# Irreducible sofic shifts and non-simple Cuntz-Krieger algebras

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# Shift spaces

Consider $\mathfrak{a}^{\mathbb{Z}}$ and

$$\sigma : \mathfrak{a}^{\mathbb{Z}} \to \mathfrak{a}^{\mathbb{Z}} \qquad \sigma(x_n) = (x_{n+1})$$

for $\mathfrak{a}$ a finite set.

**Definition** A **shift space** X is a closed and shift invariant subset of $\mathfrak{a}^{\mathbb{Z}}$. The shift is **irreducible** if for any pair of nonempty open sets $U$ and $V$, there exists $i > 0$ with $\sigma^i(U) \cap V \neq \emptyset$.

Y is a **factor** of X if there exists a surjection $\pi : \mathsf{X} \to \mathsf{Y}$ which is continuous and shift-preserving (i.e. $\sigma \circ \pi = \pi \circ \sigma$).

Y is **conjugate** to X if there exists a bijection $\chi : \mathsf{X} \to \mathsf{Y}$ which is continuous and shift-preserving. We write $\mathsf{X} \simeq \mathsf{Y}$.

**Example** Fix $S \subseteq \mathbb{N}_0$. The $S$-gap shift consists of all elements

$$\cdots 1\overbrace{0\cdots0}^{n_{-1}}1\overbrace{0\cdots0}^{n_0}1\overbrace{0\cdots0}^{n_1}1\overbrace{0\cdots0}^{n_2}1\overbrace{0\cdots0}^{n_3}1\cdots$$

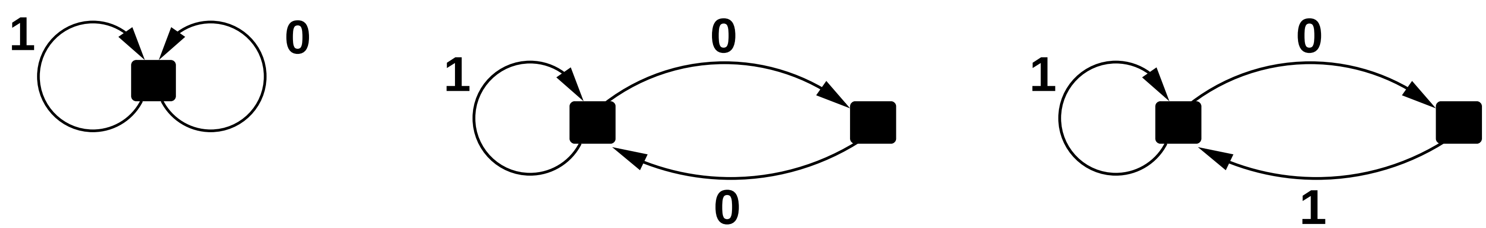
with $n_i \in S$ (infinite tails of zeros ok iff $S$ is unbounded).

$S = \mathbb{N}_0$: Full shift $(\cdots 10101010111010101101001010111 \cdots)$

$S = 2\mathbb{N}_0$: Even shift $(\cdots 10000100111000000010000100000001 \cdots)$

$S = \{0, 1\}$: Golden mean shift $(\cdots 1010111011011101111 0101 \cdots)$

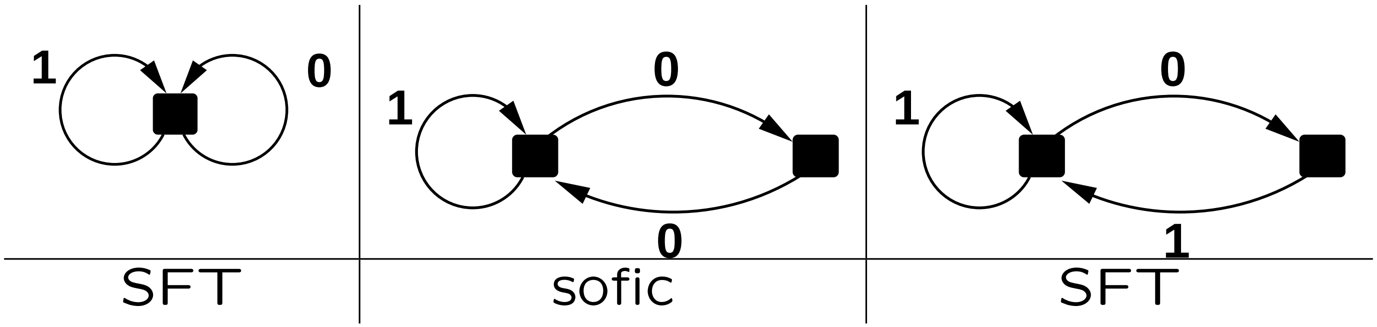
Labelled graphs $\mathcal{G} = (G, \mathcal{L})$ give rise to shift spaces $\mathsf{X}_{\mathcal{G}}$ which are irreducible when the graph is strongly connected.



Shift spaces (conjugate to) shift spaces $\mathsf{X}_{\mathcal{G}}$ given by labelled graphs are called **sofic**.

Shift spaces (conjugate to) shift spaces given by uniquely labelled graphs are called **shift of finite type (SFT)**. They are denoted $\mathsf{X}_G$ or $\mathsf{X}_A$ with $A$ the incidence matrix for $G$.

**Theorem** [Weiss]
{Factors of SFTs} = {Sofic shifts}


1
0
1
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0
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0
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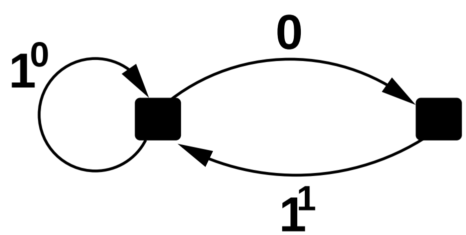
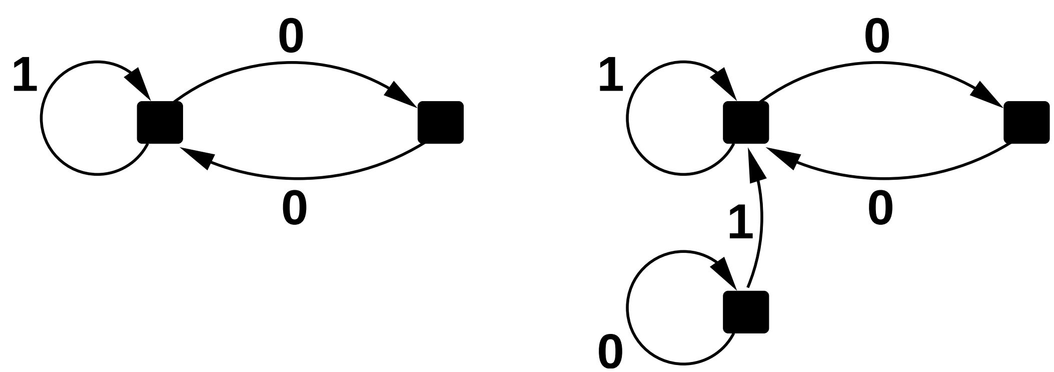
SFT
sofic
SFT


0
1
0
1
1

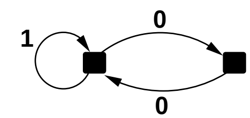
Note how a factor map $\pi : \mathsf{X}_G \to \mathsf{X}_{\mathcal{G}}$ is induced by sending each edge to its label. This is called a **cover** of $\mathsf{X}_{\mathcal{G}}$.

Several labelled graphs may represent the same sofic shift, so covers are far from unique.

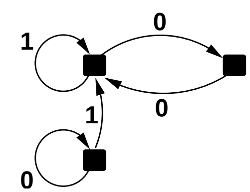


Two canonical covers are of the essence in the theory of irreducible sofic shifts:

The (right) **Fischer** cover $\overline{\overline{\mathcal{G}}}$ is the unique cover with the minimal numbers of vertices and no edges leaving the same vertex having the same label.




The (right) **Krieger** cover $\widehat{\mathcal{G}}$ identifies the right infinite words of $X_{\mathcal{G}}$ when the same finite words can be added to the left of them.



# Classification

**Problem** When is $\mathsf{X}_{\mathcal{G}} \simeq \mathsf{X}_{\mathcal{H}}$?

**Theorem** [Williams]

$\mathsf{X}_A \simeq \mathsf{X}_B$ precisely when there exist nonnegative integer matrices $D_i, E_i$ with

$$A = D_0E_0, E_0D_0 = D_1E_1, E_1D_1 = D_2E_2, \cdots, E_nD_n = B$$

**?** Is $\mathsf{X}_{\begin{bmatrix} 2 & 19 \\ 1 & 0 \end{bmatrix}} \simeq \mathsf{X}_{\begin{bmatrix} 1 & 5 \\ 4 & 1 \end{bmatrix}}$?

Generalizations to the sofic case exist but are even more hopeless to work with.

Associated to any shift space there is a **flow space** given by product topology on

$$\mathcal{S}\mathsf{X} = \frac{\mathsf{X} \times \mathbb{R}}{(x,t) \sim (\sigma(x), t+1)}$$

**Definition** X and Y are *flow equivalent* (written $\mathsf{X} \simeq_{fe} \mathsf{Y}$) when $\mathcal{S}\mathsf{X}$ and $\mathcal{S}\mathsf{Y}$ are homeomorphic in a way preserving direction in $\mathbb{R}$.

Fix $a \in \mathfrak{a}$ and $\star \notin \mathfrak{a}$ and define $\eta : \mathfrak{a}^{\mathbb{Z}} \to (\mathfrak{a} \cup \{\star\})^{\mathbb{Z}}$ as the map inserting a $\star$ after each $a$:

$$\cdots babbbaba \cdots \qquad \mapsto \qquad \cdots ba \star bbba \star ba \star \cdots$$

**Definition** The "$a \mapsto a\star$" symbol expansion of $\mathsf{X} \subseteq \mathfrak{a}\mathbb{Z}$ is the shift space $\mathsf{X}_{a \mapsto a\star} = \eta(\mathsf{X})$.

**Theorem** [Parry-Sullivan, Matsumoto]
Flow equivalence is the coarsest equivalence relation containing conjugacy and $\mathsf{X} \sim \mathsf{X}_{a \to a\star}$

**Example** $\mathsf{X}_S \simeq_{fe} \mathsf{X}_{1+S}$

**Problem** When is $\mathsf{X}_{\mathcal{G}} \simeq_{fe} \mathsf{X}_{\mathcal{H}}$?

**Theorem** [Franks]
Let $A$ and $B$ be irreducible square matrices of size $m$ and $n$, respectively. Then $\mathsf{X}_A \simeq_{fe} \mathsf{X}_B$ precisely when

$$\mathbb{Z}^m/(1-A)\mathbb{Z}^m \simeq \mathbb{Z}^n/(1-B)\mathbb{Z}^n$$

and

$$\operatorname{sgn}\det(1-A) = \operatorname{sgn}\det(1-B)$$

# Ideas from $C^*$-algebras

Any irreducible SFT is conjugate to one of the form $\mathsf{X}_A$ with $A$ an $n \times n$ $\{0,1\}$-matrix.

Such a matrix in turn defines the *Cuntz-Krieger* algebra $\mathcal{O}_A$ by generators $S_1, \cdots, S_n$ and relations

$$S_j S_j^* S_j = S_j \tag{1}$$

$$\sum_{j=1}^{n} S_j S_j^* = 1 \tag{2}$$

$$\sum_{j=1}^{n} A(i,j) S_j S_j^* = S_i^* S_i \tag{3}$$

It is a simple $C^*$-algebra when $A$ is irreducible with the so-called property (I).

The Cuntz-Krieger construction may be generalized to all (isomorphism classes of) shift spaces by the work of Matsumoto. We denote the **Matsumoto algebra** associated to the shift space X by $\mathcal{O}_{\mathsf{X}}$ (but write $\mathcal{O}_A = \mathcal{O}_{\mathsf{X}_A}$).

**Theorem** [Matsumoto, Carlsen]
$\mathsf{X} \simeq_{fe} \mathsf{Y} \Rightarrow \mathcal{O}_{\mathsf{X}} \otimes \mathbb{K} \simeq \mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}$.

**Theorem** [Rørdam]
Let $A$ and $B$ be irreducible square matrices of size $m$ and $n$, respectively. Then $\mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}$ precisely when

$$\mathbb{Z}^m/(1 - A)\mathbb{Z}^m \simeq \mathbb{Z}^n/(1 - B)\mathbb{Z}^n$$

**Theorem** [Franks]
Let $A$ and $B$ be irreducible square matrices of size $m$ and $n$, respectively. Then $\mathsf{X}_A \simeq_{fe} \mathsf{X}_B$ precisely when

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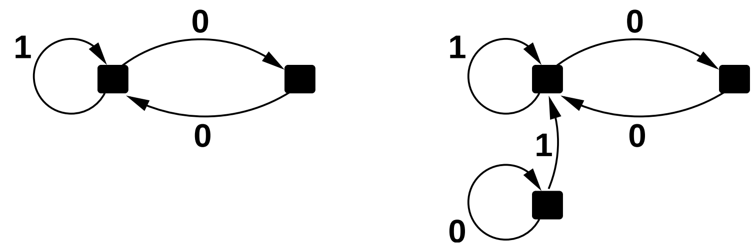
**Theorem** [Samuel; Carslen]

$\mathcal{O}_{\mathsf{X}_{\mathcal{G}}}$ is a Cuntz-Krieger algebra: $\mathcal{O}_{\mathsf{X}_{\mathcal{G}}} \simeq \mathcal{O}_{\mathsf{X}_{\widehat{\mathcal{G}}}}$

**Observation** The flow class of $\mathsf{X}_{\mathcal{G}}$ determines the flow equivalence class of $\mathsf{X}_{\widehat{\mathcal{G}}}$ and $\mathsf{X}_{\overline{\overline{\mathcal{G}}}}$. In fact, when $\mathsf{X}_{\mathcal{G}} \simeq_{fe} \mathsf{X}_{\mathcal{H}}$ we have

$$\begin{array}{ccc} S\mathsf{X}_{\overline{\overline{\mathcal{G}}}} & \dashrightarrow & S\mathsf{X}_{\overline{\overline{\mathcal{H}}}} \\ \downarrow & & \downarrow \\ S\mathsf{X}_{\mathcal{G}} & \dashrightarrow & S\mathsf{X}_{\mathcal{H}} \end{array} \qquad\qquad \begin{array}{ccc} S\mathsf{X}_{\widehat{\mathcal{G}}} & \dashrightarrow & S\mathsf{X}_{\widehat{\mathcal{H}}} \\ \downarrow & & \downarrow \\ S\mathsf{X}_{\mathcal{G}} & \dashrightarrow & S\mathsf{X}_{\mathcal{H}} \end{array}$$

# Non-simple Cuntz-Krieger algebras




Note that the even flow is an example of an irreducible sofic shift whose Krieger cover is reducible. We are hence lead to consider non-simple Cuntz-Krieger algebras. In fact:

**Proposition** [Bates-E-Pask, Johansen]
$\mathsf{X}_{\mathcal{G}}$ is non-simple for any irreducible AFT* which is not SFT. Simplicity of the Matsumoto algebra $\mathcal{O}_{\mathsf{X}_{\mathcal{G}}}$ can be determined from $\overline{\overline{\mathcal{G}}}$ and is a rather rare occurrence.

*definition supressed, but note that it is a flow invariant

Boyle and Huang classified all SFTs up to flow equivalence. And

**Theorem** [Restorff]

The collection of all six-term exact sequences

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(I) & \longrightarrow & K_0(I/J) \\ \uparrow & & & & \downarrow \\ K_1(I/J) & \longleftarrow & K_1(I) & \longleftarrow & K_0(J) \end{array} \qquad I \lhd J \lhd \mathcal{O}_A$$

provides a complete invariant for stable isomorphism of Cuntz-Krieger algebras with property (II).

Note that each ideal may occur several time, in which case the $K$-groups of the various six-term exact sequences are identified. Thus the invariant is called the "$K$-web".

Recent work by Restorff and Meyer/Nest show that the $K$-web is a complete invariant for any purely infinite $C^*$-algebra with a finite and linear ideal lattice. Furthermore, the work of Meyer/Nest indicates that the $K$-web is **not** sufficient in general.

**Project** Understand why the $K$-web is sufficient for Cuntz-Krieger algebras in spite of their ideal lattice being non-linear. Augment the invariant to arrive at a complete classification of purely infinite $C^*$-algebras with finitely many ideals.

**Theorem** [E-Restorff-Ruiz]

The automorphism group of a Cuntz-Krieger algebra $\mathcal{O}_A$ with **one** ideal $I$ up to approximate unitary equivalence is the automorphism group of

$$K_i(I) \longrightarrow K_i(\mathcal{O}_A) \longrightarrow K_i(\mathcal{O}_A/I) \longrightarrow$$

$$K_i(I;\mathbb{Z}/n) \longrightarrow K_i(\mathcal{O}_A;\mathbb{Z}/n) \longrightarrow K_i(\mathcal{O}_A/I;\mathbb{Z}/n) \longrightarrow$$

$$\mathcal{K}(\mathcal{O}_A;n)$$

where $\mathcal{K}(-;n)$ is a covariant functor defined using Kirchberg's ideal-related $KK$-theory.

# Flow classification

**Definition** The **multiplicity set** of a cover $\pi : \mathsf{Y} \to \mathsf{X}$ is

$$M(\pi) = \{y \in \mathsf{Y} \mid |\pi^{-1}(\pi(y))| > 1\}$$

When $\pi$ the Fischer cover of an AFT shift, $M(\pi)$ is a shift space in its own right.

**Theorem** [Boyle-Carlsen-E]
Let $\gamma : \mathsf{X}_{\overline{\overline{\mathcal{G}}}} \to \mathsf{X}_{\mathcal{G}}$ and $\eta : \mathsf{X}_{\overline{\overline{\mathcal{G}}}} \to \mathsf{X}_{\mathcal{G}}$ be the Fischer covers of two AFT and strictly sofic shift spaces $\mathsf{X}_{\mathcal{G}}$ and $\mathsf{X}_{\mathcal{H}}$. Then $\mathsf{X}_{\mathcal{G}}$ and $\mathsf{X}_{\mathcal{H}}$ are flow equivalent exactly when the following conditions hold:

(1) $\mathsf{X}_{\overline{\overline{\mathcal{G}}}} \simeq_{fe} \mathsf{X}_{\overline{\overline{\mathcal{H}}}}$

(2)
$$\begin{array}{ccc} SM(\gamma) & \dashrightarrow & SM(\eta) \\ \downarrow & & \downarrow \\ S\gamma(M(\gamma)) & \dashrightarrow & S\eta(M(\eta)) \end{array}$$

**Definition** The sofic shift $\mathsf{X}_{\mathcal{G}}$ with Fischer cover $\gamma : \mathsf{X}_{\overline{\overline{\mathcal{G}}}} \to \mathsf{X}_{\mathcal{G}}$ is said to be **near Markov** if $M(\gamma)$ is finite and contained in the set of periodic points of $\mathsf{X}_{\overline{\overline{\mathcal{G}}}}$.

Arranging the elements of $M(\gamma)$ according to their periods and images:

$$\begin{gathered}\gamma(x_1^1) = \cdots = \gamma(x_n^1)\\ \gamma(x_1^2) = \cdots = \gamma(x_n^1)\\ \vdots\end{gathered}$$

with $\sigma(x_1^1) = x_1^2$, $\sigma(x_1^2) = x_1^3, \ldots$ until $\sigma(x_1^k) = x_1^1$ we get a **signature** which is the multiset of pairs $(k, n)$.

**Theorem** [Boyle-Carlsen-E]

A near Markov shift $\mathcal{G}$ is classified up to flow equivalence by the flow class of $\overline{\overline{\mathcal{G}}}$ and by its signature.

The machinery of **twistwise flow equivalence** developed by Boyle and Sullivan to classify skew product extensions of irreducible SFTs give computable invariants in carefully selected AFT cases. The invariant involves $K_1(\mathbb{Z}G)$ for $G = \mathbb{Z}/2$.

**Problem** Classify the sofic $S$-gap shifts.