1-small sets

Let P be a Markov kernel on $(\mathcal{X}, \mathbb{E})$. A set $C \in \mathbb{E}$ is 1-small if

$$P_x(B) \geq \delta\, \nu(B) \qquad \text{for all } B \in \mathbb{E}, x \in C\,,$$

for some $\delta \in (0,1)$ and some probability measure ν.

If P has a 1-small set C, we can via the Athreya-Ney construction produce a lift Q on a space $(\mathcal{Z}, \mathbb{K})$ such that Q has an atom α.

$$\begin{array}{lccccccccc}
Q: & Z_0 & \longrightarrow & Z_1 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
P: & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow
\end{array}$$

Law of Large Numbers

We say that P has a law of large numbers if:

For all bounded functions $g : \mathcal{X} \to \mathbb{R}$ - and perhaps also some unbounded - there exists a limit $L(g)$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) \to L(g) \quad \text{a.e.}$$

whenever $X_0, X_1, \ldots$ is a P-Markov chain.

Important point: $L(f)$ must not depend on the distribution of X_0.

Law of Large Numbers via lifts

Q will have a law of large numbers if

1) Q is irreducible

2) the atom α is Harris-recurrent

3) the return times to α har first moment

If Q has a law of large numbers, so will P, since

$$\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) = \frac{1}{n}\sum_{i=0}^{n-1} g(h(Z_i))\,.$$

Accessible atom

We say that α is an accessible atom for Q if

$$\sum_{n=1}^{\infty} Q_z^n(\alpha) > 0 \qquad \text{for all } z \in \mathcal{Z}\,.$$

If α is accessible, then

1) Q is irreducible

2) α is a $\mathbb{K}^+$-set.

Theorem If P is irreducible and $C \in \mathbb{E}^+$ is a 1-small set , then the lift corresponding to C has an accessible atom.

Examples with 1-small sets

The random walk on $\mathbb{R}^k$,

$$X_{n+1} = X_n + W_{n+1}$$

has every compact set 1-small, if

1) W_i has a density f wrt. Lebesgue measure

2) $\inf_{x \in C} f(x) > 0$ for every compact set C

Examples with 1-small sets II

The AR(1)-proces on $\mathbb{R}$,

$$X_{n+1} = \rho X_n + W_{n+1}$$

has every compact set 1-small, if

1) W_i has a density f wrt. Lebesgue measure

2) $\inf_{x \in C} f(x) > 0$ for every compact set C

Examples without 1-small sets

The AR(2)-proces on $\mathbb{R}$,

$$X_{n+1} = \rho X_n + \gamma X_{n-1} + W_{n+1}$$

is stacked to a proces on $\mathbb{R}^2$,

$$\begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} = \begin{pmatrix} \rho & \gamma \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix} + \begin{pmatrix} W_{n+1} \\ 0 \end{pmatrix}$$

Note that $P_{x,y}$ and $P_{x',y'}$ are singular if $x \neq x'$.

Any 1-small set must be contained in a line parallel to the second axis.

There are no 1-small sets of positive Lebesgue measure.

n-small sets

Let P be a Markov kernel on $(\mathcal{X}, \mathbb{E})$. A set $C \in \mathbb{E}$ is n-small if

$$P_x^n(B) \geq \delta\, \nu(B) \qquad \text{for all } B \in \mathbb{E}, x \in C\,,$$

for some $\delta \in (0,1)$ and some probability measure ν.

A set C is small if if is n-small for some n.

If necessary, we refer to C as (n, δ, ν)-small.

Nummelins theorem

Theorem If P is irreducible and B is an $\mathbb{E}^+$-set, there is a set C such that

1) $C \subset B$

2) $C \in \mathbb{E}^+$

3) C is n-small for some n

Consequence:
if P is irreducible, there are lots and lots of small sets

Nummelins theorem - sketch of proof I

Let ψ be a maximal irreducibility measure.

Find a Lebesgue decomposition of each P_x with respect to ψ,

$$P_x = f_x \cdot \psi + \nu_x \,.$$

Here each ν_x is singular with respect to ψ.

Claim: the decomposition can chosen such that $(x, y) \mapsto f_x(y)$ is measurable.

Proof: Use that $\mathcal{X}$ is Borel, apply the convergence theorem for closed martingales. . . .

Nummelins theorem - sketch of proof II

Use this for all iterates of P,

$$P_x^n = f_x^n \cdot \psi + \nu_x^n \, .$$

Define for each x

$$\mathcal{N}_x = \left\{ y \mid \sum_{n=1}^{\infty} f_x^n(y) = 0 \right\}$$

Claim: Each $\mathcal{N}_x$-set is a ψ-nullset.

Proof: if $\psi(\mathcal{N}_x) > 0$, then due to irreducibility must be that

$$P_x^n(\mathcal{N}_x) > 0 \quad \text{for some } \; n \, .$$

But this is impossible.

Nummelins theorem - sketch of proof III

Thus we have that

$$\int_{B\times B} \sum_{n=1}^{\infty} f_x^n(y) \; d\psi \otimes \psi(x,y) > 0$$

In particular

$$\psi \otimes \psi\Big(\{(x,y) \in B \times B \mid f_x^n(y) > 0 \quad \text{for some} \;\; n\}\Big) > 0$$

This again implies that there is some n and m such that

$$\psi \otimes \psi \left(\left\{(x,y) \in B \times B \mid f_x^n(y) > \frac{1}{m}\right\}\right) > 0$$

Nummelins theorem - sketch of proof IV

False measuretheoretic claim If $\mu \otimes \mu(A) > 0$ there is a product set $B \times C \subset A$ such that $\mu \otimes \mu(B \times C) > 0$.

Pretend we believe this claim: There are sets $C, D \subset B$, both $\mathbb{E}^+$-sets, such that

$$f_x^n(y) > \frac{1}{m} \qquad \text{for all } x \in C, y \in D$$

And then we have for all $x \in C$

$$P_x^n(E) \geq \int_{E \cap D} f_x^n(y)\, d\psi(y) \geq \frac{\psi(D)}{m} \frac{\psi(E \cap D)}{\psi(D)}$$

for alle $E \in \mathbb{E}$. Hense C is n-small.

Small set covering

Corollary If P is irreducible, there is a countable cover $(C_n)_{n\in\mathbb{N}}$ of $\mathcal{X}$ using small sets.

Proof: Let $C \in \mathbb{E}^+$ be (n, δ, ν)-small. Let

$$A_{m,k} = \left\{ x \mid P_x^m(C) > \frac{1}{k} \right\}$$

Due to irreducibility, these sets cover $\mathcal{X}$. Take $x \in A_{m,k}$. For any set E

$$P_x^{n+m}(E) = \int P_y^n(E)\, dP_x^m(y) \geq \int_C P_y^n(E)\, dP_x^m(y) \geq \delta\, \nu(E)\, P_x^m(C)$$

Hence $A_{m,k}$ is $(n+m, \delta/k, \nu)$-small.

Law of large numbers?

Does Nummelins theorem imply a law of large numbers for any irreducible kernel P?

It implies a n-small set for P. And hence there is a lift Q of P^n, which has an atom.

If there is a law of large numbers for this Q, there is in fact a law of large numbers for P. Suppose $n = 2$. If we take m odd, say $m = 2\ell + 1$, then

$$\frac{1}{m}\sum_{i=0}^{m-1} g(X_i) = \frac{1}{m}\sum_{i=0}^{l} g(X_{2i}) + \frac{1}{m}\sum_{i=0}^{\ell-1} g(X_{2i+1}) \to L(g) \quad \text{a.e.}$$

since we are considering timeaverages for two P^2-Markov chains.

Law of large numbers?

Trap: There may not be a law of large numbers for Q.

Because Q may not be irreducible!

Question: When are all iterates of a Markov kernel P irreducible?

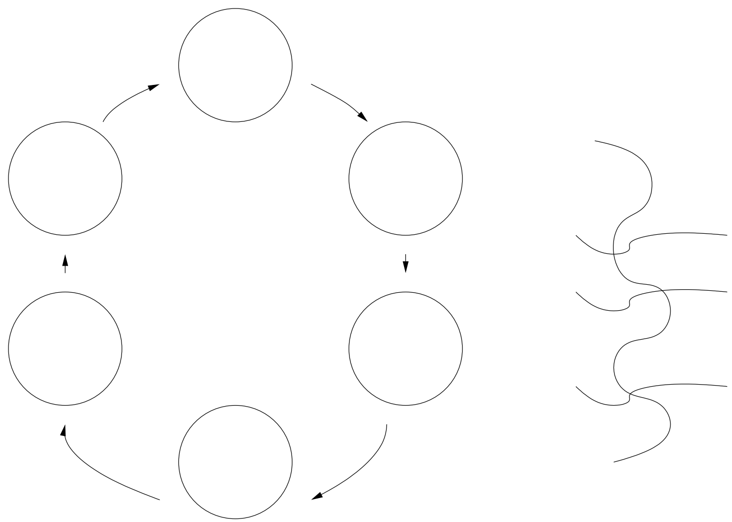
Terminology: n-skeleton = P^n.

Cycles

Let P be irreducible. A cycle of length d are disjoint sets $D_0, D_1, \ldots, D_{d-1}$ such that

$$P_x(D_{i+1}) = 1 \quad \text{for all } x \in D_i \text{ (mod d)}$$

and such that $\cup D_i$ has full irreducibility measure.



Cycles

Note that

$$P_x^j(D_{i+j}) = 1 \quad \text{for all } x \in D_i \text{ (mod d)}$$

In particular

$$P_x^d(D_i) = 1 \quad \text{for all } x \in D_i$$

so P^d has d disjoint absorbing sets.

Conclusion: If P has a cycle of lenght $d \geq 2$, the P^d can not be irreducible.

Non-repelling atoms

An atom α for P is strongly non-repelling if

$$P_x(\alpha) > 0 \quad \text{for } x \in \alpha\,.$$

An atom α for P is non-repelling if

$$P_x^n(\alpha) > 0 \quad \text{for } x \in \alpha$$

for some n.

Strongly non-repelling atoms and irreducibility

Theorem If P is irreducible and $\alpha \in \mathbb{E}^+$ is a strongly non-repelling atom, then P^n is irreducible for all n.

Proof: Let $\nu = P_x$ for some $x \in \alpha$. Let $\delta = \nu(\alpha)$. We claim that ν is an irreducibility measure for P^n.

By induction, prove that α is (m, δ^{m-1}, ν)-small: for $x \in \alpha$,

$$P_x^{m+1}(A) \geq \int_\alpha P_y^m(A)\, dP_x(y) \geq \delta^{m-1}\, \nu(A)\, P_x(\alpha) = \delta^m\, \nu(A)$$

Suppose $\nu(A) > 0$, and take some $x \in \mathcal{X}$. Since $\alpha \in \mathbb{E}^+$, there is some m such that $P_x^m(\alpha) > 0$. Find k such that $m + k \in n\mathbb{Z}$. Then

$$\sum_{i=1}^{\infty} P_x^{in}(A) \geq P_x^{m+k}(A) \geq \int_\alpha P_z^k(A)\, dP_y^m(z) \geq \delta^{k-1}\nu(A) P_y^m(\alpha) > 0$$

Non-repelling atoms and irreducibility

If P is irreducible, and $\alpha \in \mathbb{E}^+$ is an atom, α is always non-repelling.

Define

$$\mathcal{N}_\alpha = \{n \in \mathbb{N} \mid \alpha \text{ strongly non-repelling for } P^n\}$$

Lemma If $n, m \in \mathcal{N}_\alpha$ then $n + m \in \mathcal{N}_\alpha$.

Lemma Let d be the greatest common divisor in $\mathcal{N}_\alpha$. Then $\mathcal{N}_\alpha \subset d\mathbb{N}$, and there is some n_0 such that $dn \in \mathcal{N}_\alpha$ for all $n \geq n_0$

Terminology:

the periodic case: $d > 1$

the aperiodic case: $d = 1$

the strongly aperiodic case: $\mathcal{N}_\alpha = \mathbb{N}$

Non-repelling atoms and irreducibility

Lemma If α has period $d \geq 2$, we can construct a cycle.

Lemma If P is irreducible and $\alpha \in \mathbb{E}^+$ is an aperiodic atom, then P^n is irreducible for all n.

Proof: We show that α is an accessible atom for P^n. Take $x \in \mathcal{X}$ and find m such that $P_x^m(\alpha) > 0$. Find some k such that $m + k \in n\mathbb{N}$. We may take k so large that $k \in \mathcal{N}_\alpha$.

$$\sum_{i=1}^{\infty} P_x^{in}(\alpha) \geq P_x m + k(\alpha) \geq \int_\alpha P_y^k(\alpha)\, dP_x^m(y) > 0$$

Atoms and periodicity

Let P be irreducible. Assume it has an atom $\alpha \in \mathbb{E}^+$. We have proved that there are two possibilities:

1) There is a cycle of length $d \geq 2$ (periodic case)

2) All iterates P^n are irreducible (aperiodic case)

A special case of aperiodicity occurs for a strongly aperiodic atom.

Small sets and periodicity

Let P be irreducible. The previous calculations can be repeated, based on small sets instead of atoms. There are two possibilites:

1) There is a cycle of length $d \geq 2$ (periodic case)

2) All iterates P^n are irreducible (aperiodic case)

A special case of aperiodicity occurs if there is a strongly aperiodic small set. This means that C is $(1, \delta, \nu)$-small, and that $\nu(C) > 0$.

Important point: there are no relevant periodic examples. But aperiodicity is hard to prove.

Petite sets

Let P be a Markov kernel on $(\mathcal{X}, \mathbb{E})$. A set C is (a, δ, ν)-petite if

$$\sum_{n=0}^{\infty} a(n) P_x^n(A) \geq \delta \nu(A) \qquad \text{for all } A \in \mathbb{E}, x \in C\,.$$

Here a is a probability distribution on $\mathbb{N}_0$, δ is a number in $(0, 1)$, and ν is a probability measure on $\mathcal{X}$.

Observation Any small set is petite.

Theorem: If P is irreducible and aperiodic, any petite set is small.

Petite sets

Even though petite sets and small sets for reasonable models are the same thing, there are occasional uses of petiteness.

A set C can be (a, δ, ν)-petite for many different triples. In particular we can change ν, by changing the sampling distribution a.

Theorem If P is irreducible and C is a (a, δ, ν)-petite set, there is a sampling distribution b and a maximal irreducibility measure ψ such that C is (b, γ, ψ)-petite for some γ.

Feller properties

Assume $\mathcal{X}$ is a metric space - seperable, locally compact.

Let $\mathcal{M}(\mathcal{X})$ be the space of bounded measurable real functions on $\mathcal{X}$, and let $C_b(X)$ be the space of bounded countinuous real functions.

Let P be a Markov kernel. There is an associated operator $P : \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{X})$ given by

$$Ph(x) = \int h(y)\, dP_x(y) \qquad \text{for } x \in \mathcal{X}, h \in \mathcal{M}(\mathcal{X})\,.$$

Definition P is weak Feller if

$$Ph \in C_b(\mathcal{X}) \quad \text{for all } h \in C_b(\mathcal{X})$$

Definition P is strong Feller if

$$Ph \in C_b(\mathcal{X}) \quad \text{for all } h \in \mathcal{M}(\mathcal{X})$$

Feller properties

All practical examples of Markov kernels are weak Feller.

The strong Feller property is not so frequent. It is usually associated to continuous transition densities with respect to Lebesgue measure.

Definition: P is a T-chain if there is sampling distribution a, a (lower semi-)continous function $\delta : \mathcal{X} \to (0, 1)$ and a strong Feller kernel P' such that

$$\sum_{n=0}^{\infty} a(n) P_x^n(A) \geq \delta(x) P'_x(A) \qquad \text{for all } \; x \in \mathcal{X}, A \in \mathbb{E}$$

T-chains

A point x is reachable if

$$\sum_{n=0}^{\infty} P_y^n(G) > 0$$

for every $y \in \mathcal{X}$ and every open set G containing x.

Theorem If P has the T-chain property and has a reachable point, then P is irreducible.

Theorem If P is an irreducible T-chain, then all compact sets are petite.