Classification of graph algebras

Søren Eilers
eilers@math.ku.dk

Department of Mathematical Sciences
University of Copenhagen

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Program

Finitely many ideals

Question

Suppose we know how to classify the simple C^*-algebras in some class $\mathcal{C}$. What does it take to classify the C^*-algebras in $\mathcal{C}$ with finitely many ideals?

Observation (cf. Jordan-Hölder)

A finite decomposition exists

$$0 = I_0 \lhd I_1 \lhd \cdots \lhd I_n = A, \qquad I_j/I_{j-1} \text{ simple}$$

with

$$(I_1, I_2/I_1, \ldots, I_n/I_{n-1})$$

essentially unique.

Unified classification

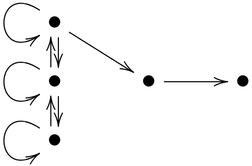
Observation

The classification of C^*-algebras has progressed rather independently for finite and infinite C^*-algebras although, at least in the real rank zero case, the classifying invariants are the same.

Question

Is it possible to give a unified proof of classification results covering both finite and infinite C^*-algebras?

Graph algebras



Graph algebras

Any countable graph G defines a C^*-algebra $C^*(G)$ which is an isomorphism invariant

Standard assumption

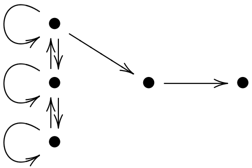
We will consider only graphs with **property (K)** and without breaking vertices. This ensures that all associated C^*-algebras have real rank zero and that their ideals are in $1-1$ correspondence with vertex sets that are

- *Hereditary* (no exit)
- *Saturated* (nothing outside points only there)

Theorem (RS; DT; CET)

For $C^*(G)$ with $M_G = \begin{bmatrix} A & \alpha & 0 & 0 \\ * & * & 0 & 0 \\ X & \xi & B & \beta \\ * & * & * & * \end{bmatrix}$ the six-term exact sequence in K-theory becomes

$$\begin{array}{ccccc} \operatorname{cok}\begin{bmatrix}A^t-1\\ \alpha^t\end{bmatrix} & \xrightarrow{I} & \operatorname{cok}\begin{bmatrix}A^t-1 & X^t\\ \alpha^t & \xi^t \\ 0 & B^t-1 \\ 0 & \beta^t\end{bmatrix} & \xrightarrow{P} & \operatorname{cok}\begin{bmatrix}B^t-1\\ \beta^t\end{bmatrix} \\ \begin{bmatrix}X^t\\ \xi^t\end{bmatrix}\Big\uparrow & & & & \Big\downarrow 0 \\ \ker\begin{bmatrix}B^t-1\\ \beta^t\end{bmatrix} & \xleftarrow[P]{} & \ker\begin{bmatrix}A^t-1 & X^t\\ \alpha^t & \xi^t \\ 0 & B^t-1 \\ 0 & \beta^t\end{bmatrix} & \xleftarrow[I]{} & \ker\begin{bmatrix}A^t-1\\ \alpha^t\end{bmatrix}\end{array}$$



$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \xi = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 \end{bmatrix} \quad \alpha = \begin{bmatrix} 1 \end{bmatrix}$$

Computation, continued

$$
\begin{array}{ccccc}
\operatorname{cok}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \xrightarrow{\;I\;} & \operatorname{cok}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \xrightarrow{\;P\;} & \operatorname{cok}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
\Big\uparrow {\scriptstyle \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]} & & & & \Big\downarrow {\scriptstyle 0} \\
\ker\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \xleftarrow[\;P\;]{} & \ker\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \xleftarrow[\;I\;]{} & \ker\begin{bmatrix} -1 \\ 1 \end{bmatrix}
\end{array}
$$

Computation, continued

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} \\ {\scriptstyle 1}\uparrow & & & & \downarrow{\scriptstyle 0} \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

Filtrated K-theory

$\mathfrak{K}(A)$:

The collection of all six term exact sequences

$$\begin{array}{ccccc} K_0(J/I) & \longrightarrow & K_0(K/I) & \longrightarrow & K_0(K/J) \\ \uparrow & & & & \downarrow \\ K_1(K/J) & \longleftarrow & K_1(K/I) & \longleftarrow & K_1(J/I) \end{array}$$

whenever $I \lhd J \lhd K \lhd A$.

Remark

Each subquotient may occur several times, in which case the K-groups of the various six-term exact sequences are identified. Thus the invariant is also called the "K-web".

Fundamental question

$\mathfrak{K}(A)_+$:

As above, but with each K_0-group

$$K_0(J/I) \longrightarrow K_0(K/I) \longrightarrow K_0(K/J)$$

considered as an **ordered** group.

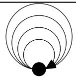
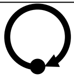
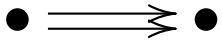
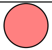
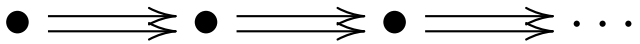
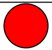
Working conjecture

$\mathfrak{K}(-)_+$ is a complete invariant for stable isomorphism of all graph algebras with finitely many ideals.

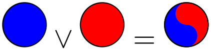
The simple case

Theorem

A simple graph algebra is either purely infinite or AF.

$\mathcal{O}_4$		
$\mathcal{O}_\infty$		
M_3		
$M_{2^\infty} \otimes \mathbb{K}$		

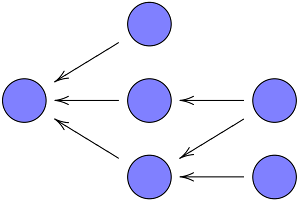
The simple case



Elliott/Kirchberg-Phillips

$\mathfrak{K}(A)_+ = K_*(A)_+$ is a complete invariant for simple graph algebras.

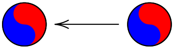
Cuntz-Krieger



Theorem [Restorff]

$\mathfrak{K}(-)$ provides a complete invariant for stable isomorphism of Cuntz-Krieger algebras with finitely many ideals.

One ideal



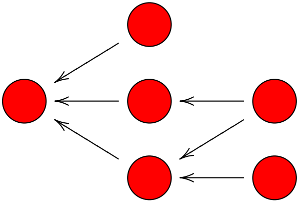
Theorem [E-Tomforde]

$\mathfrak{K}(-)_+$:

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

is a complete invariant up to stable isomorphism for the class of graph algebras with precisely one non-trivial ideal.

AF case

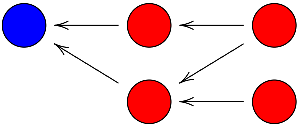


Theorem [Elliott]

$$K_*(A)_+$$

(and hence also $\mathfrak{K}(A)_+$) is a complete invariant for the class of graph algebras A which are AF.

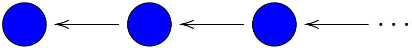
Mixed case



Theorem [E-Tomforde]

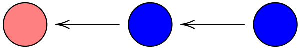
$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

(and hence also $\mathfrak{K}(A)_+$) is a complete invariant for the class of graph algebras A with a maximal nontrivial ideal I which is AF.



Theorem [Restorff (3), Meyer-Nest (n)]

$\mathfrak{K}(-)$ is a complete invariant for the class of purely infinite C^*-algebras with linear ideal lattices.

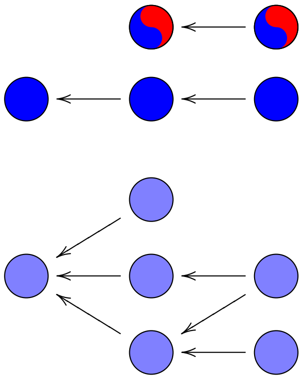


Proposition [E-Restorff-Ruiz]

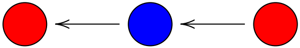
$\mathfrak{K}(-)$ is a complete invariant for the class of extension of purely infinite C^*-algebras with one ideal by finite-diemsnional C^*-algebras.

Proof approach

- Inductive
- UCT + Kirchberg classification
- Flow equivalence of SFTs



Challenge: Inductive approach



Inductive approach

Theorem [E-Restorff-Ruiz]

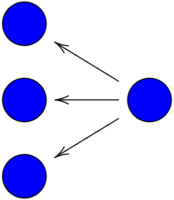
Let two C^*-algebras A, A' each have precisely one non-trivial ideal I, I'. When

- $I, A/I, I', A'/I'$ are KK-**strongly** classified by K-theory
- I, I' have the CFP
- The Busby maps $A/I \to M(I)/I$, $A'/I' \to M(I')/I'$ have full images
- $(\cdots)$

we get

$$\mathfrak{K}(A)_+ \simeq \mathfrak{K}(A')_+ \Rightarrow A \otimes \mathcal{K} \simeq A' \otimes \mathcal{K}.$$

Challenge: UCT approach



UCT approach

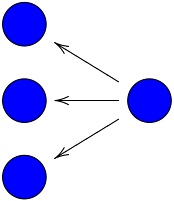
Theorem (Kirchberg)

Any $\alpha \in KK_X(A, B)^{-1}$ *induces a stable isomorphism between* A *and* B *when these are purely infinite and nuclear with* $\mathrm{Prim}(A) = \mathrm{Prim}(B) = X$.

Theorem (Meyer-Nest)

When A, B *are in the bootstrap class and* $\mathrm{p.dim}(\mathfrak{K}(A)) \leq 1$ *we have a UCT*

$$0 \longrightarrow \mathrm{Ext}(\mathfrak{K}(A), \mathfrak{K}(B)) \longrightarrow KK_X(A, B) \longrightarrow \mathrm{Hom}(\mathfrak{K}(A), \mathfrak{K}(B)) \longrightarrow 0$$



Problem

For a certain purely infinite C^*-algebra A with 7 ideals, $\operatorname{p.dim}(\mathfrak{K}(A)) > 1$. Consequently, $\mathfrak{K}(-)$ is **not** a complete invariant for all nuclear, purely infinite C^*-algebras in the bootstrap class with real rank zero.
However, the K-theory of this example is not obtainable by graph algebras.

Challenge: Flow equivalence approach