Chapter 3

Taboo probabilites

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We start out by a Markov kernel $P = (P_x)_{x \in \mathcal{X}}$ on a space $(\mathcal{X}, \mathbb{E})$. For each $n \in \mathbb{N}$ we construct an $\mathcal{X}$-Markov kernel $\mathcal{P}^n = (\mathcal{P}^n_x)_{x \in \mathcal{X}}$ on $\mathcal{X}^n$ by an inductive procedure. To start the induction we let

$$\mathcal{P}^1_x(A) = P_x(A) \qquad \text{for all } \; x \in \mathcal{X},\, A \in \mathbb{E}\,.$$

In the induction step we define the probability measure $\mathcal{P}^{n+1}_x$ as the integration of $\mathcal{P}^n$ with respect to P_x. That is, $\mathcal{P}^{n+1}_x$ is determined by the formula

$$\mathcal{P}^{n+1}_x(A \times B) = \int_A \mathcal{P}^n_y(B)\, dP_x(y) \qquad \text{for all } A \in \mathbb{E}, B \in \mathbb{E}^{\otimes n}\,. \tag{3.1}$$

It is easy to prove that the collection $\mathcal{P}^{n+1} = (\mathcal{P}^{n+1}_x)_{x \in \mathcal{X}}$ is in fact a Markov kernel, and hence the induction can proceed.

Theorem 3.1 *Let $X_0, X_1, \ldots$ be a time homogenous Markov chain, with one-step transition probability P. Then*

$$(X_1, \ldots, X_n) \mid X_0 \quad \overset{\mathcal{D}}{=} \quad \mathcal{P}^n\,.$$

Proof: Induction on n. Note that $X_1, X_2, \ldots$ is itself a time homogenous Markov chain with one-step transition probabilities P. In the induction step $n \to n+1$ we hence know that $\mathcal{P}^n$ is the conditional distribution of $(X_2, \ldots, X_{n+1})$ given X_1. But as X_0 and $(X_2, \ldots, X_{n+1})$ is conditionally independent given X_1, we may with a slight misuse of notation consider $\mathcal{P}^n$ as the conditional distribution of $(X_2, \ldots, X_{n+1})$ given (X_0, X_1). As the conditional distribution of X_1 given X_0 is P, we can integrate and get the conditional distribution of $(X_1, \ldots, X_{n+1})$ given X_0. And this conditional distribution turns out to satisfy (3.1), hence it equals $\mathcal{P}^{n+1}$.

□

Observe that if $X_0 \equiv x$, then we have that

$$P((X_1, \ldots, X_n) \in B) = \int \mathcal{P}^n_y(B)\, dX_0(P)(y) = \mathcal{P}^n_x(B)\,,$$

so we can interpret $\mathcal{P}^n_x$ as the marginal distribution of $(X_1, \ldots, X_n)$ if the chain i starteed deterministically in x. This interpretation is very useful.

Definition 3.2 *The* **taboo probabilities** *are*

$${}_AP^n_x(B) = \mathcal{P}^n_x(A^c \times A^c \times \ldots \times A^c \times B) \qquad \text{for } A, B \in \mathbb{E}\,.$$

Induction formula:

$$\begin{aligned} {}_AP^{n+1}_x(B) &= \mathcal{P}^{n+1}_x(A^c \times A^c \times \ldots \times A^c \times B) \\ &= \int_{A^c} \mathcal{P}^n_x(A^c \times A^c \times \ldots \times A^c \times B)\, dP_x(y) \\ &= \int_{A^c} {}_AP^n_x(B)\, dP_x(y)\,. \end{aligned}$$

Interpretation: if $X_0, X_1, \ldots$ is a time homogenous Markov chain with one-step transition probability P then

$$P(X_1 \notin A, \ldots, X_{n-1} \notin A, X_n \in B) = \int {}_AP^n_x(B)\, dX_0(P)(x)\,.$$

If $X_0 \equiv x$ we even get that

$$P(X_1 \notin A, \ldots, X_{n-1} \notin A, X_n \in B) = {}_AP^n_x(B)\,.$$

Definition 3.3 *The* **return probability** *to a set A is*

$$L_x(A) = \sum_{n=1}^{\infty} {}_A P_y^n(A)$$

Theorem 3.4 *If $X_0, X_1, \ldots$ is a time homogenous Markov chain, with one-step transition probability P, then*

$$P\left(\bigcup_{n=1}^{\infty} (X_n \in A)\right) = \int L_x(A)\, dX_0(P)(x)$$

Proof: Straight forward computation.

$\square$

In particular if $X_0 \equiv x$ we see that

$$P\left(\bigcup_{n=1}^{\infty} (X_n \in A)\right) = L_x(A)$$