1. Fix a simplicial set $K$, a group $G$, and a map $\varphi: K \longrightarrow B\cdot G$ of simplicial sets. (Here, $B\cdot G$ denotes the nerve of the category $B(G)$.)

(a) Fix $n \geq 2$, and consider the set of maps

$X_n = \{ f: |K| \times \Delta^n \longrightarrow BG \mid f(x, t) = |\varphi|(x) \text{ all } x \in |K|, t \in \partial\Delta^n \}.$

Set $A_n = X_n/\sim$, where $\sim$ is the relation of homotopy through elements of $X_n$. Describe $A_n$.

(b) Set $n = 1$. Let $X^*_1$ and $A^*_1$ be as in (a), but with the additional condition on maps in $X_1$ that for a choice of basepoint $* \in K_0$, $f(*, t) = *_G$ (the basepoint in $BG$) for all $t \in \Delta^n$. Determine $A^*_1$.

(Intuitively, we identify $A_n = \pi_n(map(|K|, BG), |\varphi|)$ (homotopy based at the point $|\varphi|$) and $A^*_1 = \pi_1(map(|K|, BG)_*, |\varphi|)$ (restricting to the point of basepoint preserving maps). But what we will really need to work with are the sets/groups defined in (a) and (b).)

2. Let $\mathcal{C}$ be the category of the poset $\mathbb{Z}$. (Thus $\text{Ob}(\mathcal{C}) = \mathbb{Z}$, and there is a unique morphism from $n$ to $m$ whenever $n \leq m$.)

(a) Prove that $|\mathcal{C}|$ is contractible.

(b) We embed $\mathbb{R}$ in $|\mathcal{C}|$ by identifying the interval $[n, n + 1] \subseteq \mathbb{R}$ with the edge in $\mathcal{C}$ from $n$ to $n + 1$. Prove that $\mathbb{R}$ is a strong deformation retract of $|\mathcal{C}|$.

In other words, one must construct a retraction $r: |\mathcal{C}| \longrightarrow \mathbb{R}$, together with a homotopy $|\mathcal{C}| \times I \longrightarrow |\mathcal{C}|$ from the identity to $r$ which sends $(x, t)$ to $x$ for each $x \in \mathbb{R}$ and $t \in I$.

Note: Of course, (a) follows immediately from (b). But it is much easier to do, and hence is included as a “warmup”.

3. Consider the extension of groups

$$1 \longrightarrow C_2 \longrightarrow SL_2(5) \longrightarrow PSL_2(5) \cong A_5 \longrightarrow 1.$$ 

You may assume this induces a homotopy fibration sequence

$$BC_2 \longrightarrow BSL_2(5) \longrightarrow BPSL_2(5)$$

of classifying spaces.

(a) Assuming the fiber lemma of Bousfield and Kan, prove that this sequence is still a homotopy fibration sequence after completion at 2.

(b) Prove that $\pi_2((BA_5)_2) \neq 0$. Hint: What is $\pi_1(PSL_2(5)_2)$? You can answer this by showing that $SL_2(5)$ is perfect (equal to its commutator subgroup), for which you may assume that $A_5$ is simple.