Monday December 6 2004

We recall the following important result

Theorem Suppose P is irreducible, aperiodic and Harris recurrent. Then there is a small set $C \in \mathcal{E}^+$ such that

$$(*) \qquad \sup_{x \in C} H_{C,x}(x) < \infty$$

Then P has an invariant probability measure.

□

The left hand side in $(*)$ has a probabilistic interpretation: if $X_0, X_1, \dots$ is a P-chain with $X_0 \equiv x$, and if τ_C is the first return time to C, then

$$\mathcal{E}(\tau_C) = H_{C,x}(x)$$

Hence the condition is frequently written as

$$\sup_{x \in C} \mathcal{E}_x(\tau_C) < \infty$$

where the symbol $\mathcal{E}_x$ means the expected value if the initial distribution is ε_x.

We will make this result operational through a drift criterion. The key result in this process is a triviality known as Dynkin's lemma

Dynkin's lemma: Let $\mathbb{F}_0, \mathbb{F}_1, \dots$ be a filtration. Let $(Y_n)_{n\in\mathbb{N}_0}$, $(Z_n)_{n\in\mathbb{N}_0}$ and $(W_n)_{n\in\mathbb{N}_0}$ be three non-negative processes. Suppose the Z-process is integrable and that

$$Y_n \le Z_n - E(Z_{n+1} \mid \mathbb{F}_n) + W_n \qquad \forall n$$

Then

$$E \sum_0^{\tau} Y_n \le E Z_0 + E \sum_0^{\tau} W_n$$

for every stopping time τ

Proof: By summing, we obtain

$$\sum_0^n Y_k \le \sum_0^n Z_k - E(Z_{k+1} \mid \mathbb{F}_k) + \sum_0^n W_k$$

$$= Z_0 + \sum_1^n Z_k - E(Z_k \mid \mathbb{F}_{k-1}) - E(Z_{n+1} \mid \mathbb{F}_n) + \sum_0^n W_k$$

Introduce the zero-mean martingale

$$M_n = \sum_1^n Z_k - E(Z_k \mid \mathcal{F}_{k-1})$$

Using that $Z_{n+1} \geq 0$, and thus $E(Z_{n+1} \mid \mathcal{F}_n) \geq 0$ a.e.

we obtain that

$$\sum_0^n Y_k \leq Z_0 + M_n + \sum_0^n W_k \qquad \forall n$$

and thus

$$\sum_0^\tau Y_k \leq Z_0 + M_\tau + \sum_0^\tau W_k$$

If we only consider a bounded stopping time τ

The theorem on optional sampling shows that M_τ

has mean zero. And thus

$$E \sum_0^\tau Y_k \leq E Z_0 + E \sum_0^\tau W_k$$

If τ is a general stopping time - even infinite -

we have that

$$E \sum_0^{\tau \wedge N} Y_k \leq E Z_0 + E \sum_0^{\tau \wedge N} W_k$$

for every N. As

$$\theta \sum_0^{\sigma \wedge N} Y_k \to \sigma\left(\sum_0^{\sigma} Y_k\right)$$

$$E \sum_0^{\sigma \wedge N} Z_k \to \sigma\left(\sum_0^{\sigma} Z_k\right)$$

by dominated convergence, the result follows

$\square$

<u>Theorem</u>: Let P be irreducible and aperiodic. Suppose there is a drift function $V : \mathcal{X} \to [0, \infty)$ such that

(0) $\Delta V(x) \leq -1 + b 1_C(x)$

for some small set C on which V is bounded. Then we conclude

1) P is Harris

2) P possesses an invariant probability measure

<u>Proof</u>: Written out, (0) says that

$$PV(x) \leq V(x) - 1 \qquad x \notin C$$

$$PV(x) \leq V(x) + b - 1 \qquad x \in C$$

At least

$$PV(x) \le V(x) + M \qquad \forall x$$

for some M. This implies that

$$P^k V(x) \le V(x) + kM$$

The crux of the matter is that if $X_0, X_1 \ldots$ is a P-chain with $E\, V(X_0) < \infty$, then

$$\int V(X_k)\, dP = \int P^k V(X_0)\, dP$$

$$\le \int V(X_0) + kM \, dP$$

$$< \infty$$

Note that (a) says

$$1 \le V(x) - PV(x) + b 1_C(x) \qquad \forall x$$

or

$$1 \le V(X_n) - PV(X_n) + b 1_C(X_n) \qquad \forall n$$

Note that $PV(X_n) = E(V(X_{n+1}) \mid X_0 \ldots X_n)$, so Dynkin's lemma implies that

$$(\text{II}) \quad E\, \tau + 1 \le V(X_0) + b\, E \sum_0^{\tau} 1_C(X_n)$$

This holds for any stopping time, but we will for now use it with $\tau = \sigma_C$ because that trivializes the last sum. We obtain

$$E_x \tau \leq V(x) + 2b - 1$$

As a first application, $E_x \tau < \infty$ which certainly implies $L_x(C) = 1$ for all x. We conclude from this that C is Harris recurrent and $\mathbb{E}\tau$ and this is quite sufficient to conclude that P is Harris recurrent.

Next we have that

$$\sup_{x \in C} E_x \tau \leq \sup_{x \in C} V(x) + 2b - 1 < \infty$$

and thus P is positive

$\square$

This drift criterion is very, very useful because it is frequently easy to check. It gives the existence of an invariant probability measure, but it does not really give a clue to what this measure is.

Example: Consider an autoregressive scheme on $\mathbb{R}$

$$X_{n+1} = \rho X_n + U_{n+1}$$

where $|\rho| < 1$ and where $U_1, U_2 ..$ are iid. we assume

$$E(U_i) = 0 \qquad E\, U_i^2 < \infty$$

and we assume irreducibility and aperiodicity has been checked and that compact sets are small.

Consider the drift function $V(x) = cx^2$.

$$PV(x) = E(V(\rho x + U)) = c E(\rho^2 x^2 + 2\rho x U + U^2)$$

$$= c(\rho^2 x^2 + 2\rho x EU + EU^2)$$

$$= \rho^2 Vx + cEU^2$$

Find some λ such that $\rho^2 < \lambda < 1$.

$$\rho^2 V(x) + cEU^2 \leq \lambda V(x)$$

$$\Leftrightarrow \quad cEU^2 \leq (\lambda - \rho^2) V(x)$$

$$\Leftrightarrow \quad \frac{c\, EU^2}{\lambda - \rho^2} \leq V(x)$$

$$\Rightarrow \quad \sqrt{\frac{EU^2}{\lambda - \rho^2}} \leq |x|$$

Similarly,

$$\lambda V(x) \geq V(x) - 1$$

$$1 \leq (1-\lambda) V(x)$$

$$\sqrt{\frac{1}{c(1-\lambda)}} \leq |x|$$

Hence, for $|x| \geq c = \max \left\{ \sqrt{\frac{\theta v^2}{\lambda - \rho^2}} ; \sqrt{\frac{1}{c(1-\lambda)}} \right\}$

we have that

$$\rho V(x) \leq V(x) - 1 \qquad x \notin [-c, c]$$

And clearly, both V and ρV are bounded on $[-c, c]$, so

$$\rho V(x) \leq V(x) + b \qquad x \in [1, c]$$

for some suitable b

So the AR(1) process satisfies the drift condition for positive Harris.

Alternativt kan man droppe 2. momentsbetingelsen, og se på: $V(x) = |x|$. Da er

$$\rho V(x) = E|\rho x + U| \le |\rho||x| + E|U|$$

og ettersom

$$|\rho||x| + E|U| \le |x| - 1 \iff$$

$$1 + E|U| \le (1 - |\rho|)|x| \iff$$

$$\frac{1 + E|U|}{1 - |\rho|} \le |x|$$

ser vi at

$$\rho V(x) \le V(x) - 1 \quad \text{på } [-c, c]^c$$

hvor $c = \dfrac{1 + E|U|}{1 - |\rho|}$. Begrensning på $[-c, c]$ er trivielt.

Second example: arch processes,

$$X_{n+1} = \sqrt{a^2 + b^2 X_n^2}\, U_{n+1}$$

ergodic if $b^2 < 1$, $E|U| < \infty$

Theorem: If P satisfies the criteria of the previous theorem, then C is regular, that is

$$\sup_{x \in C} \mathbb{E}_x \tau_B < \infty \qquad \forall B \in \mathcal{E}^+$$

Proof: Essentially this follows from formula (□) which reads that

$$\mathbb{E}_x \tau_B \le V(x) - 1 + b\, \mathbb{E}_x \sum_0^{\tau_B} 1_C(X_n)$$

Since $C \in \mathcal{E}^+$ is small, there is k, $\delta > 0$ such that

$$P_x^k(B) \ge \delta \qquad \forall x \in C$$

Each time ~~the~~ process visits C there is a lottery involved, whether B will be visited k steps later. If the success probability is taken to be δ, these lotteries will be independent and hence the expected number of lotteries before success will be $\frac{1}{\delta}$.

Hence

$$\mathbb{E}_x \sum_0^{\tau_B} 1_C(X_n) \le \frac{1}{\delta} \sup_{x \in C} \mathbb{E}_x \tau_C + k$$

for all $x \in C$.

(proof not entirely accurate).

Theorem if P satisfies the criterion of the previous theorem, the sublevel sets of V are all small.

proof: Let $C_n = \{x \mid V(x) \le n\}$. Using (V) we get that for $x \in C_n$

$$\delta_x \tau_c \le V(x) + b\,\delta_x \sum_0^{\tau_c} 1_{(x_n \in C)} - 1$$

$$\le n + 2b - 1$$

Hence

$$P_x(\tau_c \ge m) \le \frac{\delta_x \tau_c}{m} \le \frac{n+2b-1}{m} < \frac{1}{2}$$

if m is large enough, irrespective of x. Thus

$$P_x^k(C) \ge \frac{1}{2m} \quad \text{for at least one } k = 1, \dots, m \quad \forall x \in C_n$$

This implies that $C_n \overset{a}{\leadsto} C$ when the sampling distribution is the uniform distribution on $1 \dots m$. Hence it follows that C_n is petite. And aperiodicity implies that C_n is small.

∎

corollary: $X = \cup C_n$, each C_n regular.

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