle^{-d+1/2+|\alpha'|+m-m'} \|D^{\beta'}_{x'} x_n^m D^{m'}_{x_n} D^{\alpha'}_{\xi'}\tilde{k}\|_{L_{2,x_n}(\mathbb{R}_+)}, \end{aligned}$$

etc., are called symbol seminorms, they define the Fréchet space topology of the spaces of symbols and symbol-kernels.

From (3.6) ff. we deduce moreover:

(3.12) $$\sup_{\xi_n} |D^{\beta'}_{x'} D^{\alpha'}_{\xi'} D^{m}_{\xi_n}\xi_n^{m'} k(x',\xi)| \mathrel{\dot{\le}} \langle\xi'\rangle^{d-1-|\alpha'|-m+m'},$$

for all α', β', $m\geq m'$.

Poisson operators arise in particular from ps.d.o.s by the formula

(3.13) $$K\colon v\mapsto r^+P(v(x')\otimes\delta(x_n)) = r^+P\tilde{\gamma}_0^* v$$

(cf. (2.10) ff.), where P is a ps.d.o. satisfying the transmission property. To formulate this in a precise way for an operator P with symbol $p(x,\xi)$ in $S^d_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, we observe that P can be written in the form

(3.14) $$Pu = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} q(x',y_n,\xi)u(y)\,dyd\xi \equiv \mathrm{OP}(q(x',y_n,\xi))u,$$

where $q\in S^d_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ is related to p by

(3.15) $$q(x,\xi)\sim \textstyle\sum_{0\le j<\infty}(-\partial_{x_n})^j D^j_{\xi_n}p(x,\xi)/j!,$$

(cf. [H3, Theorem 18.5.10]); q satisfies the transmission condition. Then, writing $\tilde{q}(x',0,x_n,\xi') = \mathcal{F}^{-1}_{\xi_n\to x_n} q(x',0,\xi)$, we have that for $v\in\mathcal{S}(\mathbb{R}^{n-1})$,

(3.16)

$$\begin{aligned} r^+P(v(x')\otimes\delta(x_n)) &= (2\pi)^{-n}r^+\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}q(x',y_n,\xi)v(y')\delta(y_n)\,dyd\xi\\ &= (2\pi)^{-n}r^+\int_{\mathbb{R}^n} e^{ix\cdot\xi}q(x',0,\xi)\hat{v}(\xi')\,d\xi\\ &= (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}r^+\tilde{q}(x',0,x_n,\xi')\hat{v}(\xi')\,d\xi'\\ &= \mathrm{OPK}(r^+\tilde{q}(x',0,x_n,\xi'))v. \end{aligned}$$

Here $r^+\tilde{q}$ satisfies the estimates (3.7–9) with d replaced by $d+1$, so $r^+P\tilde{\gamma}_0^*$ is a Poisson operator of order $d+1$.

Theorem 3.1. *When K is a Poisson operator of order $d\in\mathbb{R}$ given as in (3.5), with symbol estimates uniform in $x'\in\mathbb{R}^{n-1}$, then K is continuous:*

(3.17) $$K\colon B_p^{s-1/p}(\mathbb{R}^{n-1}) \to H_p^{s-d}(\overline{\mathbb{R}}^n_+)\cap B_p^{s-d}(\overline{\mathbb{R}}^n_+), \quad \textit{for all } s\in\mathbb{R}.$$

Proof: We shall use a kind of converse of the description preceding the theorem, namely that, as observed in [BM2, (3.5)], *any* Poisson operator (3.5) can be obtained in the form (3.13): One extends the symbol-kernel $\tilde{k}(x',x_n,\xi')$ of K (considered for $x_n\geq 0$, where it satisfies (3.8)) by the method of Seeley [Se1], to a C^∞ function $\tilde{p}(x',x_n,\xi')$ of all $x_n\in\mathbb{R}$, in such a way that $\tilde{p}$ again satisfies a system of sup norm estimates

$$\sup_{x_n\in\mathbb{R}} |D^{\beta'}_{x'} x_n^m D^{m'}_{x_n} D^{\alpha'}_{\xi'}\tilde{p}(x,\xi')| \mathrel{\dot{\le}} \langle\xi'\rangle^{d-|\alpha'|-m+m'},$$

and along with it also L_2 and L_1 estimates as in (3.7), (3.9). (Actually, it suffices for the theorem to have a suitable finite set of these estimates, which is useful to know e.g. for parameter-dependent problems.) Let $p(x',\xi) = \mathcal{F}_{x_n\to\xi_n}\tilde{p}$. Then we find, using e.g. that $\sup_{\xi_n}|p|\le\|\tilde{p}\|_{L_{1,x_n}}$, that p satisfies, for all α', β', m and m',

$$\sup_{\xi_n} |D^{\beta'}_{x'} D^{\alpha'}_{\xi'} D^m_{\xi_n}\xi_n^{m'} p(x',\xi)| \mathrel{\dot{\le}} \langle\xi'\rangle^{d-1-|\alpha'|-m+m'}.$$

It follows that

(3.18) $$|D^{\beta'}_{x'} D^{\alpha}_{\xi} p(x',\xi)| \mathrel{\dot{\le}} \langle\xi'\rangle^{d-1-|\alpha|}\big(\langle\xi'\rangle/\langle\xi\rangle\big)^N$$

for all $\alpha\in\mathbb{N}^n$, $\beta'\in\mathbb{N}^{n-1}$, $N\in\mathbb{N}$;

and in particular,

(3.19) $$|D^{\beta'}_{x'} D^{\alpha}_{\xi} p(x',\xi)| \mathrel{\dot{\le}} \langle\xi\rangle^{d-1-|\alpha|} \quad \text{for all } \alpha\in\mathbb{N}^n,\ \beta'\in\mathbb{N}^{n-1}.$$

Thus p is a pseudo-differential symbol in $S^{d-1}_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$. It satisfies the transmission condition, since it is $O(\langle\xi_n\rangle^{-N})$ for all N, along with its derivatives, for each fixed ξ'. Let $P = \mathrm{OP}(p)$, then one has for $v \in \mathcal{S}(\mathbf{R}^{n-1})$,

$$r^+P\tilde{\gamma}_0^* v = r^+P(v(x')\otimes\delta(x_n)) = \mathrm{OPK}(\tilde{k})v,$$

as in (3.16). By duality, (2.10) gives the continuous operators (imbeddings):

$$\begin{aligned}\tilde{\gamma}_0^*&\colon B_p^{-s+1/p'}(\mathbf{R}^{n-1}) \to H_p^{-s}(\mathbf{R}^n),\\ \tilde{\gamma}_0^*&\colon B_p^{-s+1/p'}(\mathbf{R}^{n-1}) \to B_p^{-s}(\mathbf{R}^n),\end{aligned} \tag{3.20}$$

for $s > 1/p' = 1 - 1/p$. Then since P is of order $d-1$, K is continuous

$$\begin{aligned}K = r^+P\tilde{\gamma}_0^*&\colon B_p^{t-1/p}(\mathbf{R}^{n-1}) \to H_p^{t-d}(\overline{\mathbf{R}}^n_+),\\ K = r^+P\tilde{\gamma}_0^*&\colon B_p^{t-1/p}(\mathbf{R}^{n-1}) \to B_p^{t-d}(\overline{\mathbf{R}}^n_+),\end{aligned} \tag{3.21}$$

for $t < 1/p$. Since d is arbitrary, and we can write $K = K\,\Xi'^{-r}\,\Xi'^{r}$ for any $r \in \mathbf{R}$, where

$$\Xi'^r = \mathrm{OP}'(\langle\xi'\rangle^r)\colon B_p^{t-1/p}(\mathbf{R}^{n-1}) \xrightarrow{\sim} B_p^{t-r-1/p}(\mathbf{R}^{n-1}), \tag{3.22}$$

and $K\,\Xi'^{-r}$ is a Poisson operator of order $d-r$, the continuity statements in (3.21) extend to all $t \in \mathbf{R}$. ∎

It follows that the more general Poisson operators K, that are sums of smoothing operators and operators given as in (3.5) with non-uniform symbol estimates, have the continuity properties:

$$K\colon B^{s-1/p}_{p,\mathrm{comp}}(\mathbf{R}^{n-1}) \to H^{s-d}_{p,\mathrm{loc}}(\overline{\mathbf{R}}^n_+) \cap B^{s-d}_{p,\mathrm{loc}}(\overline{\mathbf{R}}^n_+), \quad \text{for all } s \in \mathbf{R}. \tag{3.23}$$

REMARK 3.2: When K is of order $d \geq 1$, (3.12) shows that the symbol $k(x',\xi)$ itself already has enough estimates to permit application of Lizorkin's criterion (2.1), and hence the ps.d.o. $P_K = \mathrm{OP}(k)$ satisfies (3.2). Then one can use the representation $K = r^+P_K\tilde{\gamma}_0^*$ to conclude that (3.21) holds for $t < 1/p$ and $d \geq 1$. However, we need the Poisson operators of low negative orders to lift the property (3.21) to all positive exponents $t-d$; it is here that the stronger estimates of the auxiliary symbol p are needed.

REMARK 3.3: The Poisson operator $K_{(p)}$ with symbol-kernel $\langle\xi'\rangle^{1/p}e^{-\langle\xi'\rangle x_n}$ for $x_n \geq 0$ (a basic example) is *not continuous from* $L_p(\mathbf{R}^{n-1})$ *to* $L_p(\mathbf{R}^n_+)$ *when* $p < 2$; it is only continuous from $B_p^0(\mathbf{R}^{n-1})$ to $L_p(\mathbf{R}^n_+)$, where $B_p^0(\mathbf{R}^{n-1}) \subsetneq L_p(\mathbf{R}^{n-1})$ strictly (cf. (2.6), (2.7)). In fact, $K^*_{(p)} = \Xi'^{1/p}\gamma_0\Xi^{-1}_{-,\Omega}$ (cf. (3.22) and (2.18) ff.), so continuity of $K_{(p)}$ from $L_p(\mathbf{R}^{n-1})$ to $L_p(\mathbf{R}^n_+)$ would, by duality, imply continuity of γ_0 from $H^1_{p'}(\overline{\mathbf{R}}^n_+)$ to $H^{1-1/p'}_{p'}(\mathbf{R}^{n-1})$, which is not true for $p < 2$ according to Theorem 2.3 and (2.6) ff. — The proof of the continuity of Poisson operators in L_p spaces given in [R-S1, 2.3.2.5] is insufficient in the following ways: Primarily, the estimate of $\|K_0v\|_{L_p(\mathbf{R}^n_+)}$ ($K_0 = K_{(p)}$ when $s - 1/p' = j = 0$) in line 17 from below on page 163 does not give a bound, since the coefficient of $\|v\|_{L_p(\mathbf{R}^{n-1})}$ will in general be infinite. This holds already when $p = 2$, for the simplest seminorm of $K_{(p)}$: $\int_0^\infty \sup_{\xi'} |\langle\xi'\rangle^{1/2}e^{-\langle\xi'\rangle x_n}|^2 dx_n = \infty$, since the integrand is proportional to x_n^{-1} for small x_n. Secondly, the continuity from $L_p(\mathbf{R}^{n-1})$ to $L_p(\mathbf{R}^n_+)$ asserted in line 8 of page 163 cannot be proved when $p < 2$, as noted above.

Next, we show continuity of the truncated ps.d.o.s P_Ω, following the line of thought of [BM1, Th. 2.9].

THEOREM 3.4. *Let P be a ps.d.o. with symbol in $S^d_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, satisfying the transmission condition at $x_n = 0$. Then P_Ω defines continuous mappings, for $s > 1/p - 1$,*

$$\begin{aligned}&\text{(i)} \quad P_\Omega\colon H_p^s(\overline{\mathbf{R}}^n_+) \to H_p^{s-d}(\overline{\mathbf{R}}^n_+),\\ &\text{(ii)} \quad P_\Omega\colon B_p^s(\overline{\mathbf{R}}^n_+) \to B_p^{s-d}(\overline{\mathbf{R}}^n_+).\end{aligned} \tag{3.24}$$

PROOF: In view of (2.17), (3.24) is an immediate consequence of (3.2) for $1/p - 1 < s < 1/p$ (here it holds for any ps.d.o. with symbol in $S^d_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, regardless of the transmission condition). For larger s, we show it by induction and interpolation. The induction step consists of showing that when (3.24 i) holds for some s with $s - 1/p \notin \mathbf{N}$, *for all P* of the mentioned kind, then it holds for $s+1$.

Recall the formula for differentiation of a function with a jump:

$$D_{x_n}e^+u = e^+D_{x_n}u - i\gamma_0u(x')\otimes\delta(x_n), \quad \text{for } u \in \mathcal{S}(\overline{\mathbf{R}}^n_+); \tag{3.25}$$

it is a version of Green's formula on distribution form, and extends to $u \in H_p^s(\overline{\mathbf{R}}^n_+)$ or $B_p^s(\overline{\mathbf{R}}^n_+)$ for $s > 1/p$ by Theorem 2.3. Then

$$\begin{aligned}D_{x_n}P_\Omega u &= r^+D_{x_n}Pe^+u = r^+PD_{x_n}e^+u + r^+[D_{x_n},P]e^+u\\ &= P_\Omega D_{x_n}u - iK\gamma_0u + [D_{x_n},P]_\Omega u,\end{aligned}$$

where K is the Poisson operator (3.13) of order $d+1$, and $[D_{x_n},P] = D_{x_n}P - PD_{x_n}$ is the ps.d.o. with symbol $-i\partial_{x_n}p(x,\xi) \in S^d_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, clearly satisfying the transmission condition. Moreover,

$$D_{x_j}P_\Omega u = P_\Omega D_{x_j}u + [D_{x_j},P]_\Omega u, \quad \text{when } j < n;$$

where $[D_{x_j}, P] = \mathrm{OP}(-i\partial_{x_j} p(x,\xi))$. Then we find for $u \in \mathcal{S}(\overline{\mathbf{R}}^n_+)$, denoting the norm in $H^t_p(\overline{\mathbf{R}}^n_+)$ by $\|\cdot\|_t$,

$$\begin{aligned}\|P_\Omega u\|_{s+1-d} &\dot{\le} \textstyle\sum_{1\le j\le n} \|D_{x_j} P_\Omega u\|_{s-d} + \|P_\Omega u\|_{s-d} \\ &\dot{\le} \textstyle\sum_{1\le j\le n} \|P_\Omega D_{x_j} u\|_{s-d} + \|K\gamma_0 u\|_{s-d} \\ &\quad + \textstyle\sum_{1\le j\le n} \|[D_{x_j}, P]_\Omega u\|_{s-d} + \|P_\Omega u\|_{s-d} \\ &\dot{\le} \textstyle\sum_{1\le j\le n} \|D_{x_j} u\|_s + \|u\|_s \dot{\le} \|u\|_{s+1};\end{aligned}$$

where we applied the induction hypothesis to P and $[D_{x_j}, P]$ for $j = 1, \dots, n$, and applied Theorems 2.3 and 3.1 to $K\gamma_0$.

This shows (3.24 i) for $s > 1/p - 1$, $s - 1/p \notin \mathbf{N}$. The interpolation formulas in the first line of (2.8) can now be used to carry the result over to a full scale of Besov spaces, since we can write any $s > 1/p - 1$ as a convex combination $s = (1-\theta)s_0 + \theta s_1$, where $\theta \in]0,1[$, and s_0 and $s_1 > 1/p - 1$ with $s_0 - 1/p \notin \mathbf{N}$, $s_1 - 1/p \notin \mathbf{N}$; this shows (3.24 ii). The second line in (2.8) can be used to include the spaces $H^s_p(\overline{\mathbf{R}}^n_+)$ with $s - 1/p \in \mathbf{N}$, by writing $1/p$ as a convex combination of two different numbers $1/p_0$ and $1/p_1 \in]0,1[$. Then $s - 1/p_0 \notin \mathbf{N}$ and $s - 1/p_1 \notin \mathbf{N}$ (if d is not integer, one moreover has to make sure that $s-d-1/p_0$ and $s-d-1/p_1 \notin \mathbf{N}$); and (2.8) applies with $p_2 = p$, showing that (3.24 i) holds for P also in this case. ∎

For more general P of order d, satisfying the transmission condition, this implies the continuity properties, by use of partitions of unity:

$$(3.26)\qquad \begin{aligned}&\text{(i)} \quad P_\Omega \colon H^s_{p,\mathrm{comp}}(\overline{\mathbf{R}}^n_+) \to H^{s-d}_{p,\mathrm{loc}}(\overline{\mathbf{R}}^n_+),\\ &\text{(ii)} \quad P_\Omega \colon B^s_{p,\mathrm{comp}}(\overline{\mathbf{R}}^n_+) \to B^{s-d}_{p,\mathrm{loc}}(\overline{\mathbf{R}}^n_+).\end{aligned}$$

Now let us consider trace operators. We recall from [BM2], [R-S1], [G1,2] that a trace operator of order $d \in \mathbf{R}$ and *class* $r \in \mathbf{N}$ (i.e., of "normal order" $r-1$) is an operator of the form

$$(3.27)\qquad T = \textstyle\sum_{j=0}^{r-1} S_j\gamma_j + T',$$

where the S_j are ps.d.o.s on $\mathbf{R}^{n-1}$ of order $d-j$, and T' is the adjoint of a Poisson operator of order $d+1$, it is of class 0. More precisely, T' is, up to addition of a smoothing term $T\colon r^+\mathcal{E}'(\mathbf{R}^n) \to C^\infty(\mathbf{R}^{n-1})$, of the form

(3.28)
$$\begin{aligned}(T'u)(x') &= \mathrm{OPT}\,(t'(x',\xi))u = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix'\cdot\xi'} t'(x',\xi)\widehat{e^+u}(\xi)\,d\xi \\ &= \mathrm{OPT}\,(\tilde{t}'(x,\xi'))u = (2\pi)^{1-n} \int_{\mathbf{R}^n_+} e^{ix'\cdot\xi'} \tilde{t}'(x,\xi')\acute{u}(\xi',x_n)\,dx_n d\xi',\end{aligned}$$

where $\acute{u}(\xi',x_n)$ stands for $\mathcal{F}_{x'\to\xi'}u(x',x_n)$, and the integrals are suitably interpreted. The symbol of T' is $t'(x',\xi)$, and $\tilde{t}'(x,\xi') = \overline{\mathcal{F}}^{-1}_{\xi_n\to x_n} t' = (2\pi)^{-1}\int e^{-ix_n\xi_n} t'(x',\xi)d\xi_n$ is the so-called symbol-kernel, lying in the space $S^d_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}, \mathcal{S}(\overline{\mathbf{R}}_+))$, defined above after (3.5) (when we consider uniformly estimated symbols). The full symbol of T is $t(x',\xi) = \sum_{j=0}^{r-1} s_j(x',\xi')\xi_n^j + t'(x',\xi)$, where the s_j are the symbols of the S_j, lying in $S^{d-j}_{1,0}(\mathbf{R}^{n-1}\times\mathbf{R}^{n-1})$. Applying the usual considerations for ps.d.o. adjoints in the x'-variable, one has that T' is the adjoint of a Poisson operator K with symbol $k(x',\xi) \sim \sum D^{\alpha'}_{\xi'}\partial^{\alpha'}_{x'}\bar{t}(x',\xi)/\alpha'!$ (cf. [G2, Th. 2.4.6]), in the sense that

$$(T'u,v)_{\mathbf{R}^{n-1}} = (u,Kv)_{\mathbf{R}^n_+}$$

for $u \in \mathcal{S}(\overline{\mathbf{R}}^n_+)$, $v \in \mathcal{S}(\mathbf{R}^{n-1})$, with extensions to suitable distribution spaces; K is of order $d+1$. Thus, when K is considered as a mapping

$$K\colon B^{t+d+1-1/p'}_{p'}(\mathbf{R}^{n-1}) \to H^t_{p'}(\overline{\mathbf{R}}^n_+),$$

then (cf. (2.16))

$$T'\colon H^{-t}_{p;0}(\overline{\mathbf{R}}^n_+) \to B^{-t-d-1+1/p'}_p(\mathbf{R}^{n-1}), \quad t \in \mathbf{R}.$$

It then follows moreover, in view of (2.17), that

$$(3.29)\qquad T'\colon H^s_p(\overline{\mathbf{R}}^n_+) \to B^{s-d-1/p}_p(\mathbf{R}^{n-1}), \quad 1/p - 1 < s < 1/p.$$

Another similarity to Poisson operators is that trace operators of class zero can be produced from ps.d.o.s. On one hand, when $Q = \mathrm{OP}(q(x,\xi))$, then, for $\Omega = \mathbf{R}^n_+$, $\gamma_0 Q_\Omega$ is the trace operator $\mathrm{OPT}(h^- q(x',0,\xi))$ (cf. e.g. [G1, Ex. 2.5], $h^- = I - h^+$). On the other hand, when a trace operator of order d and class zero $T' = \mathrm{OPT}(t')$ is given, and the Seeley extension idea in the proof of Theorem 3.1 is carried out for $\tilde{t}'(x,\xi')$, giving the function $\tilde{p}(x,\xi')$, then T' equals $\gamma_0 Q_\Omega$ for $Q = \mathrm{OP}(q)$ with $q(x',\xi) = \overline{\mathcal{F}}_{x_n\to\xi_n}\tilde{p}$ (of order d and satisfying the transmission condition). This can be used to give continuity for larger values of s than in (3.29): When T' is given of class 0 and order $d \in \mathbf{R}$, take s so large that $s > 1/p$ and $s - d > 1/p$. Then we find by applying Theorems 2.3 and 3.4 to the representation $T' = \gamma_0 Q_\Omega$, that T' is continuous:

$$(3.30)\qquad T'\colon H^s_p(\overline{\mathbf{R}}^n_+) \to B^{s-d-1/p}_p(\mathbf{R}^{n-1}), \quad s > \max\{1/p, 1/p+d\}.$$

When $d \le 0$, this together with (3.29) covers all values of $s > 1/p - 1$, except the value $s = 1/p$, which is included by use of the second line in (2.8) as in the proof of Theorem 3.4. The spaces on the left can be replaced by B^s_p spaces by interpolation. When $d > 0$, we can write T' as $\Xi'^t \Xi'^{-t} T'$ for some $t \ge d$ (cf. (3.22)) and apply the preceding arguments to the trace operator $\Xi'^{-t}T'$ of order ≤ 0.

To the sum over j in (3.27) we apply Theorem 2.3 and use the continuity properties of the S_j. Then we obtain altogether (cf. (2.6) ff. concerning $H^s_p \cup B^s_p$):

THEOREM 3.5. *Let T be a trace operator of order $d \in \mathbb{R}$ and class $r \in \mathbb{N}$, with uniformly estimated symbols. Then T is continuous for $s > r + 1/p - 1$:*

$$T\colon H^s_p(\overline{\mathbb{R}}^n_+) \cup B^s_p(\overline{\mathbb{R}}^n_+) \to B^{s-d-1/p}_p(\mathbb{R}^{n-1}). \tag{3.31}$$

More general trace operators T of order $d \in \mathbb{R}$ and class $r \in \mathbb{N}$ then have the continuity properties, for $s > r + 1/p - 1$,

$$T\colon H^s_{p,\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \cup B^s_{p,\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \to B^{s-d-1/p}_{p,\mathrm{loc}}(\mathbb{R}^{n-1}). \tag{3.32}$$

We shall later discuss the limitations on s in detail, determining what it takes to extend the estimates to lower values of s.

Finally, we study the singular Green operators. Here we recall that a s.g.o. of order $d \in \mathbb{R}$ and class $r \in \mathbb{N}$ is an operator of the form

$$G = \textstyle\sum_{j=0}^{r-1} K_j \gamma_j + G', \tag{3.33}$$

where the K_j are Poisson operators of order $d - j$, and G' is of class 0; more precisely, G' is, up to addition of a smoothing term $\mathcal{G}\colon r^+\mathcal{E}'(\mathbb{R}^n) \to C^\infty(\overline{\mathbb{R}}^n_+)$, of the form

$$\begin{aligned} G'u &= \mathrm{OPG}\,(g'(x',\xi,\eta_n))u \\ &= (2\pi)^{-n-1} \int_{\mathbb{R}^{n+1}} e^{ix\cdot\xi} g'(x',\xi,\eta_n)\widehat{e^+u}(\xi',\eta_n)\,d\eta_n d\xi \\ &= \mathrm{OPG}\,(\tilde g'(x,y_n,\xi'))u = (2\pi)^{1-n} \int_{\mathbb{R}^n_+} e^{ix'\cdot\xi'} \tilde g'(x,y_n,\xi')\acute u(\xi',y_n)\,dy_n d\xi'. \end{aligned} \tag{3.34}$$

The symbol of G' is $g'(x',\xi',\xi_n,\eta_n)$, and the symbol-kernel is the function (vanishing for $x_n < 0$, $y_n < 0$)

$$\begin{aligned} \tilde g'(x',x_n,y_n,\xi') &= \mathcal{F}^{-1}_{\xi_n\to x_n} \overline{\mathcal{F}}^{-1}_{\eta_n\to y_n} g' \\ &= (2\pi)^{-2} \int e^{-ix_n\xi_n + iy_n\eta_n} g'(x',\xi',\xi_n,\eta_n)\,d\xi_n d\eta_n. \end{aligned} \tag{3.35}$$

Here $g' \in S^{d-1}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \mathcal{H}^+\hat\otimes\mathcal{H}^-_{-1})$, $\tilde g' \in S^{d-1}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}^2_{++}))$ ($\mathbb{R}^2_{++} = \mathbb{R}_+ \times \mathbb{R}_+$), which means that they satisfy the respective estimates, for all indices α' and $\beta' \in \mathbb{N}^{n-1}$, m, m', k and $k' \in \mathbb{N}$:

$$\|h^+_{\xi_n} h^-_{-1,\eta_n} D^{\beta'}_{x'} D^{\alpha'}_{\xi'} D^m_{\xi_n} \xi_n^{m'} D^k_{\eta_n} \eta_n^{k'} g'(x',\xi,\eta_n)\|_{L_{2,\xi_n,\eta_n}} \dot\le \langle\xi'\rangle^{d-|\alpha'|-m+m'-k+k'}, \tag{3.36}$$

$$\|D^{\beta'}_{x'} x_n^m D^{m'}_{x_n} y_n^k D^{k'}_{y_n} D^{\alpha'}_{\xi'} \tilde g'(x,y_n,\xi')\|_{L_{2,x_n,y_n}(\mathbb{R}^2_{++})} \dot\le \langle\xi'\rangle^{d-|\alpha'|-m+m'-k+k'}. \tag{3.37}$$

The space $\mathcal{H}^-_{-1}$ equals the complex conjugate of $\mathcal{H}^+$, and $h^-_{-1}f = \overline{h^+\bar f}$; the projections $h^+_{\xi_n}$ and h^-_{-1,η_n} can be omitted from (3.36) when $m \ge m'$ and $k \ge k'$. Symbol seminorms are defined analogously to (3.11).

As in the case of Poisson symbol-kernels, (3.37) may equivalently be formulated in terms of other norms, e.g. L_1 or L_∞ norms.

The full symbol of G is $g(x',\xi,\eta_n) = \sum_{j=0}^{r-1} k_j(x',\xi)\eta_n^j + g'(x',\xi,\eta_n)$, where the k_j are the symbols of the K_j.

Let us moreover recall that every s.g.o. G' of order d and class 0 can be written as a convergent series

$$G' = \textstyle\sum_{m\in\mathbb{N}} K_m T_m, \tag{3.38}$$

where the K_m are Poisson operators of order d and the T_m are trace operators of order and class 0. More precisely, the sequence $K_m T_m$ can be chosen such that it is *rapidly decreasing*, in the sense that each symbol seminorm of the T_m is less than a constant independent of m and each symbol seminorm of the K_m is $O(m^{-N})$ for $m \to \infty$, any N; so that also the resulting operator norms of $K_m T_m$ are rapidly decreasing. One can obtain this e.g. by expanding $\tilde g'$ in a double series

$$\tilde g'(x',x_n,y_n,\xi') = \textstyle\sum_{l,m\in\mathbb{N}} c_{l,m}(x',\xi')\varphi_l(x_n,\langle\xi'\rangle)\varphi_m(y_n,\langle\xi'\rangle) \tag{3.39}$$

in terms of the orthonormal system of Laguerre functions $\{\varphi_l(x_n,\sigma)\}_{l\in\mathbb{N}}$ in $L_2(\mathbb{R}_+)$; see the details in [G1] or [G2, Ch. 2] ($\varphi_l(x_n,\langle\xi'\rangle)$ is a Poisson symbol-kernel and $\varphi_m(y_n,\langle\xi'\rangle)$ is a trace symbol-kernel).

THEOREM 3.6. *Let G be a singular Green operator of order $d \in \mathbb{R}$ and class $r \in \mathbb{N}$. When G is given as in (3.33), (3.34), with symbol estimates uniform in $x' \in \mathbb{R}^{n-1}$, then G is continuous for $s > r + 1/p - 1$:*

$$G\colon H^s_p(\overline{\mathbb{R}}^n_+) \cup B^s_p(\overline{\mathbb{R}}^n_+) \to H^{s-d}_p(\overline{\mathbb{R}}^n_+) \cap B^{s-d}_p(\overline{\mathbb{R}}^n_+). \tag{3.40}$$

PROOF: The result follows for operators of the form $\sum_{0\le j\le r-1} K_j\gamma_j$ from Theorems 2.3 and 3.1. So it remains to treat G', and here we use the representation (3.38) to get (3.40). Since the term $K_m T_m$ for each m has the claimed continuity properties, by Theorems 3.1 and 3.5, and the operator norms are controlled by a finite system of symbol seminorms (that can be determined explicitly by an inspection), it follows from the rapid convergence mentioned above that also G' satisfies (3.40). ∎

More generally, s.g.o.s G of order $d \in \mathbb{R}$ and class $r \in \mathbb{N}$ have the continuity properties, for $s > r + 1/p - 1$,

$$G\colon H^s_{p,\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \cup B^s_{p,\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \to H^{s-d}_{p,\mathrm{loc}}(\overline{\mathbb{R}}^n_+) \cap B^{s-d}_{p,\mathrm{loc}}(\overline{\mathbb{R}}^n_+). \tag{3.41}$$

REMARK 3.7: There is another, slightly more elegant point of view on singular Green operators of class 0, related to the way Poisson and trace operators are derived from ps.d.o.s, that can be used to show the Besov space estimates (but apparently not the H^s_p space estimates), and that we shall now briefly describe. Taking a Seeley extension $\tilde g_1$ of $\tilde g'(x', x_n, y_n, \xi')$ in both the x_n and the y_n variable, we obtain a pseudo-differential symbol $g_1 = \mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n}\tilde g_1$ in $n+1$ cotangent variables (see the details in [G2, Rem. 2.4.12]), such that if P_1 is the associated operator $P_1 = \mathrm{OP}^{(n+1)}(g_1)$ on $\mathbb{R}^{n+1}$, and $\tilde\gamma_0^{(n+1)}$ denotes the trace operator $\tilde\gamma_0^{(n+1)}\colon v(x, x_{n+1}) \mapsto v(x, 0)$; then

$$G' = r^+\tilde\gamma_0^{(n+1)} P_1 (\tilde\gamma_0^{(n+1)})^* e^+. \tag{3.42}$$

At first sight this gives continuity of G' from $B^s_p(\overline{\mathbb{R}}^n_+)$ to $B^{s-d}_p(\overline{\mathbb{R}}^n_+)$ for $1/p - 1 < s < 0$, $d < s$, but this can be extended to larger values of s and d by use of the order reducing operators $\Lambda^m_{-,\Omega}$ defined in Section 4. — The method is restricted to Besov spaces because they are the range spaces for $\tilde\gamma_0^{(n+1)}$.

Before going on to spaces over more general manifolds, we note that one can in some cases extend the continuity to spaces of lower order.

DEFINITION 3.8. *Let P be a ps.d.o. on $\mathbb{R}^n$ of order d, satisfying the transmission condition at $x_n = 0$, and let $G = \mathrm{OPG}(\tilde g(x', x_n, y_n, \xi'))$ and $T = \mathrm{OPT}(\tilde t(x', x_n, \xi'))$ be singular Green and trace operators on $\overline{\mathbb{R}}^n_+$ of order d and class 0. Denoting $\overline{D}_{x_j} = +i\partial_{x_j}$, we define $K_P^{(j)}$, $K_G^{(j)}$ and $S_T^{(j)}$ by*

$$\begin{aligned} K_P^{(j)} v &= r^+ iP(v(x') \otimes D^j_{x_n}\delta(x_n)) = r^+ iPD^j_{x_n}(v(x') \otimes \delta(x_n));\\ K_G^{(j)} &= \mathrm{OPK}(i\overline{D}^j_{y_n}\tilde g(x', x_n, 0, \xi')),\\ S_T^{(j)} &= \mathrm{OP}'(i\overline{D}^j_{x_n}\tilde t(x', 0, \xi')); \end{aligned} \tag{3.43}$$

here $K_P^{(j)} = \mathrm{OPK}(ih^+[\xi_n^j q(x', 0, \xi)])$ when P is as in (3.14–15). Let $m \in \mathbb{N}$.
1° $P_\Omega + G$ is said to be of class $-m$, when $K_P^{(j)} + K_G^{(j)} = 0$ for $0 \le j \le m-1$.
2° T is said to be of class $-m$, when $S_T^{(j)} = 0$ for $0 \le j \le m-1$.
If these properties hold for all $m \in \mathbb{N}$, we say that the operators are of class $-\infty$.

These definitions are introduced in [G3], where it is shown that

$$\begin{aligned} P_\Omega D^j_{x_n} u &= \textstyle\sum_{k=0}^{j-1} K_P^{(k)}\gamma_{j-1-k}u + (PD^j_{x_n})_\Omega u,\\ GD^j_{x_n} u &= \textstyle\sum_{k=0}^{j-1} K_G^{(k)}\gamma_{j-1-k}u + \mathrm{OPG}\,(\overline{D}^j_{y_n}\tilde g(x', x_n, y_n, \xi'))u,\\ TD^j_{x_n} u &= \textstyle\sum_{k=0}^{j-1} S_T^{(k)}\gamma_{j-1-k}u + \mathrm{OPT}\,(\overline{D}^j_{x_n}\tilde t(x', x_n, \xi'))u; \end{aligned} \tag{3.44}$$

for $u \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$. The formulas imply:

LEMMA 3.9. *Let $m \in \mathbb{N}$, and let G and T be of class 0.*
1° $P_\Omega + G$ is of class $-m$ if and only if

$$(P_\Omega + G)D^j_{x_n} = (PD^j_{x_n})_\Omega + G^{(j)} \quad \text{for } 1 \le j \le m, \tag{3.45}$$

with s.g.o.s $G^{(j)}$ of class 0.
2° T is of class $-m$ if and only if

$$TD^j_{x_n} = T^{(j)} \quad \text{for } 1 \le j \le m, \tag{3.46}$$

with trace operators $T^{(j)}$ of class 0.

Note that the definition (like that of operators of classes $\in \mathbb{N}$) is concerned with exact operators, not just their symbols. The definition is generalized to smooth sets Ω by use of local coordinates, or one can give global formulas in terms of a fixed choice of normal and tangential variables near $\partial\Omega$.

We observe that 1° in the definition contains the definition of separate operators P_Ω or G of classes ≤ 0 as special cases. When G is a s.g.o. of class $r > 0$ and P is a ps.d.o., we shall say that $P_\Omega + G$ is of class r. (Note that P_Ω is always of class 0.) Operators G and T of negative class are also defined in [F1,2].

THEOREM 3.10. *Let P, G and T be as in Theorems 3.4–3.6, except that we now let $r \in \mathbb{Z}$.*

1° *When $P_\Omega + G$ is of class r, it defines continuous mappings*

$$(3.47)\qquad \begin{aligned} &P_\Omega + G\colon H_p^s(\overline{\mathbb{R}}_+^n) \to H_p^{s-d}(\overline{\mathbb{R}}_+^n),\\ &P_\Omega + G\colon B_p^s(\overline{\mathbb{R}}_+^n) \to B_p^{s-d}(\overline{\mathbb{R}}_+^n), \quad \text{for all } s > r + 1/p - 1, \end{aligned}$$

all $p \in]1,\infty[$. Conversely, if $P_\Omega + G$ is continuous from $H_p^s(\overline{\mathbb{R}}_+^n)$ or $B_p^s(\overline{\mathbb{R}}_+^n)$ to $\mathcal{D}'(\mathbb{R}_+^n)$, for some $s \le r + 1/p$, some $p \in]1,\infty[$, then $P_\Omega + G$ is of class r. For G separately, the statements generalize to the mappings $G\colon H_p^s(\overline{\mathbb{R}}_+^n) \cup B_p^s(\overline{\mathbb{R}}_+^n) \to H_p^{s-d}(\overline{\mathbb{R}}_+^n) \cap B_p^{s-d}(\overline{\mathbb{R}}_+^n)$.

2° *When T is of class r, it defines continuous mappings*

$$(3.48)\quad T\colon H_p^s(\overline{\mathbb{R}}_+^n) \cup B_p^s(\overline{\mathbb{R}}_+^n) \to B_p^{s-d-1/p}(\mathbb{R}^{n-1}), \quad \text{for all } s > r + 1/p - 1,$$

all $p \in]1,\infty[$. Conversely, if T is continuous from $H_p^s(\overline{\mathbb{R}}_+^n)$ or $B_p^s(\overline{\mathbb{R}}_+^n)$ to $\mathcal{D}'(\mathbb{R}^{n-1})$, for some $s \le r + 1/p$, some $p \in]1,\infty[$, then T is of class r.

PROOF: The direct statements are proved for $r \ge 0$ in Theorems 3.4–3.6. Now let $P_\Omega + G$ be of class $-m$, $m > 0$. Then in view of (3.45), one has for all $|\alpha| \le m$,

$$(P_\Omega + G)D^\alpha = (PD^\alpha)_\Omega + G^{(\alpha)}$$

with $G^{(\alpha)}$ of class 0. Since

$$(3.49)\qquad \begin{aligned} u \in H_p^s(\overline{\mathbb{R}}_+^n) &\iff u = \textstyle\sum_{|\alpha| \le m} D^\alpha v_\alpha \text{ for some } v_\alpha \in H_p^{s+m}(\overline{\mathbb{R}}_+^n),\\ u \in B_p^s(\overline{\mathbb{R}}_+^n) &\iff u = \textstyle\sum_{|\alpha| \le m} D^\alpha v_\alpha \text{ for some } v_\alpha \in B_p^{s+m}(\overline{\mathbb{R}}_+^n), \end{aligned}$$

for any $s \in \mathbb{R}$ in view of (2.19), an application of Theorems 3.4 and 3.6 to PD^α and $G^{(\alpha)}$, $|\alpha| \le m$, shows (3.47). (In case $P = 0$, one gets a slightly better mapping property for G.) Similarly, when T is of class $-m$, (3.46) implies that

$$TD^\alpha = T^{(\alpha)}$$

is of class 0 for all $|\alpha| \le m$, so that (3.48) follows from Theorem 3.5 in view of (3.49).

For the converse statements we use the proof of Lemma 2.2, which shows that if K is a Poisson operator, resp. S is a ps.d.o. over $\mathbb{R}^{n-1}$, such that one of the following linear operators is continuous, for some $j \in \mathbb{N}$:

$$\begin{aligned} &K\gamma_j\colon H_p^{j+1/p}(\overline{\mathbb{R}}_+^n) \to \mathcal{D}'(\mathbb{R}_+^n), \quad K\gamma_j\colon B_p^{j+1/p}(\overline{\mathbb{R}}_+^n) \to \mathcal{D}'(\mathbb{R}_+^n), \text{ resp.}\\ &S\gamma_j\colon H_p^{j+1/p}(\overline{\mathbb{R}}_+^n) \to \mathcal{D}'(\mathbb{R}^{n-1}), \quad S\gamma_j\colon B_p^{j+1/p}(\overline{\mathbb{R}}_+^n) \to \mathcal{D}'(\mathbb{R}^{n-1}); \end{aligned}$$

then K resp. S must equal 0. For, one can find a sequence $w_m(x_n) \in \mathcal{S}(\overline{\mathbb{R}}_+)$ going to 0 in $H_p^{j+1/p}(\overline{\mathbb{R}}_+)$ for $m \to \infty$, such that $\partial_{x_n}^j w_m(0) = 1$ for all m. Now for example, if K is not the zero operator, there is a $v(x') \in \mathcal{S}(\mathbb{R}^{n-1})$ such that $Kv \ne 0$. We have that $u_m(x) = v(x')w_m(x_n)$ goes to 0 in $H_p^{j+1/p}(\overline{\mathbb{R}}_+^n)$, so if $K\gamma_j$ were continuous from $H_p^{j+1/p}(\overline{\mathbb{R}}_+^n)$ to $\mathcal{D}'(\mathbb{R}_+^n)$, $K\gamma_j u_m$ would go to 0 there; but it is constant equal to $Kv \ne 0$. — The proofs in the other cases are similar.

This is used as follows. First let r be an integer ≥ 0 such that $P_\Omega + G$ is continuous from $H_p^{r+1/p}(\overline{\mathbb{R}}_+^n)$ to $\mathcal{D}'(\mathbb{R}_+^n)$. If G is of class $r' > r$, then $P_\Omega + G$ is of the form

$$P_\Omega + G = P_\Omega + G' + \textstyle\sum_{j=0}^{r'-2} K_j\gamma_j + K_{r'-1}\gamma_{r'-1},$$

where the first three terms are continuous from $H_p^{r'+1/p}(\overline{\mathbb{R}}_+^n)$ to $\mathcal{D}'(\mathbb{R}_+^n)$ by Theorems 3.4 and 3.6. Hence $K_{r'-1}\gamma_{r'-1}$ also has this continuity, so $K_{r'-1}$ must equal 0. This shows that G is of class $r' - 1$. If $r' - 1 > r$, we repeat the argument; and we can continue until we reach the conclusion that G is of class r. This shows the desired statement (in H_p^s spaces) for $P_\Omega + G$ when $r \ge 0$.

Now let $r = -m$, $m \in \mathbb{N}$. If $P_\Omega + G$ is continuous from $H_p^{-m+1/p}(\overline{\mathbb{R}}_+^n)$ to $\mathcal{D}'(\mathbb{R}_+^n)$, then $(P_\Omega + G)D_{x_n}^j$ is continuous from $H_p^{1/p}(\overline{\mathbb{R}}_+^n)$ to $\mathcal{D}'(\mathbb{R}_+^n)$ for $j \le m$. By the preceding argument, $(P_\Omega + G)D_{x_n}^j$ must be of class 0 for $j \le m$, so by Lemma 3.9, $P_\Omega + G$ is of class $-m$.

The remaining proofs are very similar to this. ∎

There is now a standard procedure to carry these results over to the situation where $\mathbb{R}_+^n$ is replaced by a bounded smooth domain Ω, by means of a partition of unity and local coordinates, using that the Bessel-potential spaces and Besov spaces are invariant under such manipulations. The definition of the pseudo-differential boundary operators connected with Ω is explained e.g. in [G1,2]. Note that the class property is independent of the choice of local coordinates, because of the sharpness of Theorem 3.10. We can then simply list the consequences:

THEOREM 3.11. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary Γ (or let $\Omega = \mathbb{R}_+^n$, with $\Gamma = \mathbb{R}^{n-1}$). Let P be a pseudo-differential operator on $\mathbb{R}^n$ satisfying the transmission condition at Γ, let G be a singular Green operator on $\overline{\Omega}$, let T be a trace operator from $\overline{\Omega}$ to Γ, let K*

be a Poisson operator from Γ *to* $\overline{\Omega}$, *and let* S *be a ps.d.o. on* Γ, *all of order* $d \in \mathbb{R}$, *and* G *and* T *moreover of class* $r \in \mathbb{Z}$ *(the operators having uniform symbol estimates in case* $\Omega = \mathbb{R}^n_+$*). Then they define continuous mappings:*

$$(3.50)\quad \begin{aligned} P_\Omega &: H^s_p(\overline{\Omega}) \to H^{s-d}_p(\overline{\Omega}) \text{ and } B^s_p(\overline{\Omega}) \to B^{s-d}_p(\overline{\Omega}) \quad \text{for } s > 1/p - 1,\\ G &: H^s_p(\overline{\Omega}) \cup B^s_p(\overline{\Omega}) \to H^{s-d}_p(\overline{\Omega}) \cap B^{s-d}_p(\overline{\Omega}) \quad \text{for } s > r + 1/p - 1,\\ T &: H^s_p(\overline{\Omega}) \cup B^s_p(\overline{\Omega}) \to B^{s-d-1/p}_p(\Gamma) \quad \text{for } s > r + 1/p - 1,\\ K &: B^{s-1/p}_p(\Gamma) \to H^{s-d}_p(\overline{\Omega}) \cap B^{s-d}_p(\overline{\Omega}) \quad \text{for all } s \in \mathbb{R},\\ S &: H^s_p(\Gamma) \to H^{s-d}_p(\Gamma) \text{ and } B^s_p(\Gamma) \to B^{s-d}_p(\Gamma) \quad \text{for all } s \in \mathbb{R}. \end{aligned}$$

If $P_\Omega + G$ *is of class* $r' \in \mathbb{Z}$, *then*

$$(3.51)\quad \begin{aligned} P_\Omega + G &: H^s_p(\overline{\Omega}) \to H^{s-d}_p(\overline{\Omega}) \quad \text{and}\\ P_\Omega + G &: B^s_p(\overline{\Omega}) \to B^{s-d}_p(\overline{\Omega}) \quad \text{for } s > r' + 1/p - 1. \end{aligned}$$

The limitations on s *are sharp, in the sense that if the continuity extends to lower values of* s *than indicated, then the class is lower than indicated.*

Concerning $H^s_p \cup B^s_p$ and $H^s_p \cap B^s_p$ we recall (2.6) ff. In view of this and (2.9), we have in particular for the Sobolev-Slobodetskiĭ spaces W^s_p:

COROLLARY 3.12. *The pseudo-differential boundary operators* P_Ω, G, T, K *and* S *defined in Theorem 3.11 have the continuity properties for* $s \geq 0$*:*

(3.52)

$$\begin{aligned} P_\Omega &: W^s_p(\overline{\Omega}) \to W^{s-d}_p(\overline{\Omega}) \quad \text{for } s > 1/p - 1,\ s - d \geq 0,\\ G &: W^s_p(\overline{\Omega}) \to W^{s-d}_p(\overline{\Omega}) \quad \text{for } s > r + 1/p - 1,\ s - d \geq 0,\\ P_\Omega + G &: W^s_p(\overline{\Omega}) \to W^{s-d}_p(\overline{\Omega}) \quad \text{for } s > r' + 1/p - 1,\ s - d \geq 0,\\ T &: W^s_p(\overline{\Omega}) \to W^{s-d-1/p}_p(\Gamma) \quad \text{for } s > r + 1/p - 1,\ s - d - 1/p \geq 0,\\ &\qquad \text{under the restriction } s - d - 1/p \notin \mathbb{N}, \text{ if } p > 2,\\ K &: W^{s-1/p}_p(\Gamma) \to W^{s-d}_p(\overline{\Omega}) \quad \text{for } s - 1/p \geq 0,\ s - d \geq 0,\\ &\qquad \text{under the restriction } s - 1/p \notin \mathbb{N}, \text{ if } p < 2,\\ S &: W^s_p(\Gamma) \to W^{s-d}_p(\Gamma) \quad \text{for } s - d \geq 0. \end{aligned}$$

Moreover, the B^s_p space estimates in (3.50), (3.51) imply by interpolation, by (2.8) ff.:

COROLLARY 3.13. *The pseudo-differential boundary operators* P_Ω, G, T, K *and* S *defined in Theorem 3.11 have the continuity properties, for* $1 < p < \infty$, $1 \leq q \leq \infty$*:*

$$(3.53)\quad \begin{aligned} P_\Omega &: B^s_{p,q}(\overline{\Omega}) \to B^{s-d}_{p,q}(\overline{\Omega}) \quad \text{for } s > 1/p - 1,\\ G &: B^s_{p,q}(\overline{\Omega}) \to B^{s-d}_{p,q}(\overline{\Omega}) \quad \text{for } s > r + 1/p - 1,\\ P_\Omega + G &: B^s_{p,q}(\overline{\Omega}) \to B^{s-d}_{p,q}(\overline{\Omega}) \quad \text{for } s > r' + 1/p - 1,\\ T &: B^s_{p,q}(\overline{\Omega}) \to B^{s-d-1/p}_{p,q}(\Gamma) \quad \text{for } s > r + 1/p - 1,\\ K &: B^{s-1/p}_{p,q}(\Gamma) \to B^{s-d}_{p,q}(\overline{\Omega}) \quad \text{for all } s \in \mathbb{R},\\ S &: B^s_{p,q}(\Gamma) \to B^{s-d}_{p,q}(\Gamma) \quad \text{for all } s \in \mathbb{R}. \end{aligned}$$

These results overlap with those of [F1,2], where larger scales of spaces are treated; however, our statement on G is slightly more general, and operators $P_\Omega + G$ of negative class are not included in [F1,2].

EXAMPLE 3.14: Theorem 3.11 allows us to treat in a very simple way the generalization from L_2 to L_p of the well-known orthogonal decompositions of the space of n-vectors $L_2(\Omega)^n$ into solenoidal (divergence free) and gradient vector fields, used in the theory of the Navier-Stokes equations. Define

$$(3.54)\quad \begin{aligned} J_p(\Omega) &= \{\, u \in L_p(\Omega)^n \mid \operatorname{div} u = 0 \,\},\\ J_{p;0}(\Omega) &= \{\, u \in L_p(\Omega)^n \mid \operatorname{div} u = 0,\ \gamma_\nu u = 0 \,\},\\ G_p(\Omega) &= \{\, w = \operatorname{grad} f \mid f \in H^1_p(\overline{\Omega}) \,\},\\ G_{p;0}(\Omega) &= \{\, w = \operatorname{grad} f \mid f \in H^1_{p;0}(\overline{\Omega}) \,\}; \end{aligned}$$

here γ_ν is the trace operator $u \mapsto \gamma_0(\vec{n} \cdot u)$ where $\vec{n}$ is the normal to Γ, well-defined when u and $\operatorname{div} u$ are in L_p. Then

$$(3.55)\quad L_p(\Omega)^n = J_{p;0}(\Omega) \dot{+} G_p(\Omega), \quad L_p(\Omega)^n = J_p(\Omega) \dot{+} G_{p;0}(\Omega);$$

cf. e.g. Solonnikov [So3] and its references (and Fujiwara-Morimoto [F-M]). The projections pr_J and pr_{J_0} of $L_p(\Omega)^n$ onto $J_p(\Omega)$ resp. $J_{p;0}(\Omega)$ can be formulated explicitly as follows (cf. also [G-S2] and, for details, [G-S3]):

$$(3.56)\quad \begin{aligned} \operatorname{pr}_J &= I + \operatorname{grad} R_D \operatorname{div} = P_\Omega + G',\\ \operatorname{pr}_{J_0} &= (I - \operatorname{grad} K_N \gamma_\nu) \operatorname{pr}_J = P_\Omega + G'_0; \end{aligned}$$

here $R_D : f \mapsto u$ and $K_N : \psi \mapsto v$ are the solution operators for the following semi-homogeneous Dirichlet resp. Neumann problems for the Laplace operator:

$$(3.57)\quad \begin{cases} -\Delta u = f & \text{in } \Omega,\\ \gamma_0 u = 0 & \text{on } \Gamma; \end{cases} \quad \text{resp.} \quad \begin{cases} -\Delta v = 0 & \text{in } \Omega,\\ \gamma_1 v = \psi & \text{on } \Gamma; \end{cases}$$

under the assumptions $\int_\Omega v\, dx = 0$, $\int_\Gamma \psi\, d\sigma = 0$. The resulting operators pr_J and pr_{J_0} belong to the ps.d. boundary operator calculus, with $P =$

$\mathrm{OP}\big((\delta_{ij} - \xi_i\xi_j|\xi|^{-2})_{i,j=1,\dots,n}\big)$ of order 0 (δ_{ij} denotes the Kronecker delta), and s.g.o.s G' and G'_0 of order and class 0. Then by Theorem 3.11, the operators are continuous:

$$\text{(3.58)}\qquad \begin{aligned} &\mathrm{pr}_J, \mathrm{pr}_{J_0}\colon H_p^s(\overline{\Omega})^n \to H_p^s(\overline{\Omega})^n,\\ &\mathrm{pr}_J, \mathrm{pr}_{J_0}\colon B_p^s(\overline{\Omega})^n \to B_p^s(\overline{\Omega})^n, \text{ for all } s > 1/p - 1;\end{aligned}$$

and the decompositions (3.55) are hereby generalized to analogous decompositions of $H_p^s(\overline{\Omega})$ and $B_p^s(\overline{\Omega})$ for all $s > 1/p - 1$.

EXAMPLE 3.15: It it well known that the solution operator R_D for the Dirichlet problem recalled in (3.57) is continuous from $H_2^{-1}(\overline{\Omega})$ to $H_2^1(\overline{\Omega})$ (in fact to $H^1_{2;0}(\overline{\Omega})$); it is of the form $Q_\Omega + G$ with G of class 0. The sharpness of Theorem 3.11 shows that $Q_\Omega + G$ is of class -1. One can also prove this by calculations according to Definition 3.8. Such calculations show that $K_Q^{(0)} \neq 0$, so Q_Ω and G are not of class -1 separately. By the above theorems, we have that R_D is continuous:

$$\text{(3.59)}\qquad \begin{aligned} &R_D\colon H_p^{s-1}(\overline{\Omega}) \to H_p^{s+1}(\overline{\Omega}), \quad R_D\colon B_p^{s-1}(\overline{\Omega}) \to B_p^{s+1}(\overline{\Omega}),\\ &R_D\colon B_{p,q}^{s-1}(\overline{\Omega}) \to B_{p,q}^{s+1}(\overline{\Omega}), \quad \text{for } s > 1/p - 1,\end{aligned}$$

for any $p \in]1,\infty[$, $q \in [1,\infty]$. The corresponding solution operator for the Neumann problem is only of class 0. Similar examples for ps.d.o.s can be given by use of [G2, 1.7].

4. Elliptic systems and order-reducing operators.

Let $\overline{\Omega}$ be a compact n-dimensional C^∞ manifold with interior Ω and boundary $\partial\Omega = \Gamma$, let E and E' be vector bundles over $\overline{\Omega}$ of dimensions N and $N' \geq 1$, and let F and F' be vector bundles over Γ of dimensions M and $M' \geq 0$ (all bundles hermitean and C^∞). We can assume that $\overline{\Omega}$ is smoothly imbedded in an open n-dimensional manifold Ω_1 (so that Γ is the boundary of Ω there), and that E is the restriction to $\overline{\Omega}$ of a smooth N-dimensional bundle E_1 over Ω_1. There is a standard way to generalize the various spaces defined in Section 2 ($H_p^s(\overline{\Omega})$, etc.) to spaces of (distributional) sections of these bundles (called $H_p^s(E)$, etc.), by use of local coordinates and partitions of unity. We assume that $\overline{\Omega}$ (and Ω_1) and $\partial\Omega$ are provided with C^∞ densities dx and dx', and write the scalar product in $L_2(E)$ as $(u,v)_\Omega = \int_\Omega u \cdot \bar{v}\, dx$, etc.; the distribution dualities are taken to be extensions of this, denoted $\langle u, \bar{v}\rangle_\Omega$, etc. (One could also formulate the expressions more invariantly in terms of half-densities, as explained e.g. in Hörmander [H3, 18.1].) The pseudo-differential boundary operators are generalized as described in detail in [R-S1] and [G2]; we recall in particular that P should be given in an extending bundle E_1.

Consider a system formed of the various ps.d. boundary operators (a Green operator):

$$\text{(4.1)}\qquad \mathcal{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} \colon \begin{matrix} C^\infty(E) \\ \times \\ C^\infty(F) \end{matrix} \to \begin{matrix} C^\infty(E') \\ \times \\ C^\infty(F') \end{matrix},$$

where the operators are of order $d \in \mathbb{R}$, and $P_\Omega + G$ and T are of class $r \in \mathbb{Z}$; we say that $\mathcal{A}$ is of order d and class r. (Multi-order systems will be included later.) We assume from now on that the symbols of the operators are *polyhomogeneous*, i.e. have expansions in homogeneous terms of falling degree. The complete definition is given in [BM2], [G1,2] and [R-S1]; let us just describe the typical case of a Poisson symbol $k(x',\xi)$, here polyhomogeneity means (in local coordinates) that k has an asymptotic expansion in homogeneous terms:

$$\begin{gathered} k(x',\xi) \sim \textstyle\sum_{j\in\mathbb{N}} k_{d-1-j}(x',\xi), \text{ in the sense that}\\ k - \textstyle\sum_{j<J} k_{d-1-j} \in S_{1,0}^{d-1-J}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{H}^+) \text{ for each } J \in \mathbb{N},\\ \text{with } k_{d-1-j}(x',t\xi) = t^{d-1-j} k_{d-1-j}(x',\xi) \text{ for } t \geq 1,\ |\xi'| \geq 1.\end{gathered}$$

The principal symbol of $k(x',\xi)$ is $k^0(x',\xi) = k_{d-1}(x',\xi)$.

The system $\mathcal{A}$ (and its system of principal symbols) is called *injectively elliptic*, *surjectively elliptic*, resp. *elliptic* (also called overdetermined elliptic, underdetermined elliptic, resp. elliptic), when the following conditions (I) and (II) are satisfied (they are expressed in local coordinates):

(I) The principal symbol of P,

$$\text{(4.2)}\qquad p^0(x,\xi)\colon \mathbb{C}^N \to \mathbb{C}^{N'}$$

is injective, surjective, resp. bijective, for each x, each $|\xi| \geq 1$.

(II) The principal boundary symbol operator (the "model" operator)

$$\text{(4.3)}\qquad a^0(x',\xi',D_n) = \begin{pmatrix} p^0(x',0,\xi',D_n)_{\mathbb{R}_+} + g^0(x',\xi',D_n) & k^0(x',\xi',D_n) \\ t^0(x',\xi',D_n) & s^0(x',\xi') \end{pmatrix} \colon$$
$$\mathcal{S}(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M \to \mathcal{S}(\overline{\mathbb{R}}_+)^{N'} \times \mathbb{C}^{M'}$$

is injective, surjective, resp. bijective, for each $x' \in \Gamma$, $|\xi'| \geq 1$.

The D_n in (4.3) indicates that the operator definitions, of each kind, have been applied with respect to the x_n variable to the principal symbols.

In the elliptic case, one has necessarily $N = N'$, whereas M and M' can differ from each other (and be equal to zero). The transmission condition together with one of the ellipticity properties implies $d \in \mathbf{Z}$. Condition (I) alone defines injective ellipticity, surjective ellipticity, resp. ellipticity, of P.

Simple examples are furnished by the usual elliptic boundary value problems for elliptic differential operators P. Here $G = 0$, and $M = 0$, so that K and S are trivially zero and can be omitted; so $\mathcal{A}$ is of the form

$$\mathcal{A} = \begin{pmatrix} P_\Omega \\ T \end{pmatrix}. \tag{4.4}$$

When P is properly elliptic, Nd is even and $M' = Nd/2$. Usually, the trace operator is then written as a vector of operators of different orders $T = \{T_j\}_{0 \le j < d'}$, with T_j of order j and mapping into a bundle F_j of dimension $M'_j \ge 0$; and the above formulation is easily extended to include such multi-order systems. One can also transform all the operators to operators of order d, by replacing T_j by $\Xi'^{d-j}_{F_j} T_j$; here Ξ'^k_F (or just Ξ'^k) stands for a bijective ps.d.o. of order k in a bundle F over Γ (similar to the operator in (3.22)),

$$\Xi'^k_F \colon H^s_p(F) \xrightarrow{\sim} H^{s-k}_p(F), \quad \Xi'^k_F \colon B^s_p(F) \xrightarrow{\sim} B^{s-k}_p(F), \quad s \in \mathbf{R}; \tag{4.5}$$

such operators are constructed in the beginning of Section 5. The solution operator or parametrix for an elliptic differential boundary value problem is another example of an elliptic operator, it has the form

$$\widetilde{\mathcal{A}} = (\, \widetilde{P}_\Omega + \widetilde{G} \quad \widetilde{K} \,)\,. \tag{4.6}$$

More complicated examples occur when systems of these two types are composed. The auxiliary operators, we define further below, are truly pseudo-differential examples.

In the present section we shall consider the *elliptic* case. One of the fundamental results in the theory ([BM2], [R-S1], [G2]) is that the mere bijectiveness in Condition (II) assures that the inverse $(a^0)^{-1}$ (defined for each (x', ξ')) is a boundary symbol operator *belonging to the calculus* (in its dependence on all variables); it is of order $-d$ and elliptic too. One uses $(a^0)^{-1}$ together with $(p^0)^{-1}$ and $\mathcal{A}$ to construct a full *parametrix* $\widetilde{\mathcal{A}}$ of $\mathcal{A}$, i.e. an operator

$$\widetilde{\mathcal{A}} = \begin{pmatrix} \widetilde{P}_\Omega + \widetilde{G} & \widetilde{K} \\ \widetilde{T} & \widetilde{S} \end{pmatrix} : \begin{matrix} C^\infty(E') \\ \times \\ C^\infty(F') \end{matrix} \to \begin{matrix} C^\infty(E) \\ \times \\ C^\infty(F) \end{matrix} \tag{4.7}$$

of order $-d$ and belonging to the calculus, such that

$$\begin{aligned} &\text{(i)} \quad \mathcal{R}_1 = \widetilde{\mathcal{A}}\mathcal{A} - I \quad \text{is of order } -\infty, \\ &\text{(ii)} \quad \mathcal{R}_2 = \mathcal{A}\widetilde{\mathcal{A}} - I \quad \text{is of order } -\infty. \end{aligned} \tag{4.8}$$

To describe the mapping properties of $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ in Sobolev-type spaces in a precise way, we must take the *class* into account. The basic step in the construction of $\widetilde{\mathcal{A}}$ (found in [R-S1, 3.1.1], [G2, 3.2]) shows:

THEOREM 4.1. *When $\mathcal{A}$ is elliptic of order* 0 *and class* 0, *then $(a^0)^{-1}$ is a boundary symbol operator of order and class* 0, *and $\widetilde{\mathcal{A}}$ can be obtained to be of order and class* 0.

Operators of a more general order and class are handled by use of certain auxiliary operators, the so-called order-reducing operators, that we shall now define.

As noted earlier, the symbols $(\langle\xi'\rangle \pm i\xi_n)^m$, that define the operators $\Xi^m_\pm$ with very convenient mapping properties in relation to spaces over $\mathbf{R}^n_+$ (cf. (2.19)), are not in $S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$; for example $D^2_{\xi_1}(\langle\xi'\rangle \pm i\xi_n)$ is $O(\langle\xi'\rangle^{-1})$ but not $O(\langle\xi\rangle^{-1})$. To obtain a similar kind of symbols that do belong to our symbol classes, we let

(4.9)
$$\begin{gathered} \chi(t) \in \mathcal{S}(\mathbf{R}) \text{ with } \chi(0) = 1, \\ \kappa_1(t) \in C^\infty(\mathbf{R}, \mathbf{R}_+),\ \kappa_1(t) = |t| \text{ for } |t| \ge 1,\ \kappa_1(t) = 1/2 \text{ for } |t| \le 1/2, \\ \kappa(\xi', \mu) = \kappa_1(|(\xi', \mu)|), \end{gathered}$$

where μ is a positive parameter to be fixed later; and we set

$$\lambda^m_\pm(\xi, \mu) = \Big(\kappa(\xi', \mu)\chi\big(\frac{\xi_n}{a\kappa(\xi', \mu)}\big) \pm i\xi_n\Big)^m, \ m \in \mathbf{Z}, \tag{4.10}$$

where we take $a > 0$ so large that the negative powers are well-defined. More precisely, one has for $\delta \in]0, 1[$,

$$\begin{gathered} \Big|\frac{\kappa\chi(\xi_n/a\kappa) \pm i\xi_n}{\kappa \pm i\xi_n}\Big| = \Big|1 + \frac{(\chi(\xi_n/a\kappa) - 1)\kappa}{\kappa \pm i\xi_n}\Big| \ge 1 - \delta, \\ \text{when } \sup_{\xi_n}\Big|\frac{(\chi(\xi_n/a\kappa) - 1)\kappa}{\kappa \pm i\xi_n}\Big| \equiv \sup_t\Big|\frac{\chi(t) - 1}{1 \pm iat}\Big| \le \delta, \\ \text{which holds for } |a| \ge \delta^{-1}\sup_t|\chi'(t)|; \end{gathered} \tag{4.11}$$

and we take a to satisfy this with $\delta = 1/2$ (more conditions come in later). The functions $\lambda^m_\pm(\xi, \mu)$ are C^∞ in $(\xi, \mu) \in \mathbf{R}^{n+1}$; and since $\chi(t)$ is rapidly decreasing for $t \to \pm\infty$, the functions

$$\lambda^{m0}_\pm(\xi, \mu) = \Big(|(\xi', \mu)|\chi\big(\frac{\xi_n}{a|(\xi', \mu)|}\big) \pm i\xi_n\Big)^m, \ m \in \mathbf{Z}, \tag{4.12}$$

are C^∞ in $(\xi,\mu) \in \mathbb{R}^{n+1} \setminus \{0\}$; they are homogeneous in (ξ,μ) of degree m. Now $\chi(\xi_n/a\kappa(\xi',\mu)) - \chi(\xi_n/a|(\xi',\mu)|)$ vanishes for $|(\xi',\mu)| \ge 1$, and $\chi(\xi_n/a\kappa(\xi',\mu))$ is $O(\langle\xi_n\rangle^{-N})$ on the set $\{(\xi,\mu) \in \mathbb{R}^{n+1} \mid |(\xi',\mu)| \le 1\}$ for any N, hence is $O(\langle(\xi,\mu)\rangle^{-N})$ there. Thus $\chi(\xi_n/a\kappa(\xi',\mu))$ belongs to the space of polyhomogeneous symbols $S^0(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$, with principal symbol equal to $\chi(\xi_n/a|(\xi',\mu)|)$ for $|(\xi,\mu)| \ge 1$ (and no other homogeneous terms). It follows that the $\lambda^m_\mp(\xi,\mu)$ are in $S^m(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$, with principal symbol equal to $\lambda^{m0}_\mp(\xi,\mu)$ for $|(\xi,\mu)| \ge 1$. (In particular, in the terminology of [G2], the $\lambda^m_\mp(\xi,\mu)$ are parameter-dependent symbols of regularity $+\infty$. The idea of including the parameter μ, so that one can at some stage benefit from a parameter-dependent calculus, is found in [R-S3, 3.3].)

When we fix the parameter μ, we get symbols $\lambda^m_\mp(\xi,\mu) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, with principal symbol $\lambda^{m0}_\mp(\xi,0)$ for $|\xi| \ge 1$. They satisfy the transmission condition, since $\chi(\xi_n/a\kappa(\xi',\mu))$ is $O(\langle(\xi',\mu)\rangle^N\langle(\xi,\mu)\rangle^{-N})$ for any N.

If we in (4.9) take

$$\chi \in C_0^\infty(\mathbb{R}),\ \chi(t) = \chi(|t|),\ \chi = 1 \text{ on a neighborhood of } 0, \tag{4.13}$$

the symbols $\lambda^m_\mp$ are a slight variant of those used in [G2] (and are related to those used in [BM2], [R-S1]). Here χ has a nontrivial component in both $\mathcal{H}^+_{-1} = \mathcal{F}_{x_n\to\xi_n}e^+\mathcal{S}(\overline{\mathbb{R}}_+)$ and $\mathcal{H}^-_{-1} = \mathcal{F}_{x_n\to\xi_n}e^-\mathcal{S}(\overline{\mathbb{R}}_-)$ (cf. [G1–3] for the notation), and the same holds for all the $\lambda^m_\mp$. One can instead let the χ entering in λ^m_- be a function satisfying

$$\operatorname{supp}\mathcal{F}^{-1}\chi \subset \overline{\mathbb{R}}_-; \tag{4.14}$$

then λ^m_- is entirely in $\mathcal{H}^-$ as a function of ξ_n; it extends holomorphically to $\mathbb{C}_+$, and the estimates (4.11) (with minus) extend to ξ_n and $t \in \overline{\mathbb{C}}_+$. (A choice of this kind is used in [F2].) Similarly, one can use a χ with $\operatorname{supp}\mathcal{F}^{-1}\chi \subset \overline{\mathbb{R}}_+$ in the definition of λ^m_+, to get a symbol entirely in $\mathcal{H}^+\dot{+}\mathbb{C}[\xi_n]$, or, equivalently, one can set $\lambda^m_+(\xi) = \overline{\lambda^m_-}(\xi)$, with the just explained choice of χ for λ^m_-.

Consider the associated ps.d.o.s and boundary symbol operators

$$\begin{aligned}\Lambda^m_{\pm,\mu} &= \mathrm{OP}(\lambda^m_\pm),\\ \lambda^m_\pm(\xi',\mu,D_n)_{\mathbb{R}_+} &= \mathrm{OP}_n(\lambda^m_\pm)_{\mathbb{R}_+}\ (= \lambda^m_{\pm,\mathbb{R}_+}).\end{aligned} \tag{4.15}$$

For $m > 0$ we consider moreover

$$\begin{pmatrix}\lambda^m_+(\xi',\mu,D_n)_{\mathbb{R}_+}\\ t^{(m)}\end{pmatrix},\ \text{bdr. symbol op. for}\ \begin{pmatrix}\Lambda^m_{+,\mu,\mathbb{R}^n_+}\\ T^{(m)}\end{pmatrix}, \tag{4.16}$$

where

$$t^{(m)} = \begin{pmatrix}\kappa^m\gamma_0\\ \kappa^{m-1}\gamma_1\\ \vdots\\ \kappa\gamma_{m-1}\end{pmatrix},\ \text{bdr. symbol op. for } T^{(m)} = \begin{pmatrix}\Xi'^m_\mu\gamma_0\\ \Xi'^{m-1}_\mu\gamma_1\\ \vdots\\ \Xi'^1_\mu\gamma_{m-1}\end{pmatrix}; \tag{4.17}$$

$$\Xi'^k_\mu = \mathrm{OP}'(\kappa(\xi',\mu)^k).$$

PROPOSITION 4.2. 1° *Let $m \in \mathbb{Z}$. For sufficiently large a, $\lambda^m_-(\xi',\mu,D_n)_{\mathbb{R}_+}$ defines homeomorphisms, for any $(\xi',\mu) \in \mathbb{R}^n$,*

$$\lambda^m_{-,\mathbb{R}_+} : \begin{cases}\mathcal{S}(\overline{\mathbb{R}}_+) \xrightarrow{\sim} \mathcal{S}(\overline{\mathbb{R}}_+),\\ H_2^s(\overline{\mathbb{R}}_+) \xrightarrow{\sim} H_2^{s-m}(\overline{\mathbb{R}}_+), \text{ for } s > -1/2.\end{cases} \tag{4.18}$$

Here $\lambda^m_{-,\mathbb{R}_+}$ is of class 0, and $(\lambda^m_{-,\mathbb{R}_+})^{-1}$ is an operator of the form $\lambda^{-m}_{-,\mathbb{R}_+} + g^{(-m)}$ for a singular Green operator $g^{(-m)}$, with $\lambda^{-m}_{-,\mathbb{R}_+} + g^{(-m)}$ of class $-m$.

With the special choice (4.14), the homeomorphism property holds with the original choice of a for all m and extends to all $s \in \mathbb{R}$. Then moreover,

$$\lambda^j_{-,\mathbb{R}_+}\lambda^k_{-,\mathbb{R}_+} = \lambda^{j+k}_{-,\mathbb{R}_+} \text{ for } j,k \in \mathbb{Z}, \tag{4.19}$$

the operators $\lambda^m_{-,\mathbb{R}_+}$ are of class $-\infty$, and the operators $\mathrm{OP}_n(\lambda^m_-)$ preserve support in $\overline{\mathbb{R}}_-$.

2° *For each $m \in \mathbb{N}$, a can be taken so large that $\{\lambda^m_+(\xi',\mu,D_n)_{\mathbb{R}_+}, t^{(m)}\}$ defines homeomorphisms, for any $(\xi',\mu) \in \mathbb{R}^n$,*

$$\begin{pmatrix}\lambda^m_{+,\mathbb{R}_+}\\ t^{(m)}\end{pmatrix} : \begin{cases}\mathcal{S}(\overline{\mathbb{R}}_+) \xrightarrow{\sim} \mathcal{S}(\overline{\mathbb{R}}_+)\times\mathbb{C}^m,\\ H_2^s(\overline{\mathbb{R}}_+) \xrightarrow{\sim} H_2^{s-m}(\overline{\mathbb{R}}_+)\times\mathbb{C}^m, \text{ for } s > m-1/2.\end{cases} \tag{4.20}$$

When $\overline{\lambda^m_+} = \lambda^m_-$, the latter chosen with χ satisfying (4.14), one has for all $m \in \mathbb{Z}$, with the original choice of a,

$$\lambda^m_+ : H^s_{2;0}(\overline{\mathbb{R}}_+) \xrightarrow{\sim} H^{s-m}_{2;0}(\overline{\mathbb{R}}_+), \text{ for } s \in \mathbb{R}. \tag{4.21}$$

3° *When a is chosen as in 1° resp. 2°, $\Lambda^m_{-,\mu,\mathbb{R}^n_+}$ and $\lambda^m_-(\xi',\mu,D_n)_{\mathbb{R}_+}$, resp. $\{\Lambda^m_{+,\mu,\mathbb{R}^n_+}, T^{(m)}\}$ and $\{\lambda^m_+(\xi',\mu,D_n)_{\mathbb{R}_+}, t^{(m)}\}$, are elliptic of order m in the parameter-dependent sense [G2, Def. 3.1.3], as well as in the ordinary sense for each fixed $\mu \ge 0$.*

PROOF: We begin with showing the elementary facts that for $m > 0$, $\kappa > 0$, the differential operators $\mathrm{OP}_n((\kappa \pm i\xi_n)^m) = (\kappa \pm \partial_n)^m$ enter in homeomorphisms as follows:

$$(\kappa - \partial_n)^m_{\mathbb{R}_+} : \begin{cases}\mathcal{S}(\overline{\mathbb{R}}_+) \xrightarrow{\sim} \mathcal{S}(\overline{\mathbb{R}}_+),\\ H_2^s(\overline{\mathbb{R}}_+) \xrightarrow{\sim} H_2^{s-m}(\overline{\mathbb{R}}_+), \text{ for } s \in \mathbb{R},\end{cases} \tag{4.22}$$

with inverse $(\kappa-\partial_n)^{-m}_{\mathbf{R}_+}$; and

(4.23)
$$\begin{pmatrix}(\kappa+\partial_n)^m_{\mathbf{R}_+}\\ t^{(m)}\end{pmatrix}:\begin{cases}\mathcal{S}(\overline{\mathbf{R}}_+)\xrightarrow{\sim}\mathcal{S}(\overline{\mathbf{R}}_+)\times\mathbf{C}^m,\\ H^s_2(\overline{\mathbf{R}}_+)\xrightarrow{\sim}H^{s-m}_2(\overline{\mathbf{R}}_+)\times\mathbf{C}^m, \text{ for } s>m-1/2,\end{cases}$$

with an inverse of the form $\big((\kappa+\partial_n)^{-m}_{\mathbf{R}_+}\quad k^{(m)}\big)$. Let us show this by use of the pseudo-differential boundary operator calculus (note also that the operators are variants of (2.18)). We have, referring to [BM2] or [G2, 2.2, 2.6] for the notation and composition rules,

(4.24)
$$(\kappa-\partial_n)^j_{\mathbf{R}_+}(\kappa-\partial_n)^k_{\mathbf{R}_+}=(\kappa-\partial_n)^{j+k}_{\mathbf{R}_+} \text{ for all } j,k\in\mathbf{Z},$$

since $(\kappa-i\xi_n)^j$ and $(\kappa-i\xi_n)^k$ are in $\mathcal{H}^-$ and hence $L((\kappa-i\xi_n)^j,(\kappa-i\xi_n)^k)=0$. Since $(\kappa-\partial_n)^k_{\mathbf{R}_+}$ maps $\mathcal{S}(\overline{\mathbf{R}}_+)$ into $\mathcal{S}(\overline{\mathbf{R}}_+)$, and, for any $k\in\mathbf{Z}$, maps $H^s_2(\overline{\mathbf{R}}_+)$ into $H^{s-k}_2(\overline{\mathbf{R}}_+)$ for $s>-1/2$ (for $k<0$, use a simple case of Theorem 3.11), (4.24) implies that these mappings must be bijections, extending to all $s\in\mathbf{R}$. For (4.23) we consider the boundary value problem

(4.25)
$$(\kappa+\partial_n)^m u(x_n)=f(x_n) \text{ on } \mathbf{R}_+,\quad t^{(m)}u(0)=\varphi,$$

with $f\in\mathcal{S}(\overline{\mathbf{R}}_+)$, $\varphi\in\mathbf{C}^m$. When $m=1$, the solution equals $b*(e^+f)+\kappa b\varphi$, where $b(x_n)=H(x_n)e^{-\kappa x_n}=\mathcal{F}^{-1}(\kappa+i\xi_n)^{-1}$. For general m, the solution is found by iteration of this, after transformation to an equivalent problem with boundary condition $\{\gamma_0u,\gamma_0(\kappa+\partial_n)u,\dots,\gamma_0(\kappa+\partial_n)^{m-1}u\}=\psi$. What is interesting here is that we have, denoting by $k^{(m)}$ the Poisson operator solving (4.25) with $f=0$,

(4.26)
$$\begin{pmatrix}(\kappa+\partial_n)^m_{\mathbf{R}_+}\\ t^{(m)}\end{pmatrix}^{-1}=\big((I-k^{(m)}t^{(m)})(\kappa+\partial_n)^{-m}_{\mathbf{R}_+}\quad k^{(m)}\big)=\big((\kappa+\partial_n)^{-m}_{\mathbf{R}_+}\quad k^{(m)}\big),$$

where we in the last equality used that $t^{(m)}(\kappa+\partial_n)^{-m}_{\mathbf{R}_+}=0$, since $(\kappa+i\xi_n)^{-m}\in\mathcal{H}^+\cap\mathcal{H}_{-m}$. The operator and its inverse have the asserted continuity properties in view of Theorem 3.11.

Now consider $\lambda^m_-(\xi',\mu,D_n)_{\mathbf{R}_+}$ for $m\in\mathbf{Z}$. Since $(\kappa-i\xi_n)^{-m}\in\mathcal{H}^-$, so that the s.g.o. term $L((\kappa-i\xi_n)^{-m},\lambda^m_-)$ is 0,

(4.27)
$$(\kappa-\partial_n)^{-m}_{\mathbf{R}_+}\lambda^m_{-,\mathbf{R}_+}=\mathrm{OP}_n\big((\kappa-i\xi_n)^{-m}\lambda^m_-(\xi,\mu)\big)_{\mathbf{R}_+}.$$

In view of (4.11), we can choose a so large (depending on m) that

(4.28)
$$(\kappa\pm i\xi_n)^{-m}\lambda^m_\pm(\xi,\mu)=1+q^{(m)}_\pm(a,\xi,\mu) \text{ satisfies}$$
$$|q^{(m)}_\pm(a,\xi,\mu)|\le 1/2 \text{ for all } (\xi,\mu)\in\mathbf{R}^{n+1}.$$

Here the ps.d.o. $q^{(m)}_\pm(a,\xi',\mu,D_n)$ in $L_2(\mathbf{R})$ has norm $\le 1/2$, so a fortiori $q^{(m)}_\pm(a,\xi',\mu,D_n)_{\mathbf{R}_+}$ in $L_2(\mathbf{R}_+)$ has norm $\le 1/2$. It follows that

(4.29)
$$(\kappa-\partial_n)^{-m}_{\mathbf{R}_+}\lambda^m_{-,\mathbf{R}_+}=I+q^{(m)}_{-,\mathbf{R}_+}$$

is a homeomorphism in $L_2(\mathbf{R}_+)$, and hence $\lambda^m_{-,\mathbf{R}_+}$ is a homeomorphism

(4.30)
$$\lambda^m_{-,\mathbf{R}_+}=(\kappa-\partial_n)^m_{\mathbf{R}_+}(I+q^{(m)}_{-,\mathbf{R}_+})\colon L_2(\mathbf{R}_+)\xrightarrow{\sim}H^{-m}_2(\overline{\mathbf{R}}_+),$$

in view of (4.22). By Theorem 3.11, the continuity extends to $H^s_2(\overline{\mathbf{R}}_+)$, $s>-1/2$. The inverse of $I+q^{(m)}_{-,\mathbf{R}_+}$ is a boundary symbol operator of order and class 0 belonging to the calculus, by Theorem 4.1. Then

(4.31)
$$(\lambda^m_{-,\mathbf{R}_+})^{-1}=(I+q^{(m)}_{-,\mathbf{R}_+})^{-1}(\kappa-\partial_n)^{-m}=\lambda^{-m}_{-,\mathbf{R}_+}+g^{(-m)}$$

for a suitable s.g.o. $g^{(-m)}$; here we have used the calculus and the fact that $(\lambda^m_-)^{-1}=\lambda^{-m}_-$. The operator (4.31) is *of class* $-m$, by the sharpness of Theorem 3.11, since it is defined on $H^{-m}_2(\overline{\mathbf{R}}_+)$.

Generally, it will not be possible to extend the continuity of $\lambda^m_{-,\mathbf{R}_+}$ to $s\le -1/2$, since the part of $\chi(\xi_n/a\kappa)$ in $\mathcal{H}^+$ gives rise to a nonzero Poisson term $r^+\mathrm{OP}_n(\lambda^m_-)\delta(x_n)$, cf. Definition 3.8, so that $\lambda^m_{-,\mathbf{R}_+}$ is only of class 0. But if we choose χ with the property (4.14), then $\lambda^m_-(\xi,\mu)\in\mathcal{H}^-$ as a function of ξ_n, and here one has that $r^+\mathrm{OP}_n(\lambda^m_-)u=0$ whenever u is supported in $\overline{\mathbf{R}}_-$, so in particular, $r^+\mathrm{OP}_n(\lambda^m_-)D^N_n\delta(x_n)=0$ for all $N\in\mathbf{N}$; thus $\lambda^m_{-,\mathbf{R}_+}$ is of class $-\infty$ in this case. Then the continuity in (4.18) extends to all $s\in\mathbf{R}$. We here have, with the original choice of a,

$$\lambda^1_{-,\mathbf{R}_+}\lambda^{-1}_{-,\mathbf{R}_+}=\lambda^{-1}_{-,\mathbf{R}_+}\lambda^1_{-,\mathbf{R}_+}=I,$$
$$\lambda^j_{-,\mathbf{R}_+}\lambda^k_{-,\mathbf{R}_+}=\lambda^{j+k}_{-,\mathbf{R}_+} \text{ for } j,k\in\mathbf{N},$$

so $\lambda^{-m}_{-,\mathbf{R}_+}$ exists and equals the inverse of $\lambda^m_{-,\mathbf{R}_+}$ for all $m\in\mathbf{N}$, and (4.19) follows.

Now consider λ^m_+, for $m>0$. Since $(\kappa+i\xi_n)^{-m}\in\mathcal{H}^+$, the s.g.o.-term $L(\lambda^m_+,(\kappa+i\xi_n)^{-m})$ is 0, so

(4.32)
$$\lambda^m_+(\xi',\mu,D_n)_{\mathbf{R}_+}(\kappa+\partial_n)^{-m}_{\mathbf{R}_+}=\mathrm{OP}_n(\lambda^m_+(\kappa+i\xi_n)^{-m})_{\mathbf{R}_+}=I+\mathrm{OP}_n(q^{(m)}_+(a,\xi,\mu))_{\mathbf{R}_+},$$

is invertible in $L_2(\mathbf{R}_+)$ for large enough a, cf. (4.28). Then in view of (4.26),

$$\begin{pmatrix}\lambda^m_{+,\mathbf{R}_+}\\ t^{(m)}\end{pmatrix}=\begin{pmatrix}\lambda^m_{+,\mathbf{R}_+}\\ t^{(m)}\end{pmatrix}\big((\kappa+\partial_n)^{-m}_{\mathbf{R}_+}\quad k^{(m)}\big)\begin{pmatrix}(\kappa+\partial_n)^m_{\mathbf{R}_+}\\ t^{(m)}\end{pmatrix}=\begin{pmatrix}I+q^m_{+,\mathbf{R}_+} & k_1\\ 0 & I\end{pmatrix}\begin{pmatrix}(\kappa+\partial_n)^m_{\mathbf{R}_+}\\ t^{(m)}\end{pmatrix},$$

where $k_1 = \lambda^m_{+,\mathbb{R}_+} k^{(m)}$. Since the first factor in the last expression is invertible in $L_2(\mathbb{R}_+) \times \mathbb{C}^m$, we obtain (4.20) from (4.23) for $s = m$, and the other continuity and bijectiveness properties follow then from the calculus.

When λ^m_+ ($m \in \mathbb{Z}$) is chosen such that it lies in $\mathcal{H}^+ \dot{+} \mathbb{C}[\xi_n]$ as a function of ξ_n, then

$$\mathrm{OP}_n(\lambda^m_+)\colon \mathcal{S}_0(\overline{\mathbb{R}}_+) \to \mathcal{S}_0(\overline{\mathbb{R}}_+) \quad \text{for all } m, \tag{4.33}$$

where $\mathcal{S}_0(\overline{\mathbb{R}}_+) = \{u \in \mathcal{S}(\mathbb{R}) \mid \operatorname{supp} u \subset \overline{\mathbb{R}}_+\}$; so since $\mathrm{OP}_n(\lambda^m_+)^{-1} = \mathrm{OP}_n(\lambda^{-m}_+)$, the mapping is a bijection, and (4.21) follows by closure.

The bijectiveness properties, we have shown, imply the ellipticity statements by definition (cf. [BM2], [G2, Def. 3.1.3]). ∎

When one applies OP′ to define the full operators on $\mathbb{R}^n_+$ from these boundary operators, one finds immediately, using Theorem 3.11:

THEOREM 4.3. *For each fixed μ, the following operators are homeomorphisms, when a is chosen as indicated in Proposition 4.2:*

$$\Lambda^m_{-,\mu,\mathbb{R}^n_+}\colon H^s_p(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} H^{s-m}_p(\overline{\mathbb{R}}^n_+) \text{ for } m \in \mathbb{Z},\ s > 1/p - 1; \tag{4.34}$$

$$\begin{pmatrix} \Lambda^m_{+,\mu,\mathbb{R}^n_+} \\ T^{(m)} \end{pmatrix} : H^s_p(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} \begin{matrix} H^{s-m}_p(\overline{\mathbb{R}}^n_+) \\ \times \\ B^{s-m-1/p}_p(\mathbb{R}^{n-1}) \end{matrix} \quad \text{for} \tag{4.35}$$
$$m > 0,\ s > m + 1/p - 1.$$

Moreover, if λ^m_- is chosen with (4.14) and $\lambda^m_+ = \overline{\lambda^m_-}$, then $\Lambda^m_{-,\mu}$ preserves support in $\overline{\mathbb{R}}^n_-$, (4.34) extends to $s \in \mathbb{R}$, and

$$\Lambda^j_{-,\mu,\mathbb{R}^n_+} \Lambda^k_{-,\mu,\mathbb{R}^n_+} = \Lambda^{j+k}_{-,\mu,\mathbb{R}^n_+} \text{ for } j, k \in \mathbb{Z}; \tag{4.36}$$

$$\Lambda^m_{+,\mu}\colon H^s_{p;0}(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} H^{s-m}_{p;0}(\overline{\mathbb{R}}^n_+) \text{ for } m \in \mathbb{Z},\ s \in \mathbb{R}. \tag{4.37}$$

It is a little harder to get similar properties for operators over a manifold $\overline{\Omega}$; it is here that the parameter μ will be useful, see Section 5.

In the rest of this section, we discuss the well-known parametrix construction for general (two-sided) elliptic systems, with precisions on the mapping properties. Observe that when $\mathcal{A}$ in (4.1) is of class r, then so is a^0, since an operator cannot vanish without its principal symbol doing so. (But the converse does not hold, and for negative classes, it takes a certain effort to construct an operator of a specified class with a given principal symbol of that class, because of the adjustments to the pseudo-differential part.) So, if the elliptic operator $\mathcal{A}$ is given to be of order d and class r, a^0 has these properties too. Then a^0 is continuous:

$$a^0\colon H^r_2(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M \to H^{r-d}_2(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^{M'}. \tag{4.38}$$

On the boundary symbol level, we reduce to the case $d = r = 0$ by replacing a^0 by

$$a^0_1 = \begin{pmatrix} (\lambda^{d-r}_{-,\mathbb{R}_+})^{-1} I_{(N)} & 0 \\ 0 & \kappa^{-d+r} I_{(M')} \end{pmatrix} a^0 \begin{pmatrix} \lambda^{-r}_{-,\mathbb{R}_+} I_{(N)} & 0 \\ 0 & \kappa^{-r} I_{(M)} \end{pmatrix},$$

where $I_{(N)}$ denotes the $N \times N$ identity matrix, and we take the auxiliary operators with a fixed μ. This gives a bijection

$$a^0_1\colon L_2(\mathbb{R}_+)^N \times \mathbb{C}^M \to L_2(\mathbb{R}_+)^N \times \mathbb{C}^{M'}$$

of order and class 0 (the class is 0 since the operator is defined on $L_2(\mathbb{R}_+) \times \mathbb{C}^M$, cf. Theorem 3.11). By Theorem 4.1, $(a^0_1)^{-1}$ is of order and class 0, so

$$(a^0)^{-1} = \begin{pmatrix} \lambda^{-r}_{-,\mathbb{R}_+} I_{(N)} & 0 \\ 0 & \kappa^{-r} I_{(M)} \end{pmatrix} (a^0_1)^{-1} \begin{pmatrix} (\lambda^{d-r}_{-,\mathbb{R}_+})^{-1} I_{(N)} & 0 \\ 0 & \kappa^{-d+r} I_{(M')} \end{pmatrix}$$

is of order $-d$ and class $r - d$.

Assume in the rest of this section that $r \geq 0$, and set

$$r_1 = \max\{r, d\}, \qquad r_2 = \max\{r - d, 0\} \equiv r_1 - d. \tag{4.39}$$

(We take classes ≥ 0 here in order to refer to well-known constructions, but return to general classes in Section 5.) Observe that $(a^0)^{-1}$ is a fortiori of class r_2, and its s.g.o. part is of class r_2. First, one constructs an operator $\widetilde{\mathcal{A}}_0$ with boundary symbol operator $\sim (a^0)^{-1}$ and interior symbol $\sim (p^0)^{-1}$, of order $-d$ and class r_2. It satisfies

$$\begin{aligned} \mathcal{R}'_1 &= \widetilde{\mathcal{A}}_0 \mathcal{A} - I \quad \text{is of order } -1 \text{ and class } r_1, \\ \mathcal{R}'_2 &= \mathcal{A} \widetilde{\mathcal{A}}_0 - I \quad \text{is of order } -1 \text{ and class } r_2\,; \end{aligned} \tag{4.40}$$

the class statements are seen from the mapping properties.

Next, one constructs a full left resp. right parametrix $\widetilde{\mathcal{A}}_1$ resp. $\widetilde{\mathcal{A}}_2$ for $\mathcal{A}$, asymptotic to the series

$$\widetilde{\mathcal{A}}_1 \sim \textstyle\sum_{k \in \mathbb{N}} (-\mathcal{R}'_1)^k \widetilde{\mathcal{A}}_0, \quad \text{resp.} \quad \widetilde{\mathcal{A}}_2 \sim \widetilde{\mathcal{A}}_0 \textstyle\sum_{k \in \mathbb{N}} (-\mathcal{R}'_2)^k. \tag{4.41}$$

$\widetilde{\mathcal{A}}_1$ resp. $\widetilde{\mathcal{A}}_2$ can be obtained to be of class r_1 resp. r_2, since each term is so, which follows from the continuity of

$$\begin{aligned} \widetilde{\mathcal{A}}_0&\colon H^{r_2}_p(E') \times B^{r_2 - 1/p}_p(F') \to H^{r_1}_p(E) \times B^{r_1}_p(F), \\ \mathcal{R}'_1&\colon H^{r_1}_p(E) \times B^{r_1 - 1/p}_p(F) \to H^{r_1+1}_p(E) \times B^{r_1+1-1/p}_p(F), \\ \mathcal{R}'_2&\colon H^{r_2}_p(E') \times B^{r_2 - 1/p}_p(F') \to H^{r_2+1}_p(E') \times B^{r_2+1-1/p}_p(F'). \end{aligned}$$

Then $\widetilde{\mathcal{A}}_1$ resp. $\widetilde{\mathcal{A}}_2$ satisfy

$$\mathcal{R}_1'' = \widetilde{\mathcal{A}}_1\mathcal{A} - I \quad \text{is of order } -\infty \text{ and class } r_1,$$
$$\mathcal{R}_2'' = \mathcal{A}\widetilde{\mathcal{A}}_2 - I \quad \text{is of order } -\infty \text{ and class } r_2 .$$

Each of $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ is in fact a two-sided parametrix of $\mathcal{A}$, for example:

$$\begin{aligned}\widetilde{\mathcal{A}}_2\mathcal{A} - I &= (\widetilde{\mathcal{A}}_2 - \widetilde{\mathcal{A}}_1)\mathcal{A} + \widetilde{\mathcal{A}}_1\mathcal{A} - I\\ &= ((\widetilde{\mathcal{A}}_1\mathcal{A} - \mathcal{R}_1'')\widetilde{\mathcal{A}}_2 - \widetilde{\mathcal{A}}_1)\mathcal{A} + \mathcal{R}_1''\\ &= \widetilde{\mathcal{A}}_1\mathcal{R}_2''\mathcal{A} - \mathcal{R}_1''\widetilde{\mathcal{A}}_2\mathcal{A} + \mathcal{R}_1''\end{aligned}$$

is of class r_1 and order $-\infty$. Thus we have obtained (4.8) *with* $\mathcal{R}_1$ *resp.* $\mathcal{R}_2$ *of class* r_1 *resp.* r_2.

Let us introduce the abbreviations

$$X_p^s = H_p^s(E) \times B_p^{s-1/p}(F), \quad Y_p^s = H_p^{s-d}(E') \times B_p^{s-d-1/p}(F'),$$
$$X^\infty = C^\infty(E) \times C^\infty(F), \quad Y^\infty = C^\infty(E') \times C^\infty(F'); \tag{4.42}$$

then we are considering the mappings

$$\mathcal{A}\colon X_p^s \to Y_p^s,\ \mathcal{R}_1\colon X_p^s \to \textstyle\bigcap_{t\in\mathbb{R}} X_p^t = X^\infty,$$
$$\widetilde{\mathcal{A}}\colon Y_p^s \to X_p^s,\ \mathcal{R}_2\colon Y_p^s \to \textstyle\bigcap_{t\in\mathbb{R}} Y_p^t = Y^\infty; \quad \text{for } s > r_1 + 1/p - 1. \tag{4.43}$$

In view of (2.20), the operators $\mathcal{R}_1$ and $\mathcal{R}_2$ are compact in X_p^s resp. Y_p^s. We can therefore discuss the solvability of the equation

$$\mathcal{A}\begin{pmatrix} u \\ \varphi \end{pmatrix} = \begin{pmatrix} f \\ \psi \end{pmatrix} \tag{4.44}$$

by the help of Fredholm theory.

For convenience, we recall some standard facts on Fredholm operators. Let T be a bounded operator from a Banach space X to another Y. T is called a semi-Fredholm operator if the range $R(T)$ is closed and either the kernel (nullspace) $\operatorname{Ker}(T)$ or the cokernel $\operatorname{Coker}(T) = Y/R(T)$ has finite dimension (the latter is called the codimension of $R(T)$ in Y). Generally, $\dim \operatorname{Coker}(T) < \infty$ implies $R(T)$ closed. T is called a Fredholm operator if $\dim \operatorname{Ker}(T)$ and $\dim \operatorname{Coker}(T)$ are both finite; the index is defined as $\operatorname{ind}(T) = \dim \operatorname{Ker}(T) - \dim \operatorname{Coker}(T)$. If $X = Y$ and T is of the form $T = I + S$, where S is compact, then T is a Fredholm operator with index 0. Semi-Fredholm operators occur when $T\colon X \to Y$ is such that there exists a bounded operator $\widetilde{T}\colon Y \to X$ for which $\widetilde{T}T - I$ or $T\widetilde{T} - I$ is compact. When $T\widetilde{T} - I$ is compact, it follows from $R(T) \supset R(T\widetilde{T})$ that $\dim \operatorname{Coker}(T) \le \dim \operatorname{Coker}(T\widetilde{T}) < \infty$; then $R(T)$ is closed, so it is the annihilator in Y of $\operatorname{Ker}(T^*)$;

$$R(T) = \{\, y \in Y \mid y^*(y) = 0 \text{ for } y^* \in \operatorname{Ker}(T^*) \,\}; \tag{4.45}$$

here $\dim \operatorname{Ker}(T^*) = \dim \operatorname{Coker}(T)$. When $\widetilde{T}T - I$ is compact, it follows from $\operatorname{Ker}(T) \subset \operatorname{Ker}(\widetilde{T}T)$ that $\dim \operatorname{Ker}(T) \le \dim \operatorname{Ker}(\widetilde{T}T) < \infty$; here since $(\widetilde{T}T - I)^* = T^*\widetilde{T}^* - I$ is likewise compact (in X^*), $R(T^*)$ is closed, and it follows that $R(T)$ is closed. — In order to have the Hilbert space setting as a special case, we take sesquilinear dualities.

From (4.8 i) with (4.43) follows that the mapping $\mathcal{A}\colon X_p^s \to Y_p^s$, for each s and p with $s > r_1 + 1/p - 1$, has a finite dimensional kernel $\operatorname{Ker}_{X_p^s}(\mathcal{A})$ and a closed range

$$R_{X_p^s}(\mathcal{A}) = \mathcal{A}(H_p^s(E) \times B_p^{s-1/p}(F)). \tag{4.46}$$

(4.8 i) moreover implies, in view of (2.20), that if $\{u,\varphi\} \in X_{p_1}^t$ for some $p_1 \in]1,\infty[$, $t > r_1 + 1/p_1 - 1$, and is mapped into Y_p^s by $\mathcal{A}$, then $\{u,\varphi\} \in X_p^s$ (hypoellipticity). In particular, $\operatorname{Ker}_{X_p^s}(\mathcal{A}) \subset X^\infty$, and hence is a fixed space independent of s and p, so we may denote it $\operatorname{Ker}(\mathcal{A})$.

From (4.8 ii) with (4.43) follows that $R_{X_p^s}(\mathcal{A})$ for each s and p with $s > r_1 + 1/p - 1$ has a finite dimensional complement $N_{Y_p^s}$ in Y_p^s. The space $N_{Y_p^s}$ is by no means uniquely determined, but it can be replaced by a smooth space independent of s and p. We show this now by full use of the ellipticity (as in [R-S1, 3.1.1.1 Th. 5]), but return to show a more useful characterization of $R_{X_p^s}(\mathcal{A})$ by an s,p-independent finite set of smooth linear conditions, using only (4.8 ii), in Section 5 when some more tools are available. Since C^∞ functions are dense in Y_p^s, one can find a C^∞ subspace N with the same dimension as $N_{Y_p^s}$, that is likewise a complement, i.e.,

$$Y_p^s = R_{X_p^s}(\mathcal{A}) \dot{+} N. \tag{4.47}$$

But then one also has for $R_{C^\infty}(\mathcal{A}) = \mathcal{A}(X^\infty)$,

$$Y^\infty = R_{C^\infty}(\mathcal{A}) \dot{+} N, \tag{4.48}$$

for the spaces on the right are linearly independent since $R_{C^\infty} \subset R_{X_p^s}$, and when a section $\{f,\psi\} \in Y^\infty$ is decomposed according to (4.47), the component in $R_{X_p^s}$ is C^∞, so the hypoellipticity shows that it belongs to R_{C^∞}. In this argument, s and p were arbitrary, so it follows that the codimension of $R_{X_p^s}$ is the same for all s and p (satisfying $s > r_1 + 1/p - 1$). Since N is linearly independent of R_{X^∞}, it is by the hypoellipticity linearly independent of $R_{X^t_{p_1}}$ for any t and p_1 (with $t > r_1 + 1/p_1 - 1$), so since the codimension is constant, N must be a complement of $R_{X^t_{p_1}}$ in $Y^t_{p_1}$.

By use of (2.20) one finds moreover that $\operatorname{Ker}(\mathcal{A})$ is the kernel and N is a complement of the range of $\mathcal{A}$ in the B^s_p and $B^s_{p,q}$ scales of spaces. It follows that the index of $\mathcal{A}$ is the same in all these settings, equal to $\dim \operatorname{Ker}(\mathcal{A}) - \dim N$.

Let us formulate the results in a theorem:

THEOREM 4.4. *Let $\mathcal{A}$ (4.1) be elliptic, of order $d \in \mathbb{Z}$ and class $r \in \mathbb{N}$. Then $\mathcal{A}$ has a parametrix $\widetilde{\mathcal{A}}$ (4.7) of order d and class $r_2 = \max\{r-d, 0\}$, such that (4.8) and the mapping properties in (4.42)–(4.43) hold, with $r_1 = \max\{r, d\}$. Consequently one has:*

1° *There is a finite dimensional subspace $\operatorname{Ker}(\mathcal{A})$ of $C^\infty(E) \times C^\infty(F)$, that is the kernel of $\mathcal{A}$: $X^s_p \to Y^s_p$ for any $s \in \mathbb{R}$ and $p \in]1,\infty[$ with $s > r_1 + 1/p - 1$. Moreover, if $\{f, \psi\}$ is given in Y^s_p, and $\{u, \varphi\}$ is a solution of (4.44) in $X^t_{p_1}$ for some $p_1 \in]1,\infty[$ and $t > r_1 + 1/p_1 - 1$, then $\{u, \varphi\}$ belongs to X^s_p. The range $R_{X^s_p}(\mathcal{A})$ (4.46) is closed in Y^s_p.*

2° *There is a finite dimensional linear subspace N of Y^∞, such that N is a complement of $R_{X^s_p}(\mathcal{A})$ in Y^s_p for any s and p with $s > r_1 + 1/p - 1$.*

The statements are likewise valid with H replaced by B everywhere; and they are valid with H_p and B_p replaced by $B_{p,q}$ everywhere, for any $q \in [1,\infty]$; with the same spaces $\operatorname{Ker}(\mathcal{A})$ and N. In particular, the index of $\mathcal{A}$ is independent of which set of spaces $\mathcal{A}$ is considered in.

5. Extension to the maximal scales of spaces.

In this section we show that the mapping properties of the elliptic systems and their parametrices can be extended to the full interval for the parameter s in terms of the class of $\mathcal{A}$; and we give a precise characterization of the ranges. This will be done also for one-sided elliptic systems and for multi-order systems. The basic strategy is to reduce to cases of order and class 0 where the preceding results apply. Operators of order and class 0 moreover have the advantage that *their adjoints belong to the calculus*, cf. (3.28) ff. and, for more details, e.g. [G2, 2.4]. (Operators T and G of class > 0 contain terms with γ_j (which is not assigned an adjoint within the calculus), and for P of order > 0 such terms arise in Green's formulas.) This is useful for the treatment of one-sided elliptic systems, cf. Schulze [S], [R-S1].

For the reduction to order and class 0 we need a version of the operators from Proposition 4.2 and Theorem 4.3 for vector bundles over manifolds. Here we let Theorem 4.4 and the parameter-dependent calculus of [G2] do the work for us, in the following way.

Let $\overline{\Omega}$, Γ, Ω_1, E and E_1 be as in Section 4. Fix Riemannian metrics on Ω_1 and on Γ, such that a tubular neighborhood Σ_2 of Γ in Ω_1 is isometric with $\Gamma \times]-2,2[$, and identify E_1 over Σ_2 with the lifting of $E|_\Gamma$. The coordinate in $]-2,2[$ is denoted x_n, and we write $\Sigma_c = \Gamma \times]-c,c[$ for $c \le 2$. Then we define ps.d.o.s $\Lambda^{(m)}_{-,\mu}$ with symbols equal to λ^m_- in Σ_1 and equal to $\kappa(\xi,\mu)$ in $\Omega_1 \setminus \Sigma_{\frac{3}{2}}$, by a construction described in detail in [R-S3, 3.3], modified such that the symbol $\zeta_1(\xi') + q - i\nu$ there is replaced by our $\lambda^1_-(\xi,\mu) = \kappa(\xi',\mu)\chi(\xi_n/a\kappa(\xi',\mu)) - i\xi_n$ (q and ν correspond to μ and ξ_n in our terminology). More precisely, we use on Σ_2 the symbol

$$(5.1)\qquad \begin{gathered}\left(\kappa(\xi',\mu)\chi(\xi_n/a\kappa(\xi',\mu)) - i\xi_n\right)^{m\alpha(x_n)}\kappa(\xi,\mu)^{m(1-\alpha(x_n))}I_{E_1}, \text{ with}\\ \alpha \in C_0^\infty(\mathbb{R},[0,1]),\ \alpha(t) = 1 \text{ for } |t| \le 1,\ \alpha(t) = 0 \text{ for } |t| \ge \tfrac{3}{2},\end{gathered}$$

and use $\kappa(\xi,\mu)I_{E_1}$ on $\Omega_1 \setminus \Sigma_2$; here a is chosen according to Proposition 4.2 and I_{E_1} is the identity in E_1. The fractional power $z^{m\alpha(x_n)} = e^{m\alpha(x_n)\log z}$, $z = \kappa\chi(\xi_n/a\kappa) - i\xi_n$, is defined to depend continuously on x_n, ξ_n and κ, such that it is positive for $\xi_n = 0$ (where $z = \kappa$). The logarithm is well-defined since $\kappa\chi(\xi_n/a\kappa) - i\xi_n \ne 0$ for $\xi_n \in \mathbb{R}$ in view of (4.11). When χ is chosen according to (4.14) ff., $\kappa\chi(\xi_n/a\kappa) - i\xi_n \ne 0$ for ξ_n in the simply connected set $\overline{\mathbb{C}}_+$, so $\log z$ can be chosen to depend continuously on $(\xi_n, \kappa) \in \overline{\mathbb{C}}_+ \times \mathbb{R}_+$ (as observed in [F2]). The operator $\Lambda^{(m)}_{-,\mu}$ is pieced together by use of a finite partition of unity subordinate to a covering of Ω_1 by open sets in $\Sigma_{\frac{3}{4}}$ and open sets in $\Omega_1 \setminus \Sigma_{\frac{1}{2}}$; in the former, $\Lambda^{(m)}_{-,\mu}$ acts similarly to the operator described in Theorem 4.3. The symbols are of regularity $+\infty$ in the terminology of [G2]; in particular, they remain so under coordinate changes and multiplication with smooth functions.

In view of Proposition 4.2, the operators $\Lambda^{(m)}_{-,\mu,\Omega} = r_\Omega \Lambda^{(m)}_{-,\mu} e_\Omega$ constitute a parameter-elliptic family of operators (in the sense of [G2]), of order m, and of regularity $+\infty$ in the dependence on μ. By [G2, Th. 3.2.9], it has a μ-dependent family of parametrices $\mathcal{B}_\mu$, of regularity $+\infty$ in the dependence on μ, such that moreover, *for μ large enough*, say $\mu \ge \mu_0$, $\mathcal{B}_\mu$ is the *inverse* of $\Lambda^{(m)}_{-,\mu,\Omega}$ in $C^\infty(E)$. In particular, *the kernel dimension and range codimension are zero* for $\mu \ge \mu_0$. The value μ_0 is simply chosen to make the norm of a certain μ-dependent remainder operator $< 1/2$, so the choice allows small perturbations of the operators and the manifold. Now we take a fixed $\mu_1 \ge \mu_0$, and set

$$\Lambda^{(m)}_- = \Lambda^{(m)}_{-,\mu_1}.$$

Then since $\Lambda^{(m)}_{-,\Omega}$ is elliptic, Theorem 4.4 above with $d = m$, $r = 0$ shows that it defines homeomorphisms

$$(5.2)\qquad \Lambda^{(m)}_{-,\Omega} \colon H^s_p(E) \xrightarrow{\sim} H^{s-m}_p(E),$$

for $s > r_1 + 1/p - 1$, $r_1 = \max\{m, 0\}$.

When $m > 0$, this does not allow $s = 0$, but we shall now show how to include such values, by taking the adjoint into account. Let us define $\Lambda^{(m)}_{+,\mu}$ as the ps.d.o. adjoint to $\Lambda^{(m)}_{-,\mu}$ on E_1. Let Ξ'_μ be a family of ps.d.o.s in $E|_\Gamma$ with leading symbols equal to $\kappa(\xi',\mu)I_E$; for sufficiently large μ, say $\mu \ge \mu'_0$, it is a bijection from $B^t_p(E|_\Gamma)$ to $B^{t-1}_p(E|_\Gamma)$ for all t, and we set

$$T^{(m)}_\mu = \{\Xi'^m_\mu\gamma_0, \Xi'^{m-1}_\mu\gamma_1, \dots, \Xi'^1_\mu\gamma_{m-1}\}, \text{ with } \Xi'^k_\mu = (\Xi'_\mu)^k. \tag{5.3}$$

By Proposition 4.2, the family of operators $\{\Lambda^{(m)}_{+,\mu,\Omega}, T^{(m)}_\mu\}$ is parameter-elliptic, of order m. Then, reasoning exactly as above, we have the existence of a μ-dependent family of parametrices $\mathcal{C}_\mu$ that are true inverses for $\mu \ge \mu''_0$, say, and we then set

$$\Lambda^{(m)}_+ = \Lambda^{(m)}_{+,\mu_1}, \qquad T^{(m)} = T^{(m)}_{\mu_1},$$

where we have augmented μ_1 (if necessary) so that $\mu_1 \ge \max\{\mu_0, \mu'_0, \mu''_0\}$. Thanks to Theorem 4.4, the restriction to Ω has the mapping properties, for any $p' \in]1,\infty[$:

$$\begin{pmatrix} \Lambda^{(m)}_{+,\Omega} \\ T^{(m)} \end{pmatrix} : H^s_{p'}(E) \overset{\sim}{\to} \begin{matrix} H^{s-m}_{p'}(E) \\ \times \\ B^{s-m-1/p'}_{p'}(E|_\Gamma)^m \end{matrix} \quad \text{for } s > m + 1/p' - 1. \tag{5.4}$$

But this implies by Theorem 2.3 that $\Lambda^{(m)}_{+,\Omega}$ defines homeomorphisms

$$\Lambda^{(m)}_{+,\Omega} \colon H^{t+m}_{p';0}(E) \overset{\sim}{\to} H^t_{p'}(E) \quad \text{for } 1/p' - 1 < t < 1/p', \tag{5.5}$$

and then by (2.16), (2.17), the operator restricted in this way has an adjoint that maps $H^{-t}_p(E)$ homeomorphically onto $H^{-t-m}_p(E)$ for $1/p - 1 < -t < 1/p$. Now $\Lambda^{(m)}_-$ and its adjoint in E_1 satisfy a Green's formula

$$(\Lambda^{(m)}_- u, v)_\Omega - (u, \Lambda^{(m)}_+ v)_\Omega = (\mathbf{A}\varrho^{(m)}u, \varrho^{(m)}v)_\Gamma \quad \text{for } u \in H^m_p(E), v \in H^m_{p'}(E),$$

cf. (2.14) and [G2, 1.3], where $\mathbf{A}$ is a matrix of differential operators (invertible since $\Lambda^{(m)}_-$ is elliptic). This shows that the adjoint of the operator in (5.5) coincides with $\Lambda^{(m)}_{-,\Omega}$ on $H^m_p(E)$. As a consequence, $\Lambda^{(m)}_{-,\Omega}$ extends by continuity to a continuous operator (again denoted $\Lambda^{(m)}_{-,\Omega}$), satisfying (5.2) for $1/p-1 < s < 1/p$. Finally, the continuity extends by interpolation to all $s > 1/p - 1$. (Adjoint arguments like this are not given in [R-S1, 3.1.2].)

Note that since $(\Lambda^{(m)}_{-,\Omega})^{-1}$ is well-defined on $H^{-m}_p(E)$, it is of class $-m$, according to Theorem 3.11. It is of the form $P^{(-m)}_\Omega + G^{(-m)}$, where the principal symbol of $P^{(-m)}$ is like that of Λ^{-m}_- near Γ, cf. (5.1), and $G^{(-m)}$ has principal symbol $g^{(-m)}$ as in Proposition 4.2.

To indicate that the operators $\Lambda^{(m)}_{\pm,\Omega}$ are constructed specially for E, we sometimes write $\Lambda^{(m)}_{\pm,E}$ instead of $\Lambda^{(m)}_{\pm,\Omega}$.

The mapping properties (5.2) will not in general extend to $s \le 1/p - 1$, since $\Lambda^{(m)}_{-,\Omega}$ is generally not of negative class, as observed after (4.31). It is worth noting that this is a precise observation, valid also for the auxiliary operators used in [BM2], [G2], [R-S1] (compare with [R-S1, 2.3.2.4 Prop. 2]). But since the $(\Lambda^{(m)}_{-,\Omega})^{-1}$ are of class $-m$, these operators are sufficient for our main purposes, such as the proof of Theorem 5.4 and its corollary.

However, we shall for completeness also show what one can obtain using symbols with the special property (4.14). This gives operators having the same features as those in [F2]; and the present construction has the advantage of being stable under small perturbations. The operators are convenient for special purposes, and sometimes allow simpler formulations.

Assume that Ω_1 is a compact manifold (e.g. the "double" of $\overline{\Omega}$, and denote $\Omega = \Omega_+$, $\Omega_1 \setminus \overline{\Omega} = \Omega_-$; $r_{\Omega_\pm} = r^\pm$ and $e_{\Omega_\pm} = e^\pm$. For simplicity of notation, let $E_1 = \Omega_1 \times \mathbf{C}$. Let

$$\Xi_- = \Lambda^{(1)}_-, \qquad \Xi_+ = [\Lambda^{(1)}_-]^*. \tag{5.6}$$

When χ satisfies (4.14), Ξ_{-,Ω_+} is of class $-\infty$ (cf. Proposition 4.2), and Ξ_- has the property that it maps distributions supported in $\overline{\Omega}_-$ into distributions supported in $\overline{\Omega}_-$ (it is a "minus-operator" in the terminology of Eskin [E]). Ξ_+ has analogous properties with Ω_+ and Ω_- interchanged. (Operators based on $\Lambda^{(m)}_-$ for other m can be studied in a similar way.)

Since Ξ_- and its powers Ξ^j_- $(j \in \mathbf{N})$ are minus-operators, one has for $u \in C^\infty(\overline{\Omega}_+)$:

$$\begin{aligned} \Xi^j_{-,\Omega_+}\Xi^k_{-,\Omega_+}u &= r^+\Xi^j_- e^+ r^+ \Xi^k_- e^+ u = r^+\Xi^j_-\Xi^k_- e^+ u - r^+\Xi^j_- e^- r^- \Xi^k_- e^+ u \\ &= \Xi^{j+k}_{-,\Omega_+}u \quad \text{for all } j,k \in \mathbf{N}, \end{aligned} \tag{5.7}$$

without a singular Green term; and this identity extends by continuity to $u \in H^s_p(\overline{\mathbf{R}}^n_+)$, $s \in \mathbf{R}$, since the operators are of class $-\infty$.

Assume as we may (in view of the parameter-dependent calculus) that μ_1 has been taken so large that Ξ_- and Ξ_+ are invertible on Ω_1. By the preceding considerations, we can also assume that μ_1 is so large that we have *homeomorphisms*

$$(5.8)\qquad \begin{aligned} &\text{(i)}\quad \Xi_{-,\Omega_+}: H_p^s(\overline{\Omega}_+) \overset{\sim}{\to} H_p^{s-1}(\overline{\Omega}_+),\ s > 1/p - 1;\\ &\text{(ii)}\quad \Xi_{-,\Omega_-}: H_{p;0}^s(\overline{\Omega}_-) \overset{\sim}{\to} H_{p;0}^{s-1}(\overline{\Omega}_-),\ s = 1;\end{aligned}$$

where (5.8 ii) is a version of (5.5), with Ξ_- and Ω_- playing the role of $\Lambda_+^{(m)}$ and Ω there; here $H_{p;0}^t(\overline{\Omega}_-)$ ($t = 1$ or 0) is considered as a space of functions on Ω_-. Since Ξ_- is a minus-operator of order 1, $\Xi_{-,\Omega_-}u$ identifies with $\Xi_- e^- u$ when $u \in H_{p;0}^1(\overline{\Omega}_-)$; and (5.8 ii) shows that Ξ_- maps $e^- H_{p;0}^1(\overline{\Omega}_-)$ bijectively onto $e^- L_p(\Omega_-)$. Then the inverse Ξ_-^{-1} maps $e^- L_p(\Omega_-)$ bijectively onto $e^- H_{p;0}^1(\overline{\Omega}_-)$, and it follows by extension by continuity that Ξ_-^{-1} *is also a minus-operator.* Now Ξ_-^{-1} has similar properties as Ξ_-; in particular, (5.7) extends to negative powers, and (5.8 i) holds for Ξ_-^{-1} with $s+1$ in stead of $s-1$. Moreover,

$$\Xi_{-,\Omega_+}^{-1}\Xi_{-,\Omega_+} = I \text{ on } H^s(\overline{\Omega}_+)$$

for $s = 1$ (because they are minus-operators), so Ξ_{-,Ω_+}^{-1} is the inverse of Ξ_{-,Ω_+} there; also this identity extends by continuity to all $s \in \mathbb{R}$. Thus (5.8 i) is valid for all s. (Also (5.8 ii) extends.)

Generalizing these considerations to vector bundles and defining

$$(5.9)\qquad \Xi_-^m = (\Lambda_-^{(1)})^m,\quad \Xi_+^m = (\Lambda_-^{(1)*})^m,\quad m \in \mathbf{Z},$$

we obtain a family of ps.d.o.s in E_1 such that the restrictions to Ω satisfy

$$(5.10)\qquad \begin{aligned} &\Xi_{-,E}^m: H_p^s(E) \overset{\sim}{\to} H_p^{s-m}(E), &&\text{for all } m \in \mathbf{Z},\ s \in \mathbb{R};\\ &\Xi_+^m: H_{p;0}^s(E) \overset{\sim}{\to} H_{p;0}^{s-m}(E), &&\text{for all } m \in \mathbf{Z},\ s \in \mathbb{R};\\ &\Xi_{-,E}^j\Xi_{-,E}^k = \Xi_{-,E}^{j+k} &&\text{for all } j,k \in \mathbf{Z}.\end{aligned}$$

Altogether, we have shown:

THEOREM 5.1. *Let $\overline{\Omega}$ and E be as in Section 4.*

1° *For each $m \in \mathbf{Z}$ there is a ps.d.o. $\Lambda_-^{(m)}$ defined in an extending bundle E_1 such that its restriction $\Lambda_{-,\Omega}^{(m)}$, also denoted $\Lambda_{-,E}^{(m)}$, is elliptic of order m and class 0, and defines homeomorphisms*

$$(5.11)\qquad \Lambda_{-,E}^{(m)}: H_p^s(E) \overset{\sim}{\to} H_p^{s-m}(E),\quad \textit{for all } s > 1/p - 1.$$

The inverse $(\Lambda_{-,E}^{(m)})^{-1}$ is of the form $P_\Omega^{(-m)} + G^{(-m)}$, of order $-m$ and class $-m$.

2° *For each $m \in \mathbf{N}$ there is a ps.d.o. $\Lambda_+^{(m)}$ defined in an extending bundle E_1 (it can be taken equal to $(\Lambda_-^{(m)})^*$) such that its restriction $\Lambda_{+,\Omega}^{(m)}$, also denoted $\Lambda_{+,E}^{(m)}$, defines an elliptic system $\{\Lambda_{+,E}^{(m)}, \varrho^{(m)}\}$ with the homeomorphism properties*

$$(5.12)\qquad \begin{pmatrix}\Lambda_{+,E}^{(m)}\\ \varrho^{(m)}\end{pmatrix}: H_p^s(E) \overset{\sim}{\to} \begin{matrix} H_p^{s-m}(E)\\ \times\\ \Pi_{0\le j\le m-1} B_p^{s-j-1/p}(E|_\Gamma)\end{matrix}\qquad \textit{for } s > m+1/p-1.$$

3° *By a particular choice (with χ satisfying (4.14)), one can obtain that (5.11) extends to all $s \in \mathbb{R}$, and that the operators Ξ_-^m and Ξ_+^m defined by (5.9) satisfy (5.10); here the operators Ξ_-^m preserve support in $\Omega_1 \setminus \Omega$, and the operators $\Xi_{-,E}^m$ are of class $-\infty$.*

Analogous statements hold in B_p and $B_{p,q}$ spaces ($q \in [1,\infty]$).

In (5.12), we have removed the invertible coefficients $\Xi_{\mu_1}^{\prime k}$ from $T^{(m)}$, cf. (2.14). The mapping properties are carried over to the other scales of spaces by interpolation.

Let us introduce the short notation $\Phi_{m,E,F}$ for the system defined for $m \in \mathbf{Z}$ (cf. (4.5), (5.3)):

$$(5.13)\qquad \Phi_{m,E,F} = \begin{pmatrix}\Lambda_{-,E}^{(m)} & 0\\ 0 & \Xi_F^{\prime m}\end{pmatrix}: \begin{matrix} H_p^s(E)\\ \times\\ B_p^{s-1/p}(F)\end{matrix} \overset{\sim}{\to} \begin{matrix} H_p^{s-m}(E)\\ \times\\ B_p^{s-m-1/p}(F)\end{matrix}\qquad \text{for } s > 1/p - 1.$$

(Here one can use $\Xi_{-,E}^m$ from Theorem 5.1 3° instead of $\Lambda_{-,E}^{(m)}$; then the homeomorphism property extends to all $s \in \mathbb{R}$ and $(\Phi_{m,E,F})^{-1} = \Phi_{-m,E,F}$.)

With the help of these operators, we can now set up a solvability theory in spaces of lower regularity than Theorem 4.4 allows, get a sharp description of the classes of the operators involved, and get a precise characterization of the ranges. At the same time we include one-sided elliptic systems, and later, as a corollary, Douglis-Nirenberg elliptic systems.

Let $\mathcal{A}$ (4.1) be *injectively elliptic* or *surjectively elliptic*, of order $d \in \mathbf{Z}$ and class $r \in \mathbf{Z}$; it is continuous:

$$(5.14)\qquad \mathcal{A}: X_p^s \to Y_p^s\quad \text{for } s > r + 1/p - 1,$$

by Theorem 3.11 (cf. (4.42)). Denote

$$V_p^s = H_p^s(E') \times B_p^{s-1/p}(F') \; (= Y_p^{s+d}), \tag{5.15}$$

and consider the composed operator $\mathcal{A}_1$ defined by

$$\mathcal{A}_1 = (\Phi_{d-r,E',F'})^{-1}\, \mathcal{A}\, \Phi_{-r,E,F} \colon X_p^s \to V_p^s, \quad \text{for } s > 1/p - 1; \tag{5.16}$$

it belongs to the calculus and is likewise injectively resp. surjectively elliptic. It is of order 0, and the class is 0 in view of Theorem 3.11. Since $\mathcal{A}_1$ is of order and class 0, its adjoint $\mathcal{A}_1^*$, defined as a continuous operator from $V_{p'}^{-s}$ to $X_{p'}^{-s}$ for $-1/p < -s < -1/p + 1 = 1/p'$ (cf. (2.16), (2.17)), belongs to the calculus, and the continuity extends to higher values of $-s$;

$$\mathcal{A}_1^* \colon V_{p'}^t \to X_{p'}^t, \quad \text{for } t > 1/p' - 1, \tag{5.17}$$

moreover, $\mathcal{A}_1^*$ is surjectively resp. injectively elliptic.

Consider the case where $\mathcal{A}_1$ is *injectively elliptic*. Here $\mathcal{A}_1^*\mathcal{A}_1$ is *elliptic* (of order and class 0) and hence, by Theorem 4.1, has a parametrix $(\mathcal{A}_1^*\mathcal{A}_1)^\sim$ of order and class 0, such that

$$\mathcal{R} = (\mathcal{A}_1^*\mathcal{A}_1)^\sim \mathcal{A}_1^*\mathcal{A}_1 - I \quad \text{is of order } -\infty \text{ and class } 0; \tag{5.18}$$

in this way, $\widetilde{\mathcal{A}}_1 = (\mathcal{A}_1^*\mathcal{A}_1)^\sim \mathcal{A}_1^*$ is a left parametrix for $\mathcal{A}_1$. Then the conclusions based on (4.8 i) in Section 4 extend, showing that $\mathcal{A}_1 \colon X_p^s \to V_p^s$ for each s and p with $s > 1/p - 1$ has the same finite dimensional kernel $\operatorname{Ker}(\mathcal{A}_1) \subset X^\infty$, and has closed range in V_p^s, and that $\mathcal{A}_1$ is hypoelliptic: $\{u, \varphi\} \in X_{p_1}^t$ for some $p_1 \in]1, \infty[$ and $t > 1/p_1 - 1$, with $\mathcal{A}_1\{u, \varphi\} \in V_p^s$, imply $\{u, \varphi\} \in X_p^s$. All this carries back to

$$\mathcal{A} = \Phi_{d-r,E',F'}\, \mathcal{A}_1\, (\Phi_{-r,E,F})^{-1}, \tag{5.19}$$

showing that it has the kernel $\operatorname{Ker}(\mathcal{A}) = \Phi_{-r,E,F} \operatorname{Ker}(\mathcal{A}_1)$ and the left parametrix

$$\widetilde{\mathcal{A}} = \Phi_{-r,E,F}\, \widetilde{\mathcal{A}}_1\, \Phi_{d-r,E',F'}^{-1} \colon Y_p^s \to X_p^s \text{ for } s > r + 1/p - 1. \tag{5.20}$$

In view of Theorem 3.11, it is of class $r - d$. (Note the improvement over (4.43), when $r < 0$ or $r < d$.) The solvability properties of $\mathcal{A}_1$ carry over to solvability properties of $\mathcal{A}$ by use of the mapping properties in (5.13); then we get statements as in Theorem 4.4, but in some cases on larger spaces; we formulate this in Theorem 5.4 below.

Now consider the case where $\mathcal{A}_1$ is *surjectively elliptic*. Here $\mathcal{A}_1\mathcal{A}_1^*$ is elliptic (of order and class 0) and has a parametrix $(\mathcal{A}_1\mathcal{A}_1^*)^\sim$ of order and class 0 such that

$$\mathcal{R} = \mathcal{A}_1\mathcal{A}_1^*(\mathcal{A}_1\mathcal{A}_1^*)^\sim - I \quad \text{is of order } -\infty \text{ and class } 0; \tag{5.21}$$

in this way, $\widetilde{\mathcal{A}}_1 = \mathcal{A}_1^*(\mathcal{A}_1\mathcal{A}_1^*)^\sim$ is a right parametrix for $\mathcal{A}_1$. From this we can conclude as in the considerations in Section 4 based on (4.8 ii), that the range $R_{X_p^s}(\mathcal{A}_1) = \mathcal{A}_1(X_p^s)$ has a finite dimensional complement in V_p^s and hence is closed. But now we do not in general have any hypoellipticity to play on, and we shall not pursue the study of complements of $R_{X_p^s}(\mathcal{A}_1)$. In fact, we can do much better, namely give a precise characterization of $R_{X_p^s}(\mathcal{A}_1)$ by use of the adjoint. Since $\mathcal{A}_1^*$ (5.17) is injectively elliptic, it has a kernel $\operatorname{Ker}(\mathcal{A}_1^*) \subset Y^\infty$, that is the same for all $p' \in]1, \infty[$, $t > 1/p' - 1$. In particular, for $1/p - 1 < s < 1/p$, $R_{X_p^s}(\mathcal{A}_1)$ is the annihilator of $\operatorname{Ker}(\mathcal{A}_1^*)$, cf. (4.45):

$$R_{X_p^s}(\mathcal{A}_1) = \{\, \{f, \psi\} \in V_p^s \mid \langle f, \bar{g}\rangle_\Omega + \langle \psi, \bar{\varrho}\rangle_\Gamma = 0 \text{ for } \{g, \varrho\} \in \operatorname{Ker}(\mathcal{A}_1^*) \,\}. \tag{5.22}$$

In this formula, $\langle f, \bar{g}\rangle_\Omega$ and $\langle \psi, \bar{\varrho}\rangle_\Gamma$ are integrals over Ω resp. Γ when f and ψ are functions; otherwise we use that for $1/p - 1 < s < 1/p$, $H_p^s(E') \simeq H_{p;0}^s(E') \simeq H_{p'}^{-s}(E')^*$ and $H_p^{s-1/p}(F') \simeq H_{p'}^{-s+1/p}(F')^*$ define functionals on C^∞ functions (cf. (2.16), (2.17)).

In order to extend (5.22) to $s \geq 1/p$, we make two observations. One is that $R_{X_p^{s'}}(\mathcal{A}_1) \subset R_{X_p^s}(\mathcal{A}_1)$ for $s' > s$, so the condition in (5.22) is necessary for belonging to $R_{X_p^s}(\mathcal{A}_1)$ *for any* $s > 1/p - 1$, hence the codimension ν_s of $R_{X_p^s}(\mathcal{A}_1)$ in V_p^s satisfies $\nu_s \geq \varrho_0 = \dim \operatorname{Ker}(\mathcal{A}_1^*)$ for all $s \geq 1/p$. The other is that $R_{X_p^s}(\mathcal{A}_1)$ contains $R_{X_p^s}(\mathcal{A}_1\mathcal{A}_1^*)$, whose codimension is the fixed number $\varrho_1 = \dim \operatorname{Ker}(\mathcal{A}_1\mathcal{A}_1^*)$ for all $s > 1/p - 1$, by Theorem 4.4; so $\nu_s \leq \varrho_1$. Altogether, $\varrho_0 \leq \nu_s \leq \varrho_1$ for all s. But

$$\operatorname{Ker}(\mathcal{A}_1\mathcal{A}_1^*) = \operatorname{Ker}(\mathcal{A}_1^*),$$

since $\mathcal{A}_1\mathcal{A}_1^*\{u, \varphi\} = 0$ implies $\|\mathcal{A}_1^*\{u, \varphi\}\|^2_{L_2(E) \times L_2(F)} = 0$, so $\varrho_1 = \varrho_0$, and therefore $\nu_s = \varrho_0$ for all s. Then (5.22) describes $R_{X_p^s}(\mathcal{A}_1)$ for all $s > 1/p - 1$.

The operator $\mathcal{A}$ (5.19) now has the right parametrix $\widetilde{\mathcal{A}}$ (5.20), of class $r - d$. Here

$$R_{X_p^s}(\mathcal{A}) = \Phi_{d-r,E',F'} R_{X_p^{s-r}}(\mathcal{A}_1) \quad \text{for } s > r + 1/p - 1, \tag{5.23}$$

so $\{f,\psi\} \in Y_p^s$ lies in $R_{X_p^s}(\mathcal{A})$ if and only if

$$((\Lambda_{-,E'}^{(d-r)})^{-1} f, \bar g)_\Omega + (\Xi_{F'}^{\prime r-d}\psi, \bar\zeta)_\Gamma = 0 \text{ for } \{g,\zeta\} \in \operatorname{Ker}(\mathcal{A}_1^*), \tag{5.24}$$

where $(\Lambda_{-,E'}^{(d-r)})^{-1} = P_\Omega^{(r-d)} + G^{(r-d)}$, cf. Theorem 5.1. (5.24) can be further explicited, in a simpler way when $r \le d$ than when $r > d$ (recall that $(\Lambda_{-,E'}^{(d-r)})^{-1}$ is of order and class $r-d$). When $r \le d$, $(\Lambda_{-,E'}^{(d-r)})^{-1}$ has an adjoint *within the calculus:*

$$((\Lambda_{-,E'}^{(d-r)})^{-1})^* = (P^{(r-d)*})_\Omega + G^{(r-d)*},$$

of order and class ≤ 0, so (5.24) can be written

$$\begin{aligned} &(f,\bar g)_\Omega + (\psi,\bar\zeta)_\Gamma = 0 \text{ for } \{g,\zeta\} \in \mathcal{N}(\mathcal{A}), \text{ where} \\ &\mathcal{N}(\mathcal{A}) = (\Phi_{d-r,E',F'}^{-1})^* \operatorname{Ker}(\mathcal{A}_1^*) \subset C^\infty(E') \times C^\infty(F'). \end{aligned} \tag{5.25}$$

When $r > d$, there is a Green's formula (cf. [G2, 1.3 and 1.6])

$$((\Lambda_{-,E'}^{(d-r)})^{-1} f, \bar g)_\Omega = (f, \overline{M^{(r-d)}g})_\Omega + (\varrho^{(r-d)} f, \overline{B^{(r-d)}g})_\Gamma, \tag{5.26}$$

where $M^{(r-d)} = (P^{(r-d)*})_\Omega + G_1^{(r-d)}$, with a suitable s.g.o. $G_1^{(r-d)}$ and trace operator $B^{(r-d)}$, such that (5.24) can be written

$$\begin{aligned} &(f,\bar g)_\Omega + (\varrho^{(r-d)} f, \bar\eta)_\Gamma + (\psi,\bar\zeta)_\Gamma = 0 \text{ for } \{g,\eta,\zeta\} \in \mathcal{N}(\mathcal{A}), \\ &\mathcal{N}(\mathcal{A}) = \{M^{(r-d)}, B^{(r-d)}, (\Xi_{F'}^{\prime r-d})^*\} \operatorname{Ker}(\mathcal{A}_1^*) \\ &\qquad \subset C^\infty(E') \times C^\infty(E'|_\Gamma)^{r-d} \times C^\infty(F'). \end{aligned} \tag{5.27}$$

The use of $\Xi_{-,E'}^{r-d}$ here would spare some s.g.o. terms but not the presence of trace operators.

REMARK 5.3: The case where $r \le d$ includes for example all the usual elliptic boundary problems for differential operators, where $\mathcal{A}$ is of the form (4.4) with P of order $d > 0$ and T of order d and class $\le d$.

To *find* the space $\mathcal{N}(\mathcal{A})$ when $r \le d$, one can actually use a slightly simpler formula than (5.25), namely the one obtained by regarding $\mathcal{A}$ as an operator of order *and* class d:

$$\mathcal{N}(\mathcal{A}) = \operatorname{Ker}((\mathcal{A}\Phi_{-d,E,F})^*). \tag{5.28}$$

To consider $\mathcal{A}$ as an operator of class d instead of r here really just means that we in (5.14) only let $\mathcal{A}$ map into spaces $H_p^t(E') \times B_p^{t-1/p}(F')$ with $t > 1/p - 1$, not lower negative values. The annihilator of the range space, once it is known to be a C^∞ space independent of t, is of course the same

as before. But what is interesting about letting r take a lower value than d is, that the formula (5.25) then shows that

$$\mathcal{N}(\mathcal{A}) \subset [H_{p;0}^{d-r}(E') \cap C^\infty(E')] \times C^\infty(F'), \tag{5.29}$$

since $((\Lambda_{-,E'}^{(d-r)})^{-1})^*$ maps $H_p^0(E')$ into $H_{p;0}^{d-r}(E')$ by (5.11) for $s = 0$, $m = d - r$; this explains why the condition (5.25) makes sense for f in, say, $H_p^{r-d}(E')$: the g have some zero boundary values.

Let us formulate the results in a theorem:

THEOREM 5.4. *Let the operators $\mathcal{A}$, P, G, T, K and S be given as in Section 4, of order $d \in \mathbb{Z}$ and class $r \in \mathbb{Z}$. Consider the pseudo-differential boundary value problem*

$$\begin{aligned} (P_\Omega + G)u + K\varphi &= f \quad \text{on } \bar\Omega \\ Tu + S\varphi &= \psi \quad \text{on } \Gamma, \end{aligned} \tag{5.30}$$

where f and ψ are given with $f \in H_p^{s-d}(E')$ and $\psi \in B_p^{s-d-1/p}(F')$ for some $p \in]1,\infty[$, $s > r + 1/p - 1$.

1° If $\mathcal{A}$ is injectively elliptic, then $\mathcal{A}$ has a left parametrix $\tilde{\mathcal{A}}$ of order $-d$ and class $r - d$, such that $\tilde{\mathcal{A}}\mathcal{A} - I$ is of order $-\infty$ and class r. Any solution of (5.30) in $H_{p_1}^t(E) \times B_{p_1}^{t-1/p_1}(F)$, some $p_1 \in]1,\infty[$ and $t > r + 1/p - 1$, lies in $H_p^s(E) \times B_p^{s-1/p}(F)$; it is uniquely determined modulo a certain finite dimensional space $\operatorname{Ker}(\mathcal{A}) \subset C^\infty(E) \times C^\infty(F)$, independent of s and p. Moreover, the range $R_{X_p^s}(\mathcal{A})$ (4.46) is closed.

2° If $\mathcal{A}$ is surjectively elliptic, then $\mathcal{A}$ has a right parametrix $\tilde{\mathcal{A}}$ of order $-d$ and class $r-d$, such that $\mathcal{A}\tilde{\mathcal{A}} - I$ is of order $-\infty$ and class $r-d$. For each s and p with $s > r + 1/p - 1$, the problem (5.30) is solvable in $H_p^s(E) \times B_p^{s-1/p}(F)$ when $\{f,\psi\}$ satisfies a finite set of smooth linear conditions, described by (5.25) (cf. also (5.28)–(5.29)) in the case $r \le d$, and by (5.27) (cf. also (5.26) ff.) in the case $r > d$; here $\mathcal{N}(\mathcal{A})$ is independent of s and p.

In the elliptic case, both 1° and 2° hold, the parametrices are two-sided, and $\mathcal{A}$ is a Fredholm operator from $H_p^s(E) \times B_p^{s-1/p}(F)$ to $H_p^{s-d}(E') \times B_p^{s-d-1/p}(F')$ for all $s > r + 1/p - 1$.

The statements are likewise valid with H replaced by B everywhere; and they are valid with H_p and B_p replaced by $B_{p,q}$ everywhere, for any $q \in [1,\infty]$; with the same spaces $\operatorname{Ker}(\mathcal{A})$ and $\mathcal{N}(\mathcal{A})$.

The class result for $\tilde{\mathcal{A}}$ is best possible, in the sense that $\tilde{\mathcal{A}}$ can only be of a lower class r' than $r - d$ if $\mathcal{A}$ is of lower class than r. For, since $\tilde{\mathcal{A}}$ is elliptic of order $-d$, $\widetilde{\mathcal{A}}$ of class r' implies $\mathcal{A}$ of class $r'+d$, by Theorem 5.4. Thus, when r indicates the lowest class of $\mathcal{A}$, the interval $s \in]r+1/p-1, \infty[$, where the mapping properties hold for a given p, is maximal.

Note that when $r > d$, the theorem allows trace operators of higher "normal order" $r-1$ (class r) than in the most common requirement $r \le \max\{d,0\}$; and on the other hand that the space $H_p^{r-d}(E') \times B_p^{r-d-1/p}(F')$ is included as a range space even when $r < d$, or r is negative.

We remark that in order to *find* the spaces $\operatorname{Ker}(\mathcal{A})$ and $\mathcal{N}(\mathcal{A})$, or their dimensions, one need only apply the L_2 theory; in particular this holds for the index of $\mathcal{A}$ in the elliptic case.

We can finally easily include multi-order (Douglis-Nirenberg) systems.

COROLLARY 5.5. *Let $d \in \mathbb{Z}$ and $r \in \mathbb{Z}$. Let there be given index sets $1 \le i \le i_0$, $1 \le j \le j_0$, $1 \le k \le k_0$ and $1 \le l \le l_0$, to which there are associated integers $(s_i)_{i\le i_0}$, $(t_j)_{j\le j_0}$, $(\sigma_k)_{k\le k_0}$, and $(\tau_l)_{l\le l_0}$. Let $\mathcal{A}$ be a multi-order system going from $E\oplus F$ to $E'\oplus F'$, where $E=\oplus E_j$, $E'=\oplus E_i'$, $F=\oplus F_l$ and $F'=\oplus F_k'$; and $P=(P_{ij})$, $G=(G_{ij})$, $T=(T_{kj})$, $K=(K_{il})$ and $S=(S_{jl})$, such that the P_{ij} and G_{ij} are of orders $d+s_i+t_j$, the T_{kj} are of orders $d+\sigma_k+t_j$, the K_{il} are of orders $d+s_i+\tau_l$, and the S_{kl} are of orders $d+\sigma_k+\tau_l$. Assume moreover that the $P_{ij,\Omega}+G_{ij}$ and T_{kj} are of class t_j+r, resp. τ_l+r. In particular, $\mathcal{A}$ is continuous:*

(5.31)
$$\mathcal{A}:\ \begin{array}{c}\Pi_{j\le j_0} H_p^{s+t_j}(E_j)\\ \times\\ \Pi_{l\le l_0} B_p^{s+\tau_l-1/p}(F_l)\end{array} \to \begin{array}{c}\Pi_{i\le i_0} H_p^{s-d-s_i}(E_i')\\ \times\\ \Pi_{k\le k_0} B_p^{s-d-\sigma_k-1/p}(F_k')\end{array} \quad \text{for } s > r+1/p-1.$$

If $\mathcal{A}$ is injectively elliptic, surjectively elliptic, resp. elliptic, in the Douglis-Nirenberg sense (i.e., Conditions (I) and (II) in Section 4 are satisfied by the systems of principal parts defined according to the given orders), then $\mathcal{A}$ has a left, right or two-sided parametrix $\widetilde{\mathcal{A}}$, belonging to the calculus and continuous in the opposite direction of (5.31) for all $s > r+1/p-1$.

In particular, $\mathcal{A}$ is a semi-Fredholm resp. Fredholm operator in the respective cases; with a fixed finite dimensional C^∞ kernel, closed range and hypoellipticity in the injectively elliptic case; and with closed range defined from the annihilator of a fixed finite dimensional C^∞ space in the surjectively elliptic case (as described in (5.33) ff. below).

PROOF: One reduces the statements to corresponding statements for a case of order and class 0, by replacing $\mathcal{A}$ by $\mathcal{A}_1 = \Phi^{-1}\mathcal{A}\Psi$, where Φ and Ψ are the diagonal matrices

(5.32)
$$\Phi = \begin{pmatrix} \left(\delta_{ii'}\Lambda_{-,E_i'}^{(d-r+s_i)}\right)_{i,i'\le i_0} & 0\\ 0 & \left(\delta_{kk'}\Xi_{F_k'}^{\prime d-r+\sigma_k}\right)_{k,k'\le k_0}\end{pmatrix},$$
$$\Psi = \begin{pmatrix} \left(\delta_{jj'}\Lambda_{-,E_j}^{(-r-t_j)}\right)_{j,j'\le j_0} & 0\\ 0 & \left(\delta_{ll'}\Xi_{F_l}^{\prime -r-\tau_l}\right)_{l,l'\le l_0}\end{pmatrix}.$$

In particular, the range of $\mathcal{A}$ is in the surjectively elliptic case characterized by the condition

(5.33)
$$\langle \Phi^{-1}\{f,\varphi\}, \{\bar g, \bar\zeta\}\rangle = 0 \quad \text{for } \{g,\zeta\} \in \operatorname{Ker}(\mathcal{A}_1^*),$$

where $\operatorname{Ker}\mathcal{A}_1^* \subset \Pi_{i\le i_0}C^\infty(E_i') \times \Pi_{k\le k_0}C^\infty(F_k')$. This can be reduced to

$$\langle f,\bar g\rangle_\Omega + \langle \psi,\bar\zeta\rangle_\Gamma = 0 \text{ for } \{g,\zeta\} \in (\Phi^{-1})^*\operatorname{Ker}(\mathcal{A}_1^*),$$

if all $d-r+s_i$ are ≥ 0, and more generally to

$$\langle f,\bar g\rangle_\Omega + \textstyle\sum_{r-d-s_i>0}\langle \varrho^{(r-d-s_i)}f_i, \bar\eta_i\rangle_\Gamma + \langle\psi,\bar\zeta\rangle_\Gamma = 0 \text{ for } \{g,\eta,\zeta\} \in \mathcal{N}(\mathcal{A}),$$

for a certain finite dimensional C^∞ space $\mathcal{N}(\mathcal{A})$. ∎

For elliptic problems, the theorem and its corollary extend the known results on parametrices in L_2-type Sobolev spaces, as outlined in [BM2] and proved in detail in [R-S1] and in [G2]. Besides giving an extension to L_p-type spaces, it adds something to the L_2-type results: in [R-S1], the mapping properties are stated generally for unspecified "sufficiently large" s, and in [G2] they are stated for single-order systems, for $s \ge \max\{d,0\}$ under the assumption that $r \le \max\{d,0\}$. In particular, the inclusion of some initial spaces and range spaces with negative exponents should be of interest, possibly also for differential operator problems, where *normal* boundary conditions are treated in Lions and Magenes [L-M, Th. 2.8.2]. (In comparison, the solvability results of Roitberg for not necessarily normal problems, cf. e.g. [R1,2], in a *full scale* of spaces with $s \in \mathbb{R}$, are concerned with range spaces $H_{p;0}^{s-d}(E') \times B_p^{s-d-1/p}(F')$ when $s \in]-\infty, d[$; our parametrices easily extend to these spaces when of class 0.)

The exact characterization of the range in terms of a C^∞ annihilation, for all r and d, has not, to our knowledge, been worked out before for ps.d.o. problems, except in special cases. This is connected with the analysis of the order reducing operators in Theorem 5.1, where the presentation improves those given earlier.

For one-sided elliptic problems, the theorem and its corollary extend earlier results of Solonnikov [So1] (an L_p treatment of overdetermined elliptic differential operator problems, with interesting examples from physics), and of Schulze [S], recalled in [R-S1] (an L_2 treatment of general ps.d.o. systems, where the use of adjoints in such problems, via a reduction to order 0, is introduced); here we get the solvability properties in larger spaces and with more precision. (For the underdetermined, i.e. surjectively elliptic systems, [R-S1] just shows the existence of a smooth finite dimensional complement of the range, not its constant dimension.)

Let us also observe that the theorem extends results of Triebel [T1,2] for elliptic differential operators with normal boundary conditions in Besov and Bessel-potential spaces, to the pseudo-differential, one-sided elliptic, not necessarily normal or positive order case. In Triebel's books (in particular in [T2]), some more general spaces are treated, notably the spaces $F_{p,q}$ with $p \in]0,\infty[$ and $B_{p,q}$ with $p \in]0,\infty]$, $q \in]0,\infty]$. There is also a treatment by Solonnikov [So2], of overdetermined elliptic systems in Golovkin spaces of fractional order.

The overlap with the work of Franke [F1,2], that still awaits publication of details, has been described in the introduction.

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