November 21, 2004
EH

Markov Chains on general state spaces fall 2004

Paper 2

Formal requirements: the paper must be handed in no later that Monday December 13 2004 at noon. The paper must be given to me personally.

The paper can be written in danish or english. It is strongly encouraged that the paper is produced eletronically.

It is not probibited that participants cooperate in the problem solving phase - indeed, it is encouraged. But the final paper must be an individual piece of work.

Ernst Hansen

1. Establishing aperiodicity

Let $P = (P_x)_{x \in \mathcal{X}}$ be a Markov kernel on a space $(\mathcal{X}, \mathbb{E})$. A frequently encountered situation is where some iterate P^n of the Markov kernel consists of measures that are absolutely continuous with respect to some common reference measure μ. That is,

$$P_x^n(A) = \int_A g_x(y)\, d\mu(x) \qquad \text{for all} \quad A \in \mathbb{E}, x \in \mathcal{X}\,, \tag{1}$$

where $(x, y) \mapsto g_x(y)$ is a measurable map.

Problem 1. Show that P is irreducible if $g_x(y) > 0$ for every x and y.

Problem 2. Suppose P has a d-cycle, as described in Meyn and Tweedie, p. 117. Show that if $g_x(y) > 0$ for every x and y, then d must be a divisor in n.

Problem 3. Suppose P satisfies (1) for n **and** for $n + 1$, with densities that are strictly positive. Show that P is aperiodic.

Variations of this argument will be possible in many situations where the densities are not strictly positive.

2. AR(1)-processes in one dimension

Consider the autoregressive scheme on $\mathbb{R}$ of the form

$$X_{n+1} = \rho X_n + W_{n+1}$$

where $W_1, W_2, \ldots$ are independent and identically distributed stochastic variables. The W's have a density ϕ with respect to the Lebesgue measure m. We suppose that ϕ is continuous, strictly positive and symmetric around 0. We furthermore suppose that the W's have first moment - in that case they must necessarily have mean 0.

All these assumptions are satisfied in the classical case where the W's follows a $\mathcal{N}(0, \sigma^2)$-distribution, but they are of course also satisfied in many other situations.

Problem 4. Show that the Markov kernel P corresponding to this autoregressive scheme is irreducible. Show that all open sets are $\mathbb{B}^+$-sets. Show that P is aperiodic.

Problem 5. Show that all compact sets are small.

Problem 6. Show that P is in fact strongly aperiodic.

Problem 7. Show that if $|\rho| < 1$ then P is recurrent.
Hint: use the drift function $V(x) = |x|$.

The remaining problems in this section will establish the opposite result: if $|\rho| > 1$ then P is transient. The random walk case ($\rho = \pm 1$) is somewhat more involved, and will not be considered here.

So assume that $\rho > 1$ and consider the drift function

$$V(x) = \frac{|x|}{1 + |x|} \qquad \text{for } x \in \mathbb{R}\,.$$

Problem 8. Show that for $x > 0$

$$|PV(x) - V(\rho x)| \leq 2V'(x) + 2\int_{(\rho-1)x}^{\infty} \phi(u)\, du\,.$$

Problem 9. Show that

$$x\Big(PV(x) - V(\rho x)\Big) \to 0 \quad \text{for} \quad x \to \pm\infty\ .$$

Problem 10. Show that

$$x\Big(PV(x) - V(x)\Big) \to \frac{\rho - 1}{\rho} \qquad \text{for } x \to \pm\infty\,.$$

Problem 11. Conclude that P is transient if $|\rho| > 1$.
Hint: for the case of negative ρ, switch focus to P^2.

3. AR(1)-processes in higher dimension

Consider the autoregressive scheme on $\mathbb{R}^k$ of the form

$$X_{n+1} = RX_n + W_{n+1}$$

where R is a $k \times k$ matrix, and where $W_1, W_2, \ldots$ are independent and $\mathcal{N}(0, \Sigma)$-distributed.

Problem 12. Show that if Σ is positive definite, then the Markov kernel P associated to the autoregressive scheme is irreducible and aperiodic, and all compact sets are small.

Problem 13. Show that if there is some p such that

$$\sum_{i=0}^{p-1} R^i \Sigma (R^i)^T$$

is positive definite then P is irreducible and aperiodic, and all compact sets are small.

Hint: look at the p-skeleton - and possibly also at the $p+1$-skeleton.

Problem 14. Show that the stacking of a onedimensional AR(2)-proces gives a Markov kernel on $\mathbb{R}^2$ which is irreducible and aperiodic and with all compact sets small. (The notion of stacking in the context of AR(2)-processes is explained on p. 33 of my notes.)

Return to the proper AR(1)-processes in higher dimensions. We suppose that the condition in problem 13 is satisfied.

Problem 15. Suppose R has a real eigenvalue λ wich satisfies $|\lambda| > 1$. Show that P is transient.

Hint: let y be a **left eigenvector** for R corresponding to λ. This means that $y \neq 0$ and that $y^T R = \lambda y^T$. Consider the drift function $V(x) = V_0(y^T x)$ where $V_0(t) = \frac{|t|}{1+|t|}$.

If all eigenvalues λ of R (even the complex ones, if there are such) satisfies that $|\lambda| < 1$, it holds that $R^n \to 0$ for $n \to \infty$. This is simple to see if all eigenvalues are real and if R is diagonalizable, but it is in fact true without these extra assumptions (you do not have to show this).

Problem 16. Show that P is recurrent if all eigenvalues λ of R satisfies that $|\lambda| < 1$.

Hint: Use the drift function $V(x) = \|x\|^2$. Show that

$$P^n V(x) = \|R^n x\|^2 + \mathrm{Tr} \sum_{i=0}^{n-1} R^i \, \Sigma \, (R^i)^T \, .$$

It might come in handy to observe that for a variable Z in $\mathbb{R}^k$ with second moment and with mean 0, it holds that

$$E\|Z\|^2 = \mathrm{Tr}\, V Z \, .$$