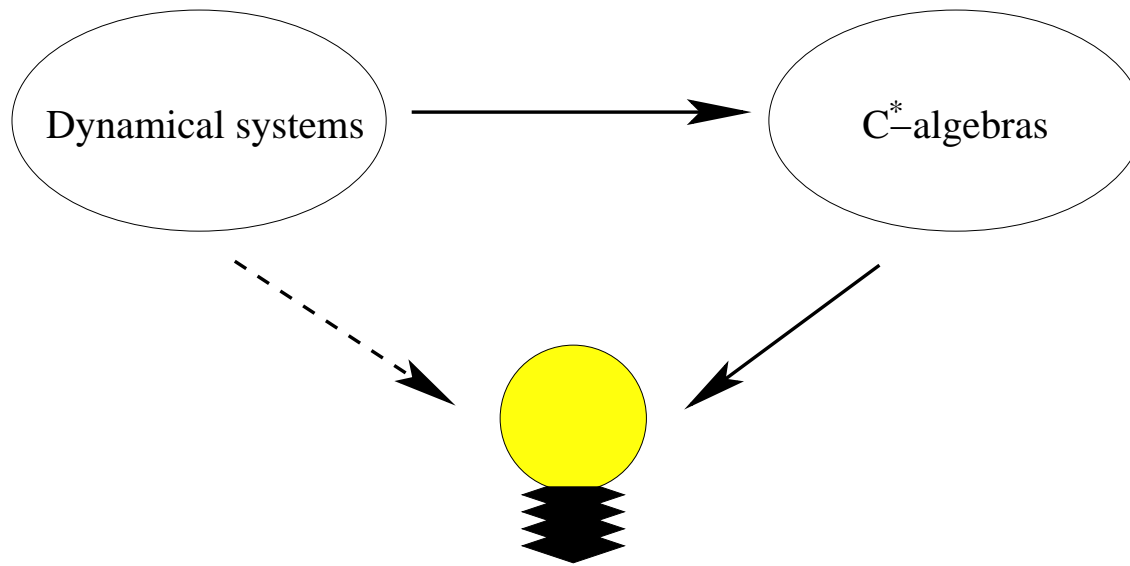


# From substitutions to tiling spaces and $C^*$ -algebras

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*M. Barge and M. Smith: Augmented Dimension Groups and Ordered Cohomology. [Private communication.]*

A substitution is a map  $\tau : \mathfrak{a} \rightarrow \mathfrak{a}^* \setminus \{\epsilon\}$  mapping letters to nonempty words in the same alphabet.

The *abelianization matrix*  $A_\tau$  counts the number of occurrences of each letter in each substitute.

**Example**  $\mathfrak{a} = \{0, 1\}$ ,  $\tau(0) = 1000$ ,  $\tau(1) = 101$ .

$$A_\tau = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Define

$$\underline{X}_\tau = \{(x_i) \in \mathfrak{a}^{\mathbb{Z}} \mid \forall i < j \exists n, a : x_{[i,j]} \text{ occurs in } \tau^n(a)\}.$$

and equip with

$$\sigma(x_n) = (x_{n+1})$$

$(\underline{X}_\tau, \sigma)$  is an example of a **shift space**. A construction by Matsumoto allows the association of a  $C^*$ -algebra  $\mathcal{O}_{\underline{X}}$  to any such shift space. We abbreviate

$$\mathcal{O}_\tau = \mathcal{O}_{\underline{X}_\tau}$$

Associated to any shift space there is a **flow space** given by product topology on

$$S\underline{X} = \frac{\underline{X} \times \mathbb{R}}{(x, t) \sim (\sigma(x), t + 1)}$$

**Definition**  $\underline{X}$  and  $\underline{Y}$  are *flow equivalent* (written  $\underline{X} \simeq_{FE} \underline{Y}$ ) when  $S\underline{X}$  and  $S\underline{Y}$  are homeomorphic in a way preserving direction in  $\mathbb{R}$ .

Since the (stabilized) Matsumoto algebra is an invariant of the shift space up to flow equivalence the process

$$\tau \curvearrowright \mathcal{O}_\tau \curvearrowright K_0(\mathcal{O}_\tau)$$

leads to a flow invariant. It is computable:

**Theorem** [Carlsen-E 02]

$$K_0(\mathcal{O}_\tau) = \varinjlim \left( \mathbb{Z}^{|\mathfrak{a}|+n_\tau} / \mathbb{Z}\tilde{\mathfrak{p}}_\tau \xrightarrow{\tilde{\mathbf{A}}} \mathbb{Z}^{|\mathfrak{a}|+n_\tau} / \mathbb{Z}\tilde{\mathfrak{p}}_\tau \xrightarrow{\tilde{\mathbf{A}}} \dots \right)$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} A_\tau & 0 \\ E_\tau & I \end{bmatrix} \quad \tilde{\mathfrak{p}}_\tau = \begin{bmatrix} 0 & \mathfrak{p}_\tau \end{bmatrix}$$

The data  $n_\tau, \mathfrak{p}_\tau, E_\tau$  is computable as exemplified below.

$$\tau(a) = abbdadcc$$

$$\tau(b) = aabdbdcc$$

$$\tau(c) = abdbcadc$$

$$\tau(d) = abbcdadc$$

$$n_{\tau} = 2, p_{\tau} = [1 \ 1], E_{\tau} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$



$$\tau(a) = abbd\mathbf{a}dcc$$

$$\tau(b) = aabd\mathbf{b}dcc$$

$$\tau(c) = abdb\mathbf{c}adc$$

$$\tau(d) = abbc\mathbf{d}adc$$

$$n_\tau = 2, p_\tau = \begin{bmatrix} 1 & 1 \end{bmatrix}, E_\tau = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

	$2-1=1$
	$2-1=1$
	$n_\tau = 2$

Crossmap:  $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} : \mathbb{Z} \rightarrow \mathbb{Z}^4$ .



$$v(a) = \text{abbcc} \color{red}{\mathbf{a}} \text{dd}$$

$$v(b) = \text{aabcc} \color{red}{\mathbf{b}} \text{dd}$$

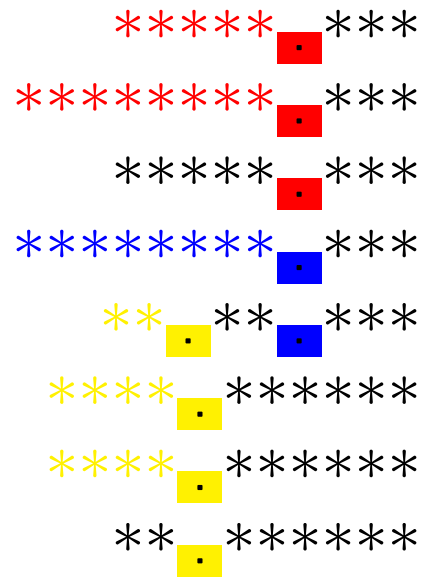
$$v(c) = \text{abbd} \color{blue}{\mathbf{c}} \text{acd}$$




$$v(d) = \text{abcb} \color{blue}{\mathbf{d}} \text{acd}$$

$$n_v = 2, p_v = \begin{bmatrix} 1 & 1 \end{bmatrix}, E_v = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

	$2-1=1$
	$2-1=1$
	$n_v = 2$

Crossmap:  $\begin{bmatrix} 0 & 0 & 2 & -1 \end{bmatrix} : \mathbb{Z} \rightarrow \mathbb{Z}^4$ .



	$3-1=2$
	$2-1=1$
	$4-1=3$
	$n_{\mathcal{T}} = 3$

$$(i) \underline{X}_\tau \simeq_{FE} \underline{X}_v$$

$$(ii) \mathcal{O}_\tau \otimes \mathbb{K} \simeq \mathcal{O}_v \otimes \mathbb{K}$$

$$(iii) [\mathbb{Z} \xrightarrow{p_\tau} \mathbb{Z}^{n_\tau} \longrightarrow K_0(\mathcal{O}_\tau)] \simeq [\mathbb{Z} \xrightarrow{p_v} \mathbb{Z}^{n_v} \longrightarrow K_0(\mathcal{O}_v)]$$

**Theorem** [Carlsen, E, Matsumoto, Restorff, Ruiz]

$$(i) \implies (ii) \iff (iii)$$

(i)  $\Rightarrow$  (ii): Matsumoto, Carlsen

(ii)  $\Rightarrow$  (iii): Carlsen, E

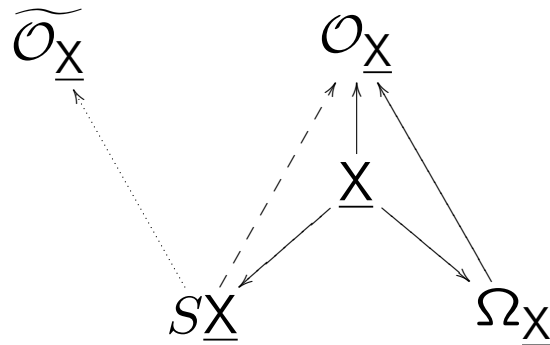
(iii)  $\Rightarrow$  (ii): E, Restorff, Ruiz

(ii)  $\not\Rightarrow$  (i): E, Restorff, Ruiz

**Theorem** [E-Restorff-Ruiz]

Let  $E_1$  and  $E_2$  be  $C^*$ -algebras each having an ideal  $B_i$  such that  $B_i$  are stable  $AF$  algebras and  $A_i = E_i/B_i$  are simple  $AT$  algebras of real rank zero. Then  $E_1 \otimes \mathbb{K} \simeq E_2 \otimes \mathbb{K}$  precisely when

$$\begin{array}{ccccccccc} K_1(E_1) & \longrightarrow & K_1(A_1) & \longrightarrow & K_0(B_1) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A_1) \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ K_1(E_2) & \longrightarrow & K_1(A_2) & \longrightarrow & K_0(B_2) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A_2) \end{array}$$



Goals:

- Explain how the data  $n_{\tau}, p_{\tau}, E_{\tau}$  can be flow invariants
- Understand the "Matsumoto relation" induced by stable isomorphism of the  $\mathcal{O}_{\underline{X}}$
- Give a better explanation of flow invariance of  $\mathcal{O}_{\underline{X}}$ ; or produce another  $C^*$ -algebra  $\widetilde{\mathcal{O}}_{\underline{X}}$  based directly on  $S\underline{X}$ .

Using Perron-Frobenius theory one can describe  $SX_\tau$  as a tiling space in  $\mathbb{R}$  with tile lengths given by the P-F eigenvector for  $A_\tau$ . For instance we have

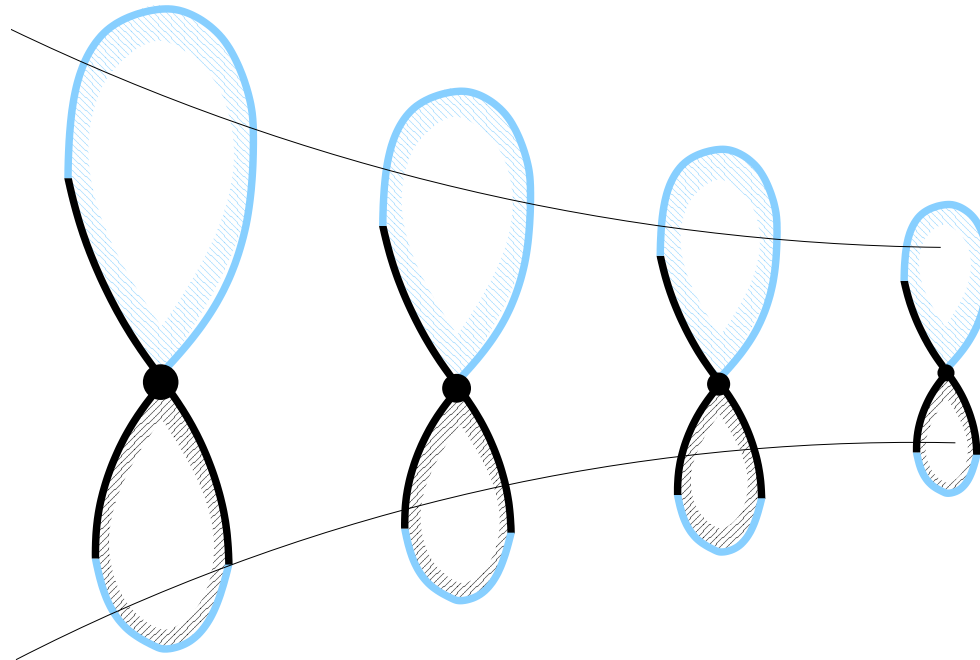
**Example**  $\alpha = \{0, 1\}$ ,  $\tau(0) = 1000$ ,  $\tau(1) = 101$ .

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ (\sqrt{5} - 1)/2 \end{bmatrix} = \frac{5 + \sqrt{5}}{2} \begin{bmatrix} 1 \\ (\sqrt{5} - 1)/2 \end{bmatrix}$$

The elements in  $SX_\tau$  can then be visualized as

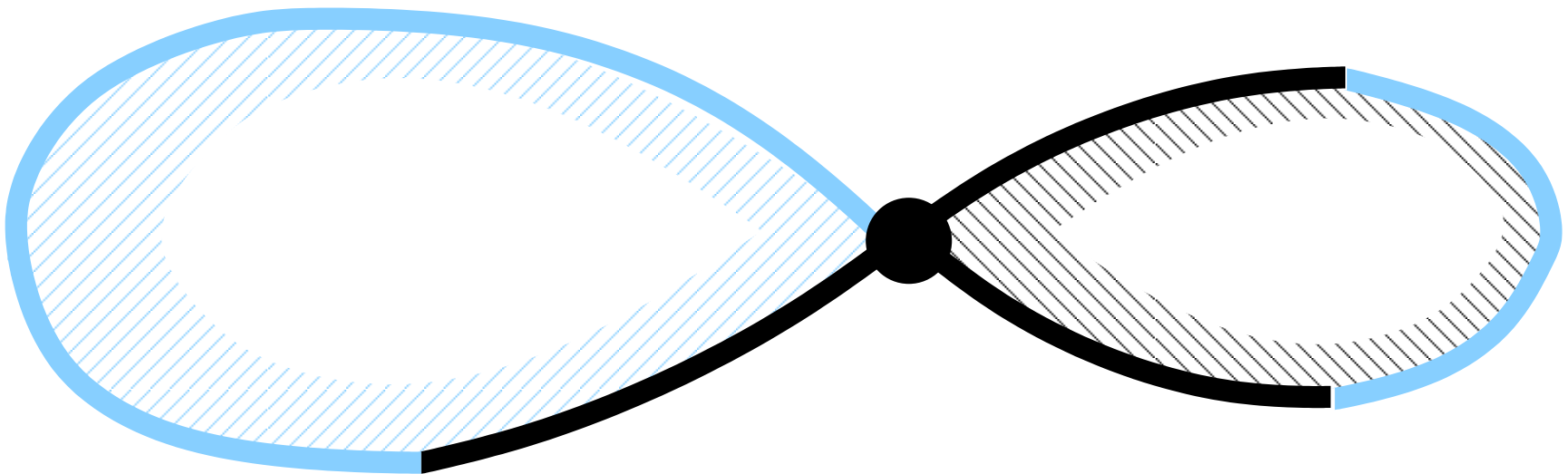


The tiling space can in turn be described as an inverse limit



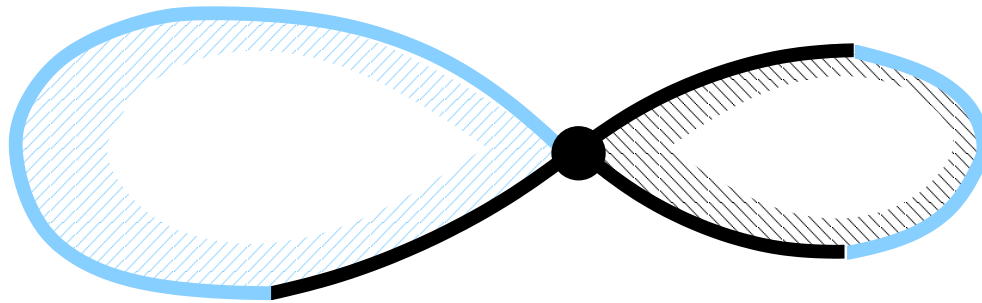
where the substitution is coded geometrically as a “map on the rose” the petals of which are precisely the tiles.

**Example**  $\alpha = \{0, 1\}$ ,  $\tau(0) = 1000$ ,  $\tau(1) = 101$ .



**Lemma**  $\varinjlim (\mathbb{Z}|a| \xrightarrow{A_\tau} \mathbb{Z}|a| \xrightarrow{A_\tau} \dots)$  is a flow invariant

**Proof** It is the first Čech cohomology group of  $S\underline{X}_\tau$ .

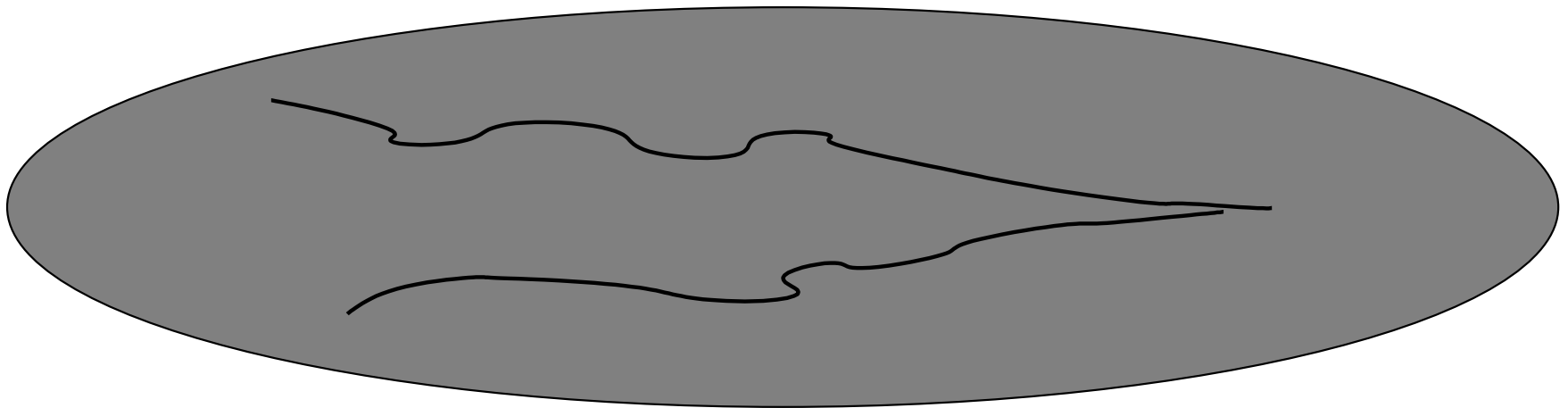


It is possible to also prove this way that the ordered group is a flow invariant.

**Forward confluency** in  $SX_{\tau}$  in the sense

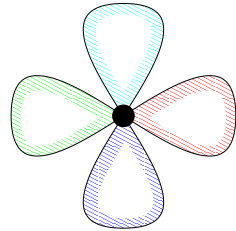
$$d([(x, a + t)], [(y, b + t)]) \rightarrow 0, \quad t \rightarrow \infty$$

is extremely rare – it only happens for finitely many  $x$  up to orbit equivalence. [The same applies to backward confluency!]



Such  $x$  are hence called **special** words.

## Example



$$\tau(a) = \text{abbd} \mathbf{a} \text{dcc}$$

$$\tau(b) = \text{aabd} \mathbf{b} \text{dcc}$$

$$\tau(c) = \text{abdb} \mathbf{c} \text{adc}$$

$$\tau(d) = \text{abbc} \mathbf{d} \text{adc}$$

$$\dots \tau^2(\text{abbd}) \tau(\text{abbd}) \text{abbd} \mathbf{a} \text{dcc} \tau(\text{dcc}) \tau^2(\text{dcc}) \dots$$

$$\dots \tau^2(\text{aabd}) \tau(\text{aabd}) \text{aabd} \mathbf{b} \text{dcc} \tau(\text{dcc}) \tau^2(\text{dcc}) \dots$$

The **forward augmented space** contains  $S\underline{X}_T$  as well as rays approximating the forward confluencies.

