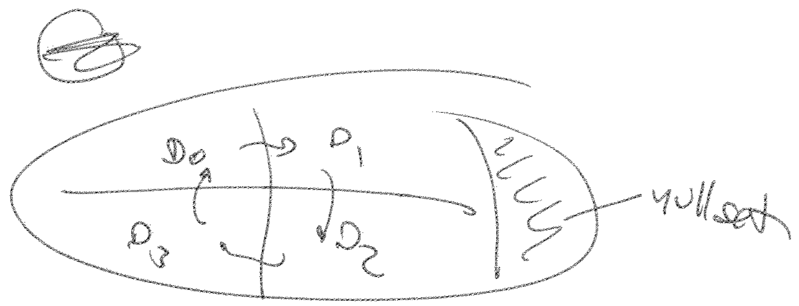
18. oktober 2004

Definition: Let P be irreducible. A cycle is a collection $D_0, D_1, \dots D_{k-1}$ of E-sets such that

1) $\bigcup_{0}^{k-1} D_i$ is full

2) $P_x(D_{i+1}) = 1 \quad \forall x \in D_i \quad (\text{mod } k)$



If there is a k-cycle, then P^k contains k disjoint absorbing sets, and hence it is not irreducible. which is clearly a menace.

<u>Lemma</u> Suppose α is a non-repelling atom for P with

$$\delta = \nu(\alpha)$$

Then α is (n, δ^{n-1}, ν)-small for any n.

<u>Proof</u>: Induction by n. For $x \in \alpha$ we have

$$P_x(A) = \nu(A) = \delta^{1-1}\nu(A)$$

so it is true for $n=1$. In the induction step

$$P_x^{n+1}(A) = \int P_y^n(A)\, dP_x(y) \qquad x \in \alpha$$

$$\geq \int_\alpha P_y^n(A)\, dP_x(y)$$

$$\geq \int_\alpha \delta^{n-1}\nu(A)\, dP_x(y)$$

$$= \delta^{n-1}\nu(A)\, P_x(\alpha)$$

$$= \delta^n \nu(A)$$

since $P_x = \nu$ for $x \in \alpha$

<u>Theorem</u>: If P is irreducible and $\alpha \in E^+$ is a non-spelling atom, then all powers P^k are irreducible.

<u>Proof</u>: We show that ν is an irreducibility measure for P^k. So take A with $\nu(A) > 0$ and take x.

Since $\alpha \in E^+$ there is $n \in \mathbb{N}$ such that $P_x^n(\alpha) > 0$.

Take $l \in \mathbb{N}$ such that $n + l \in k\mathbb{N}$. Then

$$
\begin{aligned}
\sum_{i=1}^{\infty} P_x^{ik}(A) &\geq P_x^{n+l}(A) \\
&= \int P_y^l(A)\, dP_x^n(y) \\
&\geq \int_\alpha P_y^l(A)\, dP_x^n(y) \\
&\geq \int_\alpha \delta^{l-1} \nu(A)\, dP_x^n(y) \\
&= \delta^{l-1} \nu(A)\, P_x^n(\alpha) \\
&> 0
\end{aligned}
$$

$\square$

Lemma: If α is an atom for P, then it is an atom for P^k $\forall k$.

Proof: We don't even need induction: Take $x \in \alpha$

$$P_x^n(A) = \int P_y^{n-1}(A)\, dP_x(y)$$

$$= \int P_y^{n-1}(A)\, d\nu(y)$$

So this is not a choice left of x.

$\square$

Let P be irreducible, and let $\alpha \in E^+$ be an atom. Let

$$\mathcal{O} = \{ n \mid P_x^n(\alpha) > 0 \quad \forall x \in \alpha \}$$

Periodic case:

$$\mathcal{O} \subseteq d\mathbb{Z} \qquad \text{for some } d \geq 2$$

Aperiodic case

$$\{n_0, n_0+1, n_0+2, \dots\} \subseteq \mathcal{O} \qquad \text{for some } n_0.$$

Since $\mathcal{O}$ is a semigroup.

Theorem : let P be irreducible, and let $\alpha \in \mathcal{E}^+$ be an aperiodic atom. Then P^k is irreducible for each k.

Proof : Take $x \in \mathcal{X}$. Find n such that

$$P_x^n(\alpha) > 0$$

take some $m \geq n_0$ such that $n+m \in k\mathbb{N}$.
Then

$$\sum_{i=1}^{\infty} P_x^{ki}(\alpha) \geq P_x^{n+m}(\alpha) = \int P_y^m(\alpha)\, dP_x^n(y)$$

$$\geq \int_\alpha P_y^m(\alpha)\, dP_x^n(y)$$

$$\geq \nu_m(\alpha)\, P_x^n(\alpha)$$

where ν_m is the constant value of $P_\cdot^m$ across α. This is positive. So we have shown that α is an accessible atom for P^k. Hence P^k is irreducible.

$\Box$

The periodic case leads to a cycle.

Lemma: If $(Z, f, Q, *)$ is a lift of (Z, P) then $(Z, f, Q^n, *)$ is a lift of (Z, P^n).

Proof: Trivial

Lemma: Let $(\mathcal{Z}, f, Q, *)$ be a lift of $(\mathcal{X}, P)$

If Q is irreducible, so is P.

Proof: Let φ be an irreducibility measure for Q

We claim $f(\varphi)$ is an irreducibility measure for P
Take A such that $f(\varphi)(A) > 0$
Take $x \in \mathcal{X}$, let $\lambda = \delta_x$ and let λ_*^* be the

lift of λ.

Let K_z be the resolvent for Q. Since

$\varphi(f^{-1}(A)) > 0$ we know that

$$K_z(f^{-1}(A)) > 0 \qquad \forall z$$

Thus

$$\int K_z(f^{-1}(A))\, d\lambda^*(z) > 0$$

and in particular there is some n such that

$$\int Q_z^n(f^{-1}(A))\, d\lambda^*(z) > 0$$

But this equals

$$P_x^n(A)$$

elaborate?

Theorem: If P is irreducible and $C \in \mathbb{E}^+$ is a non-repelling 1-small set, then all powers P^k are irreducible.

Proof: Construct the Athreya-Ney lift Q on $X \times \{0,1\}$ be splitting on C. We claim that $C \times \{1\}$ is a non-repelling atom for Q. We know it to be an atom, so it is just the question of non-repellingness. If the (X,Y)-chain is started in $C \times \{1\}$, then next x is drawn after ν and the next y is drawn after $\xi(x)$. That is

$$Q_{x,1}(C \times \{1\}) = \int_C \xi(x)\, d\nu(x)$$
$$\geq \int_C \delta\, d\nu(x) = \delta\, \nu(C) > 0$$

Hence all powers Q^n are irreducible. As Q^n is a lift of P^n, this implies that P^n is irreducible.

□

Let $\mathcal{P}$ be irreducible. Let C be small.
We may assume that C is (m, δ, ν)-small
when $\nu(C) > 0$.

$$\mathfrak{A}_\nu = \{ m \mid C \text{ is } (m, \delta, \nu)\text{-small for some } \delta \}$$

It turns out that $\mathfrak{A}_\nu$ is a semigroup. Hence
there is a d such that

$$\mathfrak{A}_\nu \subseteq d\mathbb{Z}$$

and such that

$$\mathfrak{A}_\nu \cap \{x \mid x \geq n\} = d\mathbb{Z} \cap \{x \mid x \geq n\}$$

for some n.

<u>Theorem</u>: there is a d-cycle. $D_0 \ldots D_d$

<u>Theorem</u>: Any other cycle $K_0, \ldots, K_k$ satisfies
that k is a divisor in d, and that
each ~~[illegible]~~ D_i is contained in a single K_i