Chapter 4

Atoms

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Definition 4.1 *Let $P = (P_x)_{x \in \mathcal{X}}$ be an $\mathcal{X}$-kernel on $(\mathcal{X}, \mathbb{E})$. A set $\alpha \in \mathbb{E}$ is called an* **atom** *for P if there is a probability measure ν such that*

$$P_x(A) = \nu(A) \qquad \text{for all } x \in \alpha\ ,\ A \in \mathbb{E}\,. \tag{4.1}$$

Another way to state the definition is to say that $P_{x_1} = P_{x_2}$ whenever $x_1, x_2 \in \alpha$ - the introduction of ν is just to give a way to refer to this common value of P_x for $x \in \alpha$ without introducing an irrelevant point x.

A single point x is always an atom, but it is usually quite irrelevant: Markov chains on general spaces will typically **never** hit a prespecified singleton. The concept becomes interesting if we can find non-negligeable atoms. If P is irreducible, a non-negligeable atom could mean $\alpha \in \mathbb{E}^+$ - this is usually refered to as an **accessible atom**. Or we can even introduce the concept of a **Harris recurrent atom**, wich is an atom α that satisfies that

$$L_x(\alpha) = 1 \qquad \text{for all } x \in \mathcal{X}\,.$$

We say that an update function $\phi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ and an error distribution μ on $\mathcal{Y}$ generate P if the Markov chain

$$X_{n+1} = \phi(X_n, Y_{n+1}) \qquad \text{for } n = 0, 1, \ldots$$

gets one-step transition proability P whenever $Y_1, Y_2, \ldots$ are iid. variables with distribution μ. If the update function satisfies

$$\phi(x_1, y) = \phi(x_2, y) \qquad \text{for all } x_1, x_2 \in \alpha, y \in \mathcal{Y}$$

then α clearly is an atom for P. We tend to use the word **vissible atom** in this case, because the atomic property immediatly signifies itself. But an atom need not be vissible in a given update scheme - whether a set is an atom or not is a property of the kernel, not of the update scheme.

If $X_0, X_1, \ldots$ is a time homogenous Markov chain on $\mathcal{X}$ with one-step transition probability P, and if α is a Harris recurrent atom for P, then the first return time τ to α is almost surely finite.

Theorem 4.2 *If $X_0, X_1, \ldots$ is a time homogenous Markov chain on $\mathcal{X}$ with one-step transition probability P, and if α is a Harris recurrent atom for P, then the first return time τ to α satisfies that*

$$(X_{\tau+1}, X_{\tau+2}, \ldots) \perp\!\!\!\perp \mathbb{F}_\tau \,. \tag{4.2}$$

Proof: From the strong Markov property wer know that

$$X_{\tau+1}, X_{\tau+2}, \ldots \perp\!\!\!\perp \mathbb{F}_\tau \mid X_\tau \,. \tag{4.3}$$

Let us find the conditional distribution of $X_{\tau+1}$ given X_τ. We might focus on the fact that $X_\tau, X_{\tau+1}, X_{\tau+2}, \ldots$ is a time homogenous Markov chain with one-step transition probability P, and say that this conditional distribution is P. But we may also consider the trivial kernel $Q = (Q_x)_{x \in \mathcal{X}}$ given by

$$Q_x(B) = \nu(B) \qquad \text{for all } x \in \mathcal{X},\ B \in \mathbb{E}\,,$$

where ν is the measure satisfying (4.1). The two kernels seem rather different, but they do agree on α. Note that $P(X_\tau \in \alpha) = 1$ by definition. Hence we might consider Q to be the conditional distribution of $X_{\tau+1}$ given x_τ.

It is a well-known fact (and it is trivial to demonstrate) that if the conditional distribution of Y given X is constant, then X and Y are independent. So we have demonstrated that

$$X_{\tau+1} \perp\!\!\!\perp X_\tau \,.$$

This can be rephrased as

$$X_{\tau+1} \perp\!\!\!\perp X_\tau \mid \{\emptyset, \Omega\} \,.$$

Combining with (4.3) we obtain

$$X_{\tau+1} \perp\!\!\!\perp \mathbb{F}_\tau \vee \mathbb{F}(X_\tau) \mid \{\emptyset, \Omega\} \,.$$

Or, slightly easier to read,

$$X_{\tau+1} \perp\!\!\!\perp \mathbb{F}_\tau \,. \tag{4.4}$$

From here it is easy to prove

$$(X_{\tau+1}, \ldots, X_{\tau+k}) \perp\!\!\!\perp \mathbb{F}_\tau \tag{4.5}$$

by induction on k. Suppose it is true for k. Using the strong Markov property we have that

$$X_{\tau+k+1} \perp\!\!\!\perp \mathbb{F}_{\tau+k} \mid X_{\tau+k} \,.$$

As $X_{\tau+1}, \ldots, X_{\tau+k-1}$ are all $\mathbb{F}_{\tau+k}$-measureable, we can shift them to the conditioning side, and then we can reduce $\mathbb{F}_{\tau+k}$ to $\mathbb{F}_\tau$. Hence we get that

$$X_{\tau+k+1} \perp\!\!\!\perp \mathbb{F}_\tau \mid (X_{\tau+1}, \ldots, X_{\tau+k}) \,.$$

Combining with the induction assumption (4.5) we get that

$$(X_{\tau+1}, \ldots, X_{\tau+k+1}) \perp\!\!\!\perp \mathbb{F}_\tau \tag{4.6}$$

□

Note the conclusion of the theorem: at time τ the future is independent of the past. Not merely conditional independent given the present, but truely and unconditionally independent. We refer to this independence as **regeneration**. Regeneration occurs, because there is no relevant information on the future in knowing the specific value of X_τ - all points in the atom α are in a sense the same.

Looking closer at the proof, we see that it does not really matter that τ is the first return time to α. What is needed is that τ is an almost surely finite stopping time, which satisfies $P(X_\tau \in \alpha) = 1$. So regeneration will also take place at the second return time, the third return time and so on, if it can be demonstratetd that these return times are almost surely finite.

Regeneration is crucial in the analysis of Markov chains. As is explained in example 2.28, it leads to quite transparent proofs of laws of large numbers, central limit theorems and so forth, because it allows us to break the Markov chain down into iid pieces.

Regeneration also explains the prominent role that renewal processes have in general Markov chain theory. They are not merely examples to which the theory can be applied - they are actually essential in the analysis of Markov chains with an atom. The iid. waiting times that the renewal proces is build on, will be waiting times between visits to the atom.

The reader may of course wonder if the Markov chains occurring in real life have atoms - it seems like a rather exotic property. And indeed almost no naturally occuring Markov chain has an atom. But one of the great theoretical insights in the late 20th century was that there are lots of 'unnatural' Markov chains with an atom, and that any irreducible Markov chain can be considered to be subordinate to one of these artificial chains. The natural Markov chain does perhaps not regenerate, but the artificial chain in the background does, and knowledge of this is enough to make strong conclusions about the natural chain.