1 Homotopy colimits

* Exercise 1. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be three spaces and two maps. We may form the double mapping cylinder $C$ of these data by setting $C = (X \times I) \coprod (Y \times I) \coprod Z/\sim$, where $(x, 1) \sim (f(x), 0)$ and $(y, 1) \sim (g(y))$ for all $x \in X$ and $y \in Y$.

What is the homotopy type of $C$? Show that we can realize $C$ as being homotopy equivalent to the homotopy colimit of the diagram $B[2] \to \text{TOP}$ corresponding to this picture.

* Exercise 2. Given an infinite sequence of spaces and maps $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots$, we can form the infinite mapping telescope, generalizing the notion of a mapping cylinder. Describe how we might view these data as a diagram in $\text{TOP}$ indexed on the poset of natural numbers, and show that the mapping telescope is homotopic to the homotopy colimit of this diagram.

Exercise 3. Let $G$ be a discrete group, and let $X$ be a space with a left $G$-action, viewed as a functor $X : BG \to \text{TOP}$. What is colim $BG X$?

Show that $\text{hocolim} BG X \simeq EG \times_G X$. Here, $EG \times_G X$ is the Borel construction of the $G$-space, defined to be the quotient of $EG \times X$ by the diagonal $G$-action.

2 Another model of the homotopy colimit

Definition 2.1. If $F : C \to \text{CAT}$ is a $C$-diagram in small categories, the Grothendieck category of $F$ is the category $\mathcal{G}(F)$. The objects of $\mathcal{G}(F)$ are pairs $(c, x)$ where $c \in \text{ob}(C)$ and $x \in \text{ob}(F(c))$, and a morphism $(c, x) \to (c', x')$ is a pair $(\varphi, \gamma)$ where $\varphi \in C(c, c')$ and $\gamma \in F(c')(F(\varphi)(x), x')$.

* Exercise 4. Work out explicitly what the composition of morphisms of $\mathcal{G}(F)$ is.

Exercise 5. If $G$ is a discrete group, $BG$ its classifying category, and $F : BG \to \text{CAT}$ sends the unique object of $BG$ to $BH$ for $H$ some other discrete group, describe $\mathcal{G}(F)$.

Exercise 6. If $G$ is a small groupoid and $X : G \to \text{SET}$ is a left $G$-set, we may think of $X$ as a functor to $\text{CAT}$, where any set is identified with the small category whose objects are its elements and where all morphisms are the identity. What is $\mathcal{G}(X)$ in this case?

Exercise 7. Show that the Grothendieck category on $F : C \to \text{CAT}$ comes with a natural “forgetful functor” $\mathcal{G}(F) \to C$. What is this functor in the case that $F$ is $X : G \to \text{SET}$ for $G$ a small groupoid?

Let us assume the following result of Thomason:

Theorem 2.2. Given the diagram $FC \to \text{CAT}$, there is a weak equivalence of space $\text{hocolim}_C |NF(c)| \simeq |NG(F)|$

In other words, the homotopy colimit of a functor taking value in spaces that are realizations of categories can itself be seen as the realization of a category.

* Exercise 8. Find a category whose nerve’s realization is homotopy equivalent to $S^2$. 
3 Cohomology of colimits

* Exercise 9. Let \( C \) be the category determining a pushout, \( C = \{ x_1 \leftarrow x_0 \rightarrow x_2 \} \). Describe a homotopy colimit of a diagram indexed on \( C \) explicitly. What is another name for the spectral sequence of the cohomology of the homotopy colimit?

* Exercise 10. If \( X \) is a \( G \)-space, write out the spectral sequence for computing the cohomology of \( \text{hocolim}_{BG} X \) and identify it with another well-known spectral sequence.