November 16 2004

Supplementary proofs on recurrence / transience

Recall the definition

$$H_{Ax}(B) = \sum_{n=1}^{\infty} {}_A P_x^n(B)$$

The taboo-potential $H_{Ax}(\cdot)$ is a kernel related to the taboo probabilities ${}_A P_x^n(\cdot)$ in the same way the potential

$$U_x(B) = \sum_{n=1}^{\infty} P_x^n(B)$$

is related to the ordinary n-step probabilities.

<u>Lemma</u>: Let P be irreducible. If $A, B \in \mathcal{E}^+$, then

$$\{x \in A \mid H_{Ax}(B) > 0\} \in \mathcal{E}^+$$

<u>Proof</u>: Let us denote

$$\tilde{B} = \{x \in A \mid H_{Ax}(B) > 0\} .$$

Recall the last exit decomposition

$$U_x(B) = \int_A H_{Ay}(B)\, \partial U_x(y) + H_{Ax}(B)$$

Take $x \in \mathcal{X}$. Due to irreducibility, we know that $U_x(B) > 0$, and hence one of the two terms in the above last exit decomposition must be positive. The two possibilities correspond to

$x \in B$ (the latter term is positive)

$U_x(A \cap B) > 0$ (the integral is positive)

In particular $x \in B$ or $U_x(B) > 0$.

Now consider the trivial decomposition

$$U_x(B) = P_x(B) + \int_{B^c} U_y(B)\, d P_x(y)$$

The integrand in the integral is positive by the above argument, so the integral is positive unless $P_x(B^c) = 0$. In which case of course $P_x(B) = 1$

This argument shows that $U_x(B) > 0 \quad \forall x$.

Let ψ be a maximal irreducibility measure. It follows from Meyn and Tweedie's Prop 4.2.2 (iii) that if $\mathcal{O}$ is a ψ-nullset, then

$$\{x \mid U_x(\mathcal{O}) > 0\}$$

is also a ψ-null-set. But we have just shown the above set to be the entire space $\mathcal{X}$ which is certainly <u>not</u> a nullset. Thus $\psi(\mathcal{O}) > 0$, which was what we set out to prove.

$\square$

<u>Theorem 1</u>: Let P be irreducible. If there are sets $C, D \in \mathbb{E}^+$ such that

$$L_x(D) < 1 \qquad \forall x \in C$$

then P is transient.

<u>Proof</u>: If we let

$$C_n = \{x \in C \mid L_x(D) \leq 1 - \tfrac{1}{n}\}$$

Then $C_1 \subseteq C_2 \subseteq \dots$ and $\bigcup_1^\infty C_n = C$. In particular we must have that $C_n \in \mathbb{E}^+$ for n large enough. By replacing C with C_n for such an n, we may assume that $C, D \in \mathbb{E}^+$ and that

$$L_x(D) \leq 1-\varepsilon \qquad \forall x \in C$$

for a suitable $\varepsilon > 0$.

Suppose for a minute that $\psi(C \cap D) > 0$, where ψ is a maximal irreducibility measure. Then for any $x \in C \cap D$ we in particular have that $x \in C$ and so

$$L_x(C \cap D) \leq L_x(D) \leq 1-\varepsilon$$

Meyn and Tweedie prop 8.3.1 (iii) shows that $C \cap D$ must be uniformly transient, and as we have assumed $C \cap D \in \mathbb{E}^+$, this shows that P can not be recurrent. Hence it must be transient, which is what we wanted to prove.

So the interesting case is when $\psi(C \cap D) = 0$.
In that case we may replace C by $C \setminus D$ and obtain the situation: C, D are disjoint $\mathbb{E}^+$-sets, and

$$L_x(D) \leq 1-\varepsilon \qquad \forall x \in C.$$

According to the lemma we have that

$$\psi\{x \in D \mid H_{Dx}(C) > 0\} > 0$$

It follows from the definition of the taboo kernel that

$$\{x \in D \mid H_{Dx}(C) > 0\} = \bigcup_{n \geq 1}^{\infty} \{x \in D \mid {}_D P_x^n(C) > 0\}$$

so there must be an n such that

$$\psi\{x \in D \mid {}_D P_x^n(C) > 0\} > 0$$

By taking m large enough, and appealing to the continuity properties of ψ, we find that

$$D_0 = \{x \in D \mid {}_D P_x^n(C) \geq \tfrac{1}{m}\}$$

is an $\mathbb{E}^+$-set. The remains of the proof will establish that D_0 is uniformly transient.

We have the general decomposition

$$L_x(A) = (1 - {}_A P_x^n(A^c)) + \int_{A^c} L_y(A)\, d\, {}_A P_x^n(y)$$

where the first term represents the possibility of hitting A prior to time n and the integral represents the possibility of hitting A after time n without a hit before. We may rephrase this as

$$1 - L_x(A) = \int_{A^c} (1 - L_y(A))\, d\, {}_A P_x^n(y)$$

Using this on D_0 we have that

$$1 - L_x(D_0) = \int_{D_0^c} (1 - L_y(D_0))\, d\, {}_{D_0} P_x^n(y)$$

$$\geq \int_C (1 - L_y(D_0))\, d\, {}_{D_0} P_x^n(y)$$

$$\geq \varepsilon \; {}_{D_0} P_x^n(C)$$

$$\geq \varepsilon \; {}_D P_x^n(C) \geq \frac{\varepsilon}{m} \qquad \forall x \in D_0$$

phrased differently: $L_x(D_0) \leq 1 - \frac{\varepsilon}{m} \quad \forall x \in D_0$

and so D_0 is uniformly transient.

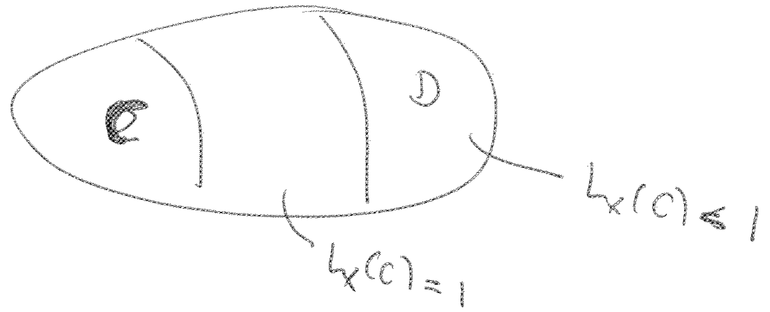
$\square$

Theorem 2   If P is irreducibl and transient and if $C \in \mathcal{B}^+$ is small, thn

$$D = \{ x \notin C \mid L_x(C) < 1 \}$$

is an $\mathcal{B}^+$-set.

proof:

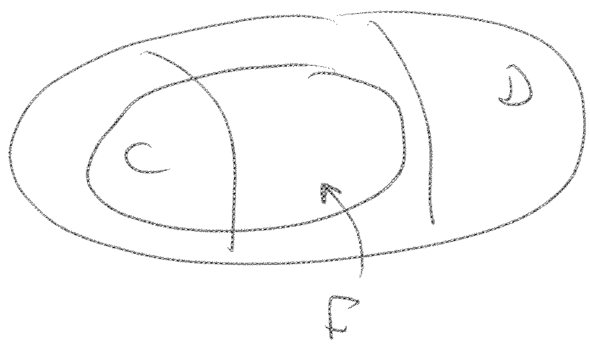


Assum $\psi(D) = 0$ whr ψ is a maximal irreducibility measure. Thn

$$\bar{D} = \{ x \mid U_x(D) > 0 \}$$

must also have ψ-measure zero, according to Meyn and Tweedie prop. 4.2.2 (iii).

This means that $\bar{D}^c$ is full, and thus it contains an absorbing set F.



Let $C_0 = C \cap F$. Since F has full ψ-measure, we have that $\psi(C_0) = \psi(C) > 0$. Take $x \in C_0$.

$$L_x(C_0) = P_x(C_0) + \int_{C_0^c} L_y(C_0)\, dP_x(y)$$

$$= P_x(C_0) + \int_{F \setminus C_0} L_y(C_0)\, dP_x(y)$$

since F is absorbing. On $F \setminus C_0$ we have that $L_y(C) = 1$. But as F is absorbing, this must in fact mean that $L_y(C_0) = 1$. And so

$$L_x(C_0) = P_x(C_0) + \int_{F \setminus C_0} 1\, dP_x(y) = P_x(F)$$

And this is 1, as F is absorbing.

The conclusion is that C_0 is a Harris-recurrent $\mathbb{E}^+$-set.

It is also small, since C is small. Meyn and Tweedie's Theorem 8.3.5 (i) shows that this is in conflict with the assumption of transience. Hence the extra assumption that $\psi(D) = 0$ must be wrong, and we conclude $D \in \mathbb{E}^+$.

$\square$