

Irreducible sofic shifts and non-simple Cuntz-Krieger algebras

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Texas A&M, August 5, 2008

Shift spaces

Consider $\mathfrak{a}^{\mathbb{Z}}$ and

$$\sigma : \mathfrak{a}^{\mathbb{Z}} \rightarrow \mathfrak{a}^{\mathbb{Z}} \quad \sigma(x_n) = (x_{n+1})$$

for \mathfrak{a} a finite set.

Definition A **shift space** X is a closed and shift invariant subset of $\mathfrak{a}^{\mathbb{Z}}$. The shift is **irreducible** if for any pair of nonempty open sets U and V , there exists $i > 0$ with $\sigma^i(U) \cap V \neq \emptyset$.

Y is a **factor** of X if there exists a surjection $\pi : X \rightarrow Y$ which is continuous and shift-preserving (i.e. $\sigma \circ \pi = \pi \circ \sigma$).

Y is **conjugate** to X if there exists a bijection $\chi : X \rightarrow Y$ which is continuous and shift-preserving. We write $X \simeq Y$.

Example Fix $S \subseteq \mathbb{N}_0$. The S -gap shift consists of all elements

$$\dots 1 \overbrace{0 \dots 0}^{n_{-1}} 1 \overbrace{0 \dots 0}^{n_0} 1 \overbrace{0 \dots 0}^{n_1} 1 \overbrace{0 \dots 0}^{n_2} 1 \overbrace{0 \dots 0}^{n_3} 1 \dots$$

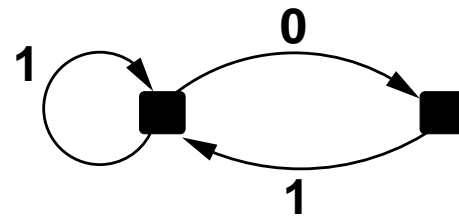
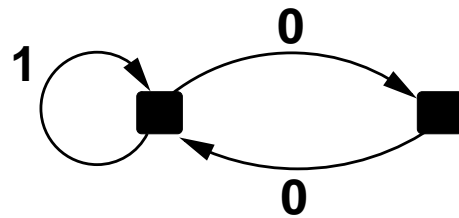
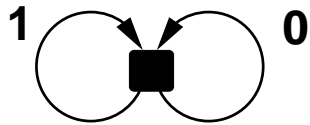
with $n_i \in S$ (infinite tails of zeros ok iff S is unbounded).

$S = \mathbb{N}_0$: Full shift $(\dots 1010101011101010110100101011 \dots)$

$S = 2\mathbb{N}_0$: Even shift $(\dots 100001001110000001000010000001 \dots)$

$S = \{0, 1\}$: Golden mean shift $(\dots 10101110110111011110101 \dots)$

Labelled graphs $\mathcal{G} = (G, \mathcal{L})$ give rise to shift spaces $X_{\mathcal{G}}$ which are irreducible when the graph is strongly connected.

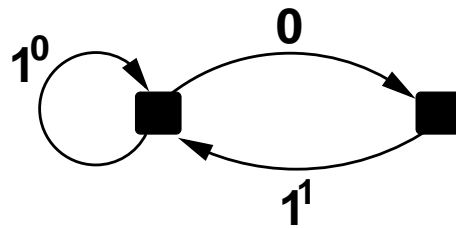
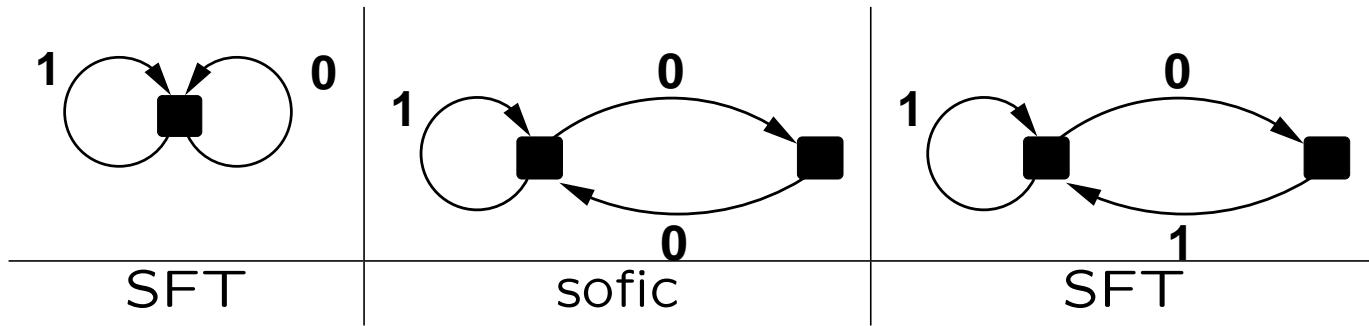


Shift spaces (conjugate to) shift spaces X_G given by labelled graphs are called **sofic**.

Shift spaces (conjugate to) shift spaces given by uniquely labelled graphs are called **shift of finite type (SFT)**. They are denoted X_G or X_A with A the incidence matrix for G .

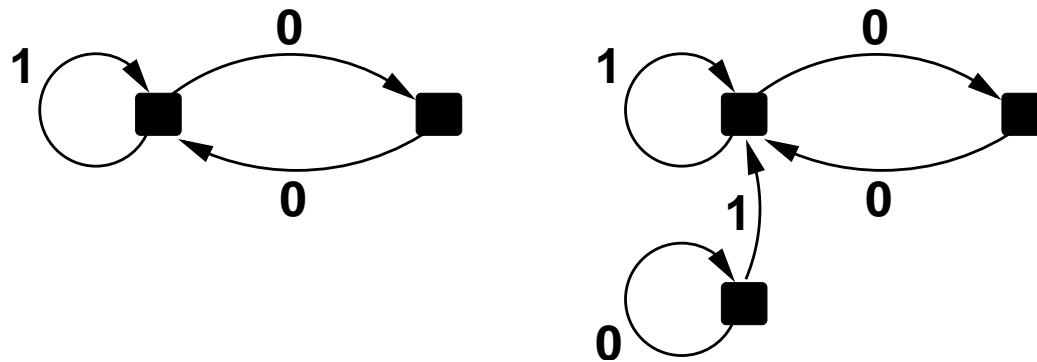
Theorem [Weiss]

$\{\text{Factors of SFTs}\} = \{\text{Sofic shifts}\}$



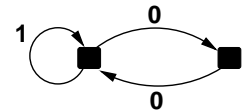
Note how a factor map $\pi : X_G \rightarrow X_{\mathcal{G}}$ is induced by sending each edge to its label. This is called a **cover** of $X_{\mathcal{G}}$.

Several labelled graphs may represent the same sofic shift, so covers are far from unique.

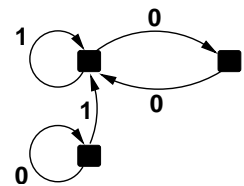


Two canonical covers are of the essence in the theory of irreducible sofic shifts:

The (right) **Fischer** cover $\overline{\mathcal{G}}$ is the unique cover with the minimal numbers of vertices and no edges leaving the same vertex having the same label.



The (right) **Krieger** cover $\widehat{\mathcal{G}}$ identifies the right infinite words of $X_{\mathcal{G}}$ when the same finite words can be added to the left of them.



Classification

Problem When is $X_G \simeq X_H$?

Theorem [Williams]

$X_A \simeq X_B$ precisely when there exist nonnegative integer matrices D_i, E_i with

$$A = D_0 E_0, E_0 D_0 = D_1 E_1, E_1 D_1 = D_2 E_2, \dots, E_n D_n = B$$

? Is $X \begin{bmatrix} 2 & 19 \\ 1 & 0 \end{bmatrix} \simeq X \begin{bmatrix} 1 & 5 \\ 4 & 1 \end{bmatrix}$?

Generalizations to the sofic case exist but are even more hopeless to work with.

Associated to any shift space there is a **flow space** given by product topology on

$$SX = \frac{X \times \mathbb{R}}{(x, t) \sim (\sigma(x), t + 1)}$$

Definition X and Y are *flow equivalent* (written $X \simeq_{fe} Y$) when SX and SY are homeomorphic in a way preserving direction in \mathbb{R} .

Fix $a \in \mathfrak{a}$ and $\star \notin \mathfrak{a}$ and define $\eta : \mathfrak{a}^{\mathbb{Z}} \rightarrow (\mathfrak{a} \cup \{\star\})^{\mathbb{Z}}$ as the map inserting a \star after each a :

$$\dots babbbaba \dots \quad \mapsto \quad \dots ba \star bbba \star ba \star \dots$$

Definition The “ $a \mapsto a\star$ ” symbol expansion of $X \subseteq \mathfrak{a}^{\mathbb{Z}}$ is the shift space $X_{a \mapsto a\star} = \eta(X)$.

Theorem [Parry-Sullivan, Matsumoto]

Flow equivalence is the coarsest equivalence relation containing conjugacy and $X \sim X_{a \rightarrow a\star}$

Example $X_S \simeq_{fe} X_{1+S}$

Problem When is $X_G \simeq_{fe} X_H$?

Theorem [Franks]

Let A and B be irreducible square matrices of size m and n , respectively. Then $X_A \simeq_{fe} X_B$ precisely when

$$\mathbb{Z}^m / (1 - A)\mathbb{Z}^m \simeq \mathbb{Z}^n / (1 - B)\mathbb{Z}^n$$

and

$$\text{sgn det}(1 - A) = \text{sgn det}(1 - B)$$

Ideas from C^* -algebras

Any irreducible SFT is conjugate to one of the form X_A with A an $n \times n$ $\{0, 1\}$ -matrix.

Such a matrix in turn defines the *Cuntz-Krieger* algebra \mathcal{O}_A by generators S_1, \dots, S_n and relations

$$S_j S_j^* S_j = S_j \quad (1)$$

$$\sum_{j=1}^n S_j S_j^* = 1 \quad (2)$$

$$\sum_{j=1}^n A(i, j) S_j S_j^* = S_i^* S_i \quad (3)$$

It is a simple C^* -algebra when A is irreducible with the so-called property (I).

The Cuntz-Krieger construction may be generalized to all (isomorphism classes of) shift spaces by the work of Matsumoto. We denote the **Matsumoto algebra** associated to the shift space X by \mathcal{O}_X (but write $\mathcal{O}_A = \mathcal{O}_{X_A}$).

Theorem [Matsumoto, Carlsen]

$$X \simeq_{fe} Y \Rightarrow \mathcal{O}_X \otimes \mathbb{K} \simeq \mathcal{O}_Y \otimes \mathbb{K}.$$

Theorem [Rørdam]

Let A and B be irreducible square matrices of size m and n , respectively. Then $\mathcal{O}_A \otimes \mathbb{K} \simeq \mathcal{O}_B \otimes \mathbb{K}$ precisely when

$$\mathbb{Z}^m / (1 - A)\mathbb{Z}^m \simeq \mathbb{Z}^n / (1 - B)\mathbb{Z}^n$$

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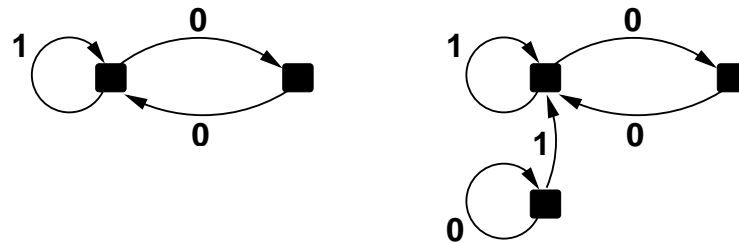
Theorem [Samuel; Carlsen]

\mathcal{O}_{X_G} is a Cuntz-Krieger algebra: $\mathcal{O}_{X_G} \simeq \mathcal{O}_{X_{\widehat{G}}}$

Observation The flow class of X_G determines the flow equivalence class of $X_{\widehat{G}}$ and $X_{\overline{G}}$. In fact, when $X_G \simeq_{fe} X_H$ we have

$$\begin{array}{ccc}
 SX_{\overline{G}} \dashrightarrow SX_{\overline{H}} & & SX_{\widehat{G}} \dashrightarrow SX_{\widehat{H}} \\
 \downarrow & & \downarrow \\
 SX_G \dashrightarrow SX_H & & SX_G \dashrightarrow SX_H
 \end{array}$$

Non-simple Cuntz-Krieger algebras



Note that the even flow is an example of an irreducible sofic shift whose Krieger cover is reducible. We are hence lead to consider non-simple Cuntz-Krieger algebras. In fact:

Proposition [Bates-E-Pask, Johansen]

$X_{\mathcal{G}}$ is non-simple for any irreducible AFT* which is not SFT. Simplicity of the Matsumoto algebra $\mathcal{O}_{X_{\mathcal{G}}}$ can be determined from $\overline{\mathcal{G}}$ and is a rather rare occurrence.

*definition supressed, but note that it is a flow invariant

Boyle and Huang classified all SFTs up to flow equivalence. And

Theorem [Restorff]

The collection of all six-term exact sequences

$$\begin{array}{ccccc}
 K_0(J) & \longrightarrow & K_0(I) & \longrightarrow & K_0(I/J) & & I \triangleleft J \triangleleft \mathcal{O}_A \\
 & & & & \downarrow & & \\
 & & & & K_1(I) & \longleftarrow & K_0(J) \\
 & & & & \uparrow & & \\
 K_1(I/J) & \longleftarrow & K_1(I) & \longleftarrow & K_0(J) & &
 \end{array}$$

provides a complete invariant for stable isomorphism of Cuntz-Krieger algebras with property (II).

Note that each ideal may occur several time, in which case the K -groups of the various six-term exact sequences are identified. Thus the invariant is called the “ K -web”.

Recent work by Restorff and Meyer/Nest show that the K -web is a complete invariant for any purely infinite C^* -algebra with a finite and linear ideal lattice. Furthermore, the work of Meyer/Nest indicates that the K -web is **not** sufficient in general.

Project Understand why the K -web is sufficient for Cuntz-Krieger algebras in spite of their ideal lattice being non-linear. Augment the invariant to arrive at a complete classification of purely infinite C^* -algebras with finitely many ideals.

Theorem [E-Restorff-Ruiz]

The automorphism group of a Cuntz-Krieger algebra \mathcal{O}_A with **one** ideal I up to approximate unitary equivalence is the automorphism group of

$$K_i(I) \longrightarrow K_i(\mathcal{O}_A) \longrightarrow K_i(\mathcal{O}_A/I) \longrightarrow$$

$$K_i(I; \mathbb{Z}/n) \longrightarrow K_i(\mathcal{O}_A; \mathbb{Z}/n) \longrightarrow K_i(\mathcal{O}_A/I; \mathbb{Z}/n) \longrightarrow$$

$$\mathcal{K}(\mathcal{O}_A; n)$$

where $\mathcal{K}(-; n)$ is a covariant functor defined using Kirchberg's ideal-related KK -theory.

Flow classification

Definition The **multiplicity set** of a cover $\pi : Y \rightarrow X$ is

$$M(\pi) = \{y \in Y \mid |\pi^{-1}(\pi(y))| > 1\}$$

When π the Fischer cover of an AFT shift, $M(\pi)$ is a shift space in its own right.

Theorem [Boyle-Carlsen-E]

Let $\gamma : X_{\overline{\mathcal{G}}} \rightarrow X_{\mathcal{G}}$ and $\eta : X_{\overline{\mathcal{H}}} \rightarrow X_{\mathcal{H}}$ be the Fischer covers of two AFT and strictly sofic shift spaces $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$. Then $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$ are flow equivalent exactly when the following conditions hold:

$$(1) \quad X_{\overline{\mathcal{G}}} \simeq_{fe} X_{\overline{\mathcal{H}}}$$

$$(2) \quad \begin{array}{ccc} SM(\gamma) & \dashrightarrow & SM(\eta) \\ \downarrow & & \downarrow \\ S\gamma(M(\gamma)) & \dashrightarrow & S\eta(M(\eta)) \end{array}$$

Definition The sofic shift $X_{\mathcal{G}}$ with Fischer cover $\gamma : X_{\overline{\mathcal{G}}} \rightarrow X_{\mathcal{G}}$ is said to be **near Markov** if $M(\gamma)$ is finite and contained in the set of periodic points of $X_{\overline{\mathcal{G}}}$.

Arranging the elements of $M(\gamma)$ according to their periods and images:

$$\begin{aligned} \gamma(x_1^1) &= \dots = \gamma(x_n^1) \\ \gamma(x_1^2) &= \dots = \gamma(x_n^2) \\ &\vdots \end{aligned}$$

with $\sigma(x_1^1) = x_1^2, \sigma(x_1^2) = x_1^3, \dots$ until $\sigma(x_1^k) = x_1^1$ we get a **signature** which is the multiset of pairs (k, n) .

Theorem [Boyle-Carlsen-E]

A near Markov shift \mathcal{G} is classified up to flow equivalence by the flow class of $\overline{\mathcal{G}}$ and by its signature.

The machinery of **twistwise flow equivalence** developed by Boyle and Sullivan to classify skew product extensions of irreducible SFTs give computable invariants in carefully selected AFT cases. The invariant involves $K_1(\mathbb{Z}G)$ for $G = \mathbb{Z}/2$.

Problem Classify the sofic S -gap shifts.