September 20, 2004
EH

Markov Chains on general state spaces fall 2004

Paper 1

Formal requirements: the paper must be handed in no later that Monday October 11 2004 at noon. Note that there is no lecture on that day, as it is in the week of the autumn break. The paper must be given to me personally.

The paper can be written in danish or english. It is strongly encouraged that the paper is produced eletronically.

It is not probibited that participants cooperate in the problem solving phase - indeed, it is encouraged. But the final paper must be an individual piece of work.

Ernst Hansen

1. A Markov chain after a stopping time

Let $X_0, X_1, \ldots$ be a time homegenous Markov chain on a space $(\mathcal{X}, \mathbb{E})$ with one-step transition probabilities $(\hat{P}_x)_{x \in \mathcal{X}}$. Let τ be an almost surely finite stopping time.

PROBLEM 1. Show that $\tau + k$ is a stopping time for each $k = 1, 2, \ldots$.

PROBLEM 2. Show that $X_\tau, X_{\tau+1}, X_{\tau+2}, \ldots$ is a Markov chain.

PROBLEM 3. Show that $X_\tau, X_{\tau+1}, X_{\tau+2}, \ldots$ is time homogenous with the same one-step probabilities as the original chain.
Hint: There is a simple argument via update schemes. The proof of 'strong homogeneity implies weak homogeneity' might be inspirational.

2. A renewal proces which is markovian

In general a renewal proces does **not** have the Markov property. But there are certain situations where the notorious 'memoryless property' of exponential and geometric distributions chime in, and guarantees Markovian properties.

Let $Z_1, Z_2, \ldots$ be independent Bernoulli random variables, all with the same succesprobability $p \in (0, 1)$. Define the associated random walk

$$M_n = \sum_{k=1}^{n} Z_k \qquad \text{for } n = 0, 1, 2, \ldots$$

Define the random times

$$T_n = \inf\{m \in \mathbb{N}_0 \mid M_m \geq n\} \qquad \text{for } n = 0, 1, 2, \ldots$$

and the 'waiting times'

$$Y_n = T_n - T_{n-1} \qquad \text{for } n = 1, 2, \ldots$$

PROBLEM 4. Show that T_n is an almost surely finite stopping time, adapted to the random walk $M_0, M_1, \ldots$. Show also that $T_0 \leq T_1 \leq \ldots$

PROBLEM 5. Show that $Y_1, Y_2, \ldots$ are independent.

PROBLEM 6. Show that $Y_1, Y_2, \ldots$ are identically distributed, and that they follow a geometric distribution with succesprobability p.

PROBLEM 7. Show that the renewal proces generated by the waiting times $Y_1, Y_2, \ldots$ is simply $M_0, M_1, M_2, \ldots$

PROBLEM 8. Collect the pieces to a proof of the general statement: if $Y_1, Y_2, \ldots$ are independent variables, all geometrically distributed with succesprobability p, then the associated renewal proces is a Markov chain.