Categories and Topology: Exercises 2

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1 Kan complexes and nerves of categories

Definition 1.1. For \( n \geq 1 \) and \( 0 \leq k \leq n \), the \( k \text{th } n \)-horn is the simplicial set \( \Lambda^k_n \subset \Delta_n \) whose \( i \)-simplices consist of those maps \( \varphi \in \Delta([i],[n]) \) whose images do not contain the complement of \( \{k\} \) in \([n]\). A simplicial set \( K \) is a Kan complex if “every horn can be filled,” i.e., if every simplicial map \( \Lambda^k_n \rightarrow K \) extends to a simplicial map \( \Delta_n \rightarrow K \).

Exercise 1. Describe the geometric realization of \( \Lambda^k_n \), and explain why this is called a “horn.”

Exercise 2. Is \( \Delta_n \) a Kan complex for all \( n \)?

Exercise 3. If \( X \) is a topological space, show that the singular simplicial set \( S_n(X) \) is a Kan complex.

Exercise 4. If \( C \) is a small category, is the nerve of \( C \) a Kan complex? If not, which horns can be filled?

Exercise 5. Give a condition on a simplicial set \( K \) that is equivalent to \( K \)'s being the nerve of a small category.

Definition 1.2. A path in a simplicial set \( K \) is a simplicial map \( \gamma : \Delta^1 \rightarrow K \). Let \( I = [0,1] \) denote the unique nondegenerate 1-simplex of \( \Delta^1 \). The 0-simplex \( d^1_1 \gamma = \gamma([0]) \in K_0 \) is the initial point of \( \gamma \), and \( d^0_0 \gamma = \gamma([1]) \in K_0 \) is the terminal point of \( \gamma \).

Two 0-simplices \( a,b \in K_0 \) lie in the same path component if there exists a path in \( K \) with initial point \( a \) and terminal point \( b \).

Exercise 6. Show that if \( K \) is a Kan complex, “lying in the same path component” is an equivalence relation. Find a category \( C \) in whose nerve this statement is false.

Exercise 7. Suppose that \( C \) is a small category with a “beginning” object, i.e., some \( c \in \text{ob} C \) such that for all other objects \( c' \in \text{ob} C \), \( C(c,c') \neq \emptyset \) (similarly one could define an “ending” object as one that accepts a morphism from every object of \( C \)). Suppose further that for every \( c' \in \text{ob} C \) we have chosen an “inclusion” morphism \( c^c' \in C(c,c) \) such that \( c^c' = \text{id}_c \).

Let \( \text{Mor}(C) \) denote the set of all morphisms of \( C \). Construct a map \( \text{Mor}(C) \rightarrow \pi_1(|C|) \) that sends composition to multiplication and “inclusion” to the identity, and which is universal with regards to these properties. Equivalently, construct a functor \( C \rightarrow \mathcal{B}(\pi_1(|C|)) \) that sends “inclusion” to the identity. Show that this actually gives a computation of \( \pi_1(|C|) \).

Exercise 8.

Generalize the previous exercise by computing \( \pi_1(|K|) \), for \( K \) any connected simplicial set, in terms of \( K_1 \) and \( K_2 \).

\[ ^1 \text{This is not at all standard terminology.} \]
2 Covering spaces of groupoids

Let \( \mathcal{G}, \mathcal{H} \) be small groupoids (categories all of whose morphisms are invertible).

**Exercise 9.** Show that if \( \mathcal{G} \) is a connected groupoid (i.e., all objects are isomorphic), for any \( g \in \text{ob} \mathcal{G} \) we have \( B \text{Aut}_\mathcal{G}(g) \simeq B\mathcal{G} \) as spaces. Show moreover that this homotopy is actually induced by an equivalence of categories \( B \text{Aut}_\mathcal{G}(g) \simeq \mathcal{G} \).

**Definition 2.1.** A (left) \( \mathcal{G} \)-set is a functor \( X : \mathcal{G} \to \text{SET} \) (a right \( \mathcal{G} \)-set is a functor \( Y : \mathcal{G}^{\text{op}} \to \text{SET} \)). A morphism of \( \mathcal{G} \)-sets is a natural transformation.

**Definition 2.2.** For \( g \in \text{ob} \mathcal{G} \), the star at \( g \) is the set \( \star(g) \) of morphisms whose source is \( g \).

A map of groupoids \( F : \mathcal{H} \to \mathcal{G} \) is a cover if for every \( g \in \mathcal{G} \) and \( h \in F^{-1}(g) \), \( F \) induces a bijection \( \star(h) \to \star(g) \).

**Exercise 10.** Show that if \( F : \mathcal{H} \to \mathcal{G} \) is a cover of groupoids, then \( B\mathcal{H} \to B\mathcal{G} \) is a cover of topological spaces.

**Exercise 11.** Show that there is an isomorphism of categories of \( \mathcal{G} \)-sets and covers of \( \mathcal{G} \).

**Hint:** It might be easier to start with the case that \( \mathcal{G} = B\mathcal{G} \) is the classifying category of a discrete group. In this case, a \( B\mathcal{G} \)-set can be identified in a natural way with a left \( \mathcal{G} \)-set. What is the corresponding cover of \( B\mathcal{G} \)?

**Exercise 12.** Show that an inclusion \( H \leq G \) of discrete groups induces a map \( \iota : BH \to BG \) that is homotopy equivalent to a cover of topological spaces (i.e., there is some space \( T \), a homotopy equivalence \( BH \simeq T \) and a cover \( T \to BG \) such that \( \iota \) is homotopic to the composite \( BH \to T \to BG \)).

3 Mapping spaces

Let \( G \) and \( H \) be discrete groups.

**Exercise 13.** For \( K \) a simplicial set, compute \( |K|, BG \) and \( ||K||, BG \) (basepoint-preserving homotopy classes of basepoint-preserving maps and homotopy classes of all maps, respectively).

**Exercise 14.** If \( \varphi : G \to H \) is a morphism of groups, identify the homotopy type of the component of the mapping space \( \text{Map}(BG, BH) \) that contains \( B\varphi \), denoted \( \text{Map}(BG, BH)_{B\varphi} \). Again, do this also in the case that \( BG \) is the realization of a simplicial set.

**Definition 3.1.** A map of spaces \( f : X \to Y \) is centric if \( f \) induces a homotopy equivalence \( \text{Map}(X, X)_{\text{id}} \to \text{Map}(X, Y)_{f} \).

**Exercise 15.** Show that if \( \iota : H \leq G \) is an inclusion of discrete groups, then \( B\iota : BH \to BH \) is centric if and only if \( C_G(H) = Z(H) \).