1 Mapping spaces revisited

Let $G$ and $H$ be discrete groups, and $K$ a simplicial set.

**Exercise 1.** Assume that $|K|$ is connected, and let $T \subset K$ be a maximal tree. That $T$ is a tree means that $T$ is a simplicial subset of $K$ with no nondegenerate simplices above degree 1 and that $T$ contains no simplicial subsets whose realizations are homeomorphic to a circle ("there are no loops in $T"$). That $T$ is maximal means that $T_0 = K_0$ and $|T|$ is connected.

For any vertex $x_0 \in K_0$, use these data to construct a map $K_1 \to \pi_1(|K|, x_0)$ with the obvious universal property (refer to our discussion of the exercise concerning the nerve of a category with a "beginning" object), and use this to describe $\pi_1(|K|, x_0)$ in terms of $K_1$ and $K_2$.

**Exercise 2.** Recall that we have seen in class that $[BG, BH]_* \cong \text{Hom}(G, H)$ and $[BG, BH] \cong \text{Rep}(G, H)$. Show that if $BG$ is replaced by an arbitrary simplicial set $K$, we can identify $|K|, H$ with $\text{Hom}(\pi_1(|K|), H)$, and similarly in the unpointed case.

*Hint:* Begin by showing that for a chosen maximal tree $T \subset K$, any map $|K| \to BH$ is not just homotopic to the realization of a simplicial map $K \to \mathcal{B}_\ast(G)$, but that this simplicial map may be chosen so that its realization is constant on $|T|$. Use this to reduce the problem to the case where $K$ has a single vertex by considering the simplicial set $K/T$. Do the proof in the one vertex case.

**Exercise 3.** If $\varphi: G \to H$ is a morphism of groups, identify the homotopy type of the component of the mapping space $\text{Map}(BG, BH)$ that contains $B\varphi$, denoted $\text{Map}(BG, BH)_{B\varphi}$. Show that in fact we have $\text{Map}(BG, BH)_{B\varphi} \cong BC_H(\varphi(G))$.

*Hint:* It is enough to compute the homotopy groups of the mapping space. What is another way of thinking of a loop in $\text{Map}(|K|, BH)_{B\varphi}$ that is based at $B\varphi$?

**Definition 1.1.** A map of spaces $f : X \to Y$ is called centric if $f$ induces a homotopy equivalence $\text{Map}(X, X)_{id} \to \text{Map}(X, Y)_f$.

**Exercise 4.** Show that if $\iota: H \leq G$ is an inclusion of discrete groups, then $B\iota: BH \to BH$ is centric if and only if $C_G(H) = Z(H)$.

2 $p$-completion

**Exercise 5.** (From class)

- Show that a finite group $G$ is $p$-perfect if and only if $G = O^p(G)$. (Especially in light of the next part, why are we assuming $G$ is finite here?)

- Show that if $X$ is a space such that $\pi_1(X)$ contains a $p$-perfect subgroup of finite index, then $X$ is $p$-good and $\pi_1(X^p)$ can be described as a quotient of $\pi(X)$.

**Exercise 6.** Show that the $p$-completion of a point is (weakly homotopic to) a point.
**Exercise 7.** If $G$ is a finite $p'$-group (so $p \nmid |G|$), what is $B G_p^\wedge$?

* Exercise 8. Show that the $p$-completion of a product is (weakly homotopic to) the product of the $p$-completions.

**Exercise 9.** Find two nonisomorphic nontrivial finite groups $G$ and $H$ such that $B G_p^\wedge \simeq B H_p^\wedge$.

**Exercise 10.** If $H$ is a normal subgroup of the finite group $G$ and $p \nmid |H|$, show that $B(G/H)_p^\wedge \simeq B G_p^\wedge$. 