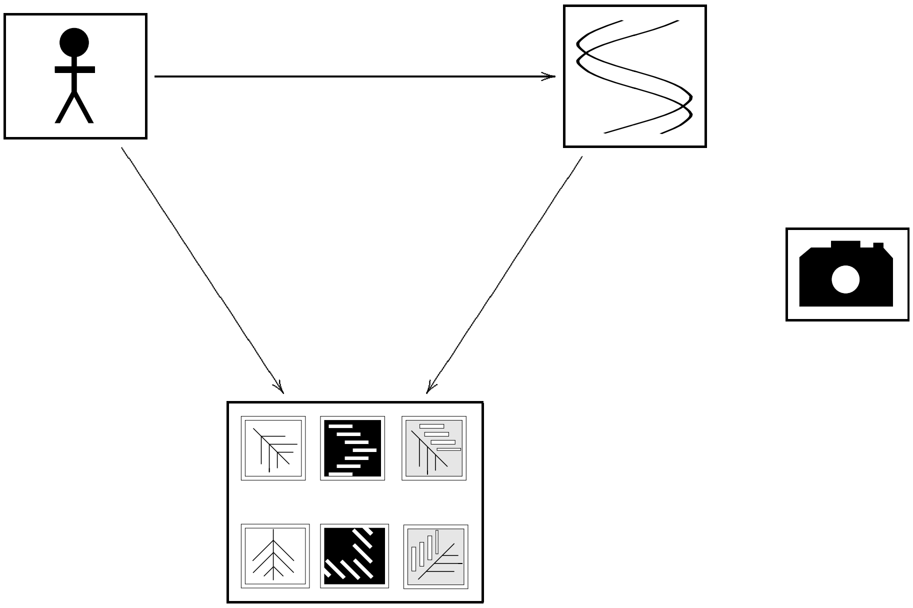


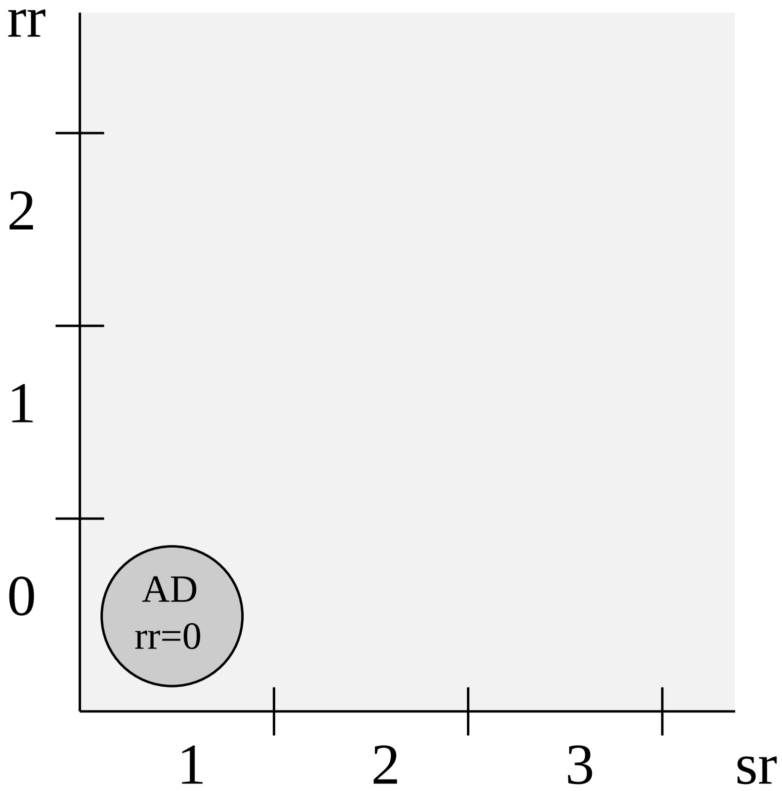
$$\begin{tikzcd}
\begin{array}{c} AD \\ \text{algebras} \end{array} \arrow[rr] \arrow[dr] & & \begin{array}{c} \text{shape} \\ \text{theory} \end{array} \arrow[dl] & \\
& \begin{array}{c} K\text{-theory} \\ \text{with} \\ \text{coefficients} \end{array} & & \begin{array}{c} KK\text{-} \\ \text{theory} \end{array}
\end{tikzcd}$$

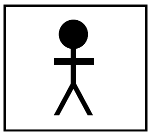
C^*-algebras (A, B):

$$+, \cdot, ^*, \|\cdot\|, \text{ axioms} \ldots$$

If A is commutative $[a \cdot b = b \cdot a]$, $A = C_0(X)$ for some locally compact Hausdorff space X.

Dimension:





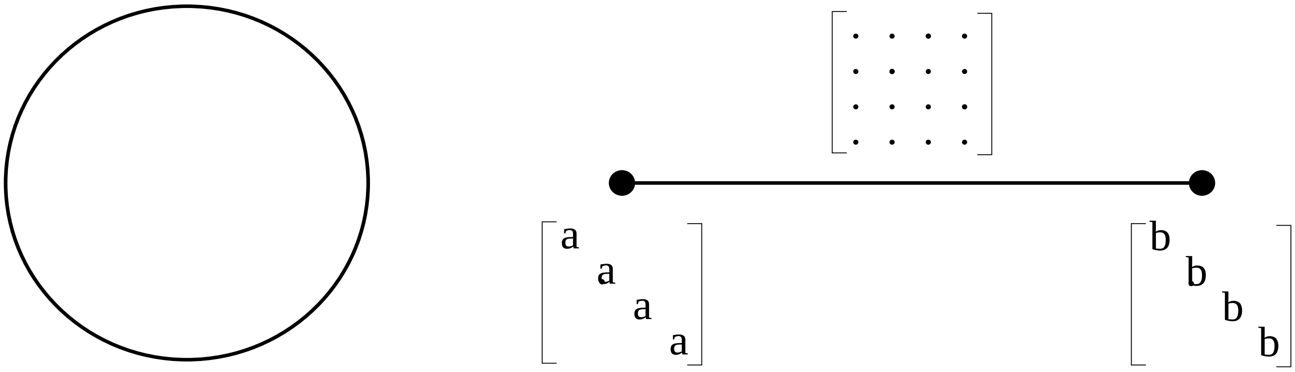
AD algebras:

- $\mathbb{I}_n^{\sim} = \{f \in M_n(C([0,1])) | f(0), f(1) \in \mathbb{C}\}$

- $B_j \in \{C(S^1), \mathbb{I}_n^{\sim}\}$

- $A_i = M_{m_1} \otimes B_1 \oplus \cdots \oplus M_{m_N} \otimes B_N$

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \longrightarrow \cdots \longrightarrow A = \varinjlim (A_i, \varphi_i)$$



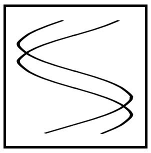
KK-theory:

$$KK(A,B) = \{\alpha \mid A \overset{\alpha}{\rightsquigarrow} B\}$$

- $A \xrightarrow{f} B$ yields $A \overset{[f]}{\rightsquigarrow} B$

- $f \mapsto [f]$ is neither injective nor surjective

- $A \overset{\alpha}{\rightsquigarrow} B \overset{\beta}{\rightsquigarrow} C$ yields $A \overset{\alpha\beta}{\rightsquigarrow} C$



KK-shape equivalence: $A \sim_{sh} B$ when

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[dr] & A_2 \arrow[r] \arrow[dr] & A_3 \arrow[r] & \cdots \arrow[r] & A \\
B_1 \arrow[r] \arrow[u] & B_2 \arrow[r] \arrow[u] & B_3 \arrow[r] \arrow[u] & \cdots \arrow[r] & B
\end{tikzcd}$$

$$A_i \rightsquigarrow A_{i+1} \quad = \quad \begin{tikzcd}
A_i \arrow[dr, squiggly] & A_{i+1} \\
 & B_{i+1} \arrow[u, squiggly]
\end{tikzcd}$$

$$B_i \rightsquigarrow B_{i+1} \quad = \quad \begin{tikzcd}
A_i \arrow[dr, squiggly] & \\
B_i \arrow[u, squiggly] & B_i + 1
\end{tikzcd}$$

Theorem [Elliott] If A and B are real rank zero AD algebras, then

$$A \simeq B \Longleftrightarrow A \sim_{sh} B$$

$KK(-,-)$

$$
\begin{array}{llcl}
\boxed{0} & KK(\mathbb{C},-) & = & K_0(-) \\
\boxed{n} & KK(\mathbb{I}_n,-) & = & K_0(-;\mathbb{Z}/n) \\
\boxed{I} & KK(C_0(\mathbb{R}),-) & = & K_1(-)
\end{array}
$$

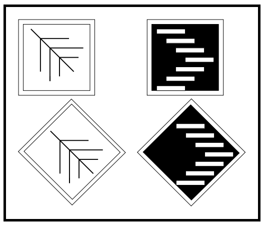
$$K_0(A;\mathbb{Z}/n) = K_0(A)\otimes\mathbb{Z}/n \oplus K_1(A)[n]$$

A	$K_0(A)$	$K_1(A)$	$K_0(A;\mathbb{Z}/n)$
$C(S^1)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/n$
$\mathbb{I}_n^{\sim}$	$\mathbb{Z}$	$\mathbb{Z}/n$	$\mathbb{Z}/n\oplus\mathbb{Z}/n$
$\mathbb{I}_m^{\sim}$	$\mathbb{Z}$	$\mathbb{Z}/m$	$\mathbb{Z}/n\oplus\mathbb{Z}/(m,n)$

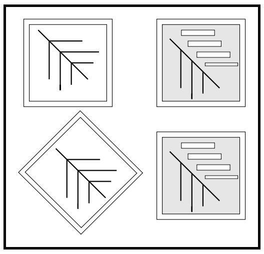
$\boxed{\textbf{!}}$ [Elliott] $K_0(A)$ is an ordered group

$$K_\star(A) = K_0(A) \oplus K_1(A)$$

$\boxed{\textbf{!}}$ [Dădărlat-Nemethi/Elliott] $K_\star(A)$ is an ordered group



$\boxed{\textbf{!}}$ [Dădărlat-Loring] $K_0(A) \oplus K_0(A; \mathbb{Z}/n)$ is an ordered group



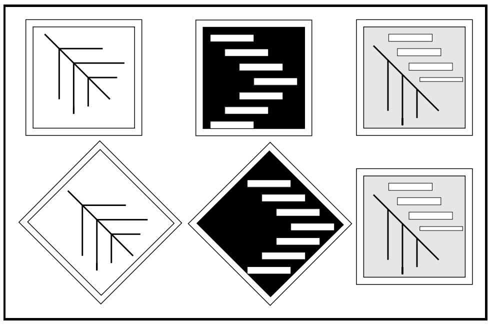
Theorem [Elliott] Let A, B be AD algebras of real rank zero. Assume that $K_1(A)$ is torsion free. Then

$$[K_\star(A), K_\star(A)^+] \simeq [K_\star(B), K_\star(B)^+] \Longleftrightarrow A \simeq B$$

$\boxed{!}$ [Dădărlat-Loring] There exist AD algebras A, B of real rank zero with

$$[K_\star(A), K_\star(A)^+] \simeq [K_\star(B), K_\star(B)^+]$$

yet $A \not\simeq B$**.**



$$KK(A,B) \curvearrowright \mathrm{Hom}(K_0(A),K_0(B)) = \boxed{\mathbf{0}} \to \boxed{\mathbf{0}}$$

$$A \rightsquigarrow B \curvearrowright \left[\mathbb{C} \rightsquigarrow A \mapsto \mathbb{C} \rightsquigarrow B\right]$$

$$KK(A,B) \longrightarrow \begin{array}{c} \boxed{\mathbf{0}} \to \boxed{\mathbf{0}} \\ \\ \boxed{\mathbf{l}} \to \boxed{\mathbf{l}} \end{array}$$ Not injective

---

$$KK(A,B) \longrightarrow \begin{array}{c} \boxed{\mathbf{0}} \to \boxed{\mathbf{0}} \\ \\ \boxed{\mathbf{n}} \to \boxed{\mathbf{n}} \\ \\ \boxed{\mathbf{l}} \to \boxed{\mathbf{l}} \end{array}$$ Not surjective

---

$$KK(A,B) \longrightarrow \begin{tikzcd} \boxed{\mathbf{0}} \arrow[r] \arrow[d] & \boxed{\mathbf{0}} \arrow[d] \\ \boxed{\mathbf{n}} \arrow[r] \arrow[d] & \boxed{\mathbf{n}} \arrow[d] \\ \boxed{\mathbf{l}} \arrow[r] & \boxed{\mathbf{l}} \end{tikzcd}$$ Bijective

---

$$A \longrightarrow B \curvearrowright A \rightsquigarrow B \curvearrowright \begin{array}{c} \xrightarrow{+} \\ \xrightarrow{+} \\ \xrightarrow{+} \end{array}$$

I $\quad KK(\mathbb{C}, -) = K_0(-)$

n $\quad KK(\mathbb{I}_n, -) = K_0(-; \mathbb{Z}/n)$

0 $\quad KK(C_0(\mathbb{R}), -) = K_1(-)$

$$
\begin{array}{ccc}
 & & K_0(-) \\
 & & \downarrow \times n \\
\square & & \\
 & \downarrow & \downarrow \rho_n \\
\square & & \\
 & \downarrow & \downarrow \beta_n \\
\square & & \\
 & & \downarrow \times n \\
 & & K_0(-)
\end{array}
$$

$\mathbf{K}(A;n) \simeq \mathbf{K}(B;n)$:

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\rho_n} & K_0(A;\mathbb{Z}/n) & \xrightarrow{\beta_n} & K_1(A) \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_1} \\
K_0(B) & \xrightarrow{\rho_n} & K_0(B;\mathbb{Z}/n) & \xrightarrow{\beta_n} & K_1(B)
\end{array}$$

- (φ_0, φ_1) is an order isomorphism

- (φ_0, φ) is an order isomorphism

- [φ_0 is scaled]

$\boxed{\textbf{Theorem}}$ Let A, B be AD algebras of real rank zero. Assume that n annihilates the torsion of $K_1(A)$. Then

$$\mathbf{K}(A;n) \simeq \mathbf{K}(B;n) \Longleftrightarrow A \simeq B$$

$\boxed{\textbf{Sketch}}$ $\mathbf{K}(A;n) \simeq \mathbf{K}(B;n) \Longrightarrow A \sim_{sh} B.$

- Groups are finitely generated:

$$\begin{tikzcd}
\mathbf{K}(A_1;n) \arrow[r, Rrightarrow] \arrow[dr, Rrightarrow] & \mathbf{K}(A_2;n) \arrow[r, Rrightarrow] \arrow[dr, Rrightarrow] & \mathbf{K}(A_3;n) \arrow[r, Rrightarrow] & \cdots \arrow[r, Rrightarrow] & \mathbf{K}(A;n) \\
\mathbf{K}(B_1;n) \arrow[r, Rrightarrow] \arrow[u, Rrightarrow] & \mathbf{K}(B_2;n) \arrow[r, Rrightarrow] \arrow[u, Rrightarrow] & \mathbf{K}(B_3;n) \arrow[r, Rrightarrow] \arrow[u, Rrightarrow] & \cdots \arrow[r, Rrightarrow] & \mathbf{K}(B;n)
\end{tikzcd}$$

- Existence:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[dr] & A_2 \arrow[r] \arrow[dr] & A_3 \arrow[r] & \cdots \arrow[r] & A \\
B_1 \arrow[r] \arrow[u] & B_2 \arrow[r] \arrow[u] & B_3 \arrow[r] \arrow[u] & \cdots \arrow[r] & B
\end{tikzcd}$$

- Uniqueness:

$$\begin{tikzcd}
A_i \arrow[dr] & \\
B_i \arrow[u] \arrow[r] & B_{i+1}
\end{tikzcd}
\curvearrowright
\begin{tikzcd}
A_i \arrow[dr, rightsquigarrow] & \\
B_i \arrow[u, rightsquigarrow] \arrow[r, rightsquigarrow] & B_{i+1}
\end{tikzcd}
\curvearrowright
\begin{tikzcd}
A_i \arrow[dr, Rrightarrow] & \\
B_i \arrow[u, Rrightarrow] \arrow[r, Rightarrow] & B_{i+1}
\end{tikzcd}$$

Further topics

(I) Unbounded torsion

(II) Classes of C^*-algebras

(III) Reduction/Minimality

(IV) Automorphisms

$\boxed{\mathbf{I}}$

- n tor $K_1(-) = 0$.

$\surd$ Complete invariant: $K_\star$, $K_\star^+$, and

  - $K_0(-;\mathbb{Z}/n), K_0(-;\mathbb{Z}\oplus\mathbb{Z}/n)^+$
  - ρ_n, β_n

- 2^∞ tor $K_1(-) = 0$.

$\surd$ Complete invariant: $K_\star$, $K_\star^+$, and

  - $K_0(-;\mathbb{Z}/2^k), K_0(-;\mathbb{Z}\oplus\mathbb{Z}/2^k)^+$
  - ρ_{2^k}, β_{2^k}
  - $\kappa_{2^{k+1},2^k} : K_0(A;\mathbb{Z}/2^k) \to K_0(A;\mathbb{Z}/2^{k+1})$

- $K_0(A;\mathbb{Z}(2^\infty)) = \varinjlim_k (K_0(A;\mathbb{Z}/2^k), \kappa_{2^{k+1},2^k})$
  - $K_0(A)\otimes\mathbb{Z}[\frac{1}{2}] \oplus K_0(A;\mathbb{Z}(2^\infty))^+ = \varinjlim K_0(A;\mathbb{Z}\oplus\mathbb{Z}/2^k)^+$
  - $\rho = \varinjlim \rho_{2^k}$, $\beta = \varinjlim \beta_{2^k}$
- $K_0(A)\otimes\mathbb{Z}[\frac{1}{2}] \xrightarrow{\rho} K_0(A,\mathbb{Z}(2^\infty)) \xrightarrow{\beta} K_1(A)$

$\boxed{\textbf{Theorem}}$ Let A, B be AD algebras of real rank zero. Assume that tor $K_1(A)$ is a 2-group. Then

$$\overline{\mathbf{K}}(A;2^\infty) \simeq \overline{\mathbf{K}}(B;2^\infty) \Longleftrightarrow A \simeq B$$

$\boxed{\textbf{!}}$ $\overline{\mathbf{K}}(A)$:

$$K_0(A) \otimes \mathbb{Q} \xrightarrow{\rho} K_0(A, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} K_1(A)$$

Theorem Let A, B be AD algebras of real rank zero. Then

$$\overline{\mathbf{K}}(A) \simeq \overline{\mathbf{K}}(B) \Longleftrightarrow A \simeq B$$

$\boxed{\textbf{II}}$

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \longrightarrow \cdots \longrightarrow A = \varinjlim(A_i, \varphi_i)$$

- $A_i = M_{m_1} \otimes B_1 \oplus \cdots \oplus M_{m_N} \otimes B_N$

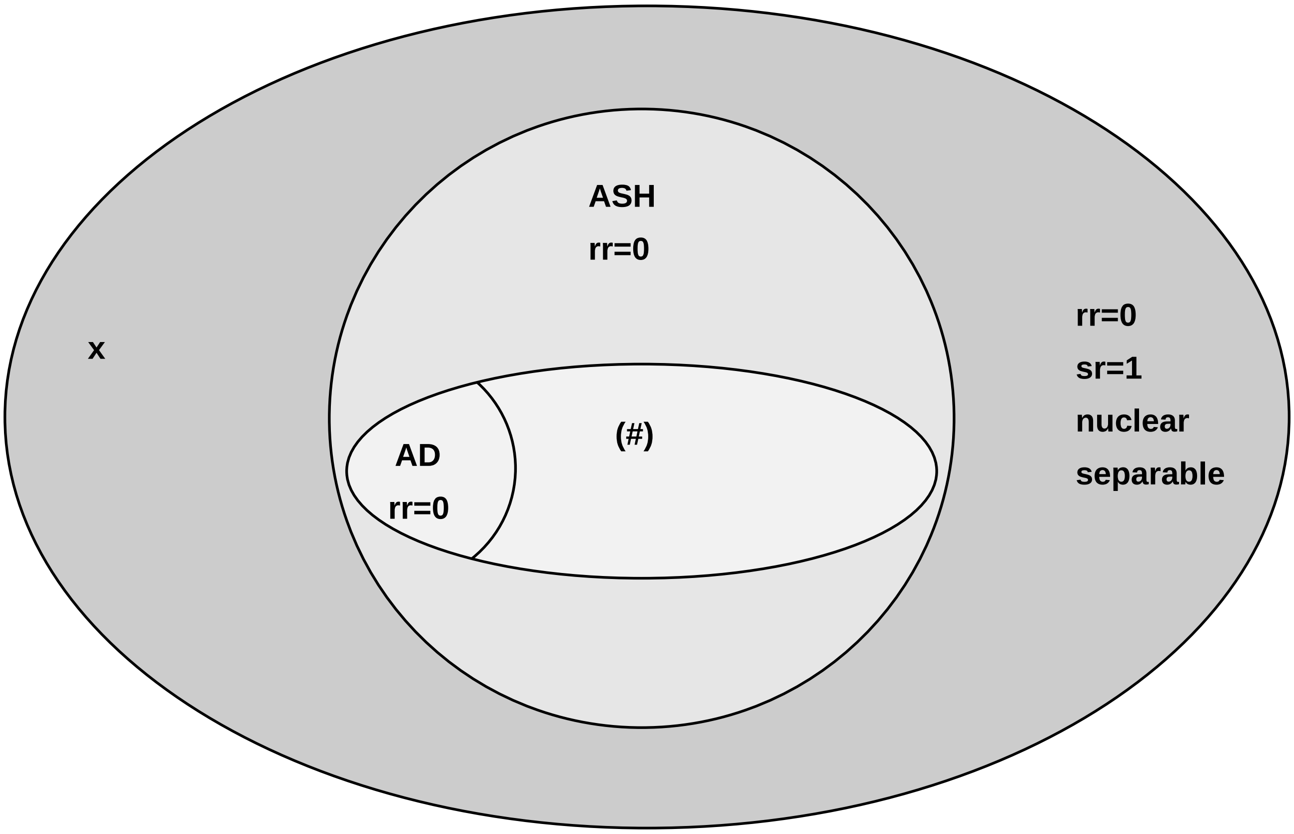
- $B_j \in \{C(S^1), \mathbb{I}_n^{\sim}\}$

$\boxed{!}$ $AD[n]$: $B_j \in \{C(S^1), \mathbb{I}_m^{\sim}\}$, $m \uparrow n$

$$n \,\text{tor}\, K_1(A) = 0 \Longleftrightarrow A \text{ is } AD[n]$$

$\boxed{!}$ $AD[2^\infty]$: $B_j \in \{C(S^1), \mathbb{I}_{2^k}^{\sim}\}$, $k \in \mathbb{N}$

$$2^\infty \,\text{tor}\, K_1(A) = 0 \Longleftrightarrow A \text{ is } AD[2^\infty]$$



$$\boxed{(\#)\text{: } \mathrm{tor}(K_0(A), K_1(A)) = 0}$$

$\boxed{\textbf{Theorem}}$ (with Dădărlat) The sequence

$$\begin{tikzcd}
K_0(A)\otimes\mathbb{Q} \arrow[r] & K_0(A,\mathbb{Q}/\mathbb{Z}) \arrow[r] & K_1(A) \arrow[d] \\
K_0(A) \arrow[u] & K_1(A,\mathbb{Q}/\mathbb{Z}) \arrow[l] & K_1(A)\otimes\mathbb{Q} \arrow[l]
\end{tikzcd}$$

is complete for the class of ASH algebras with (#).

$\boxed{\textbf{III}}$

Say n tor $K_1(A) = 0$. For a complete invariant, we need

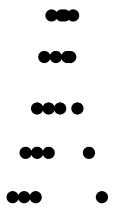
$\boxed{\mathbf{0}}$ $K_0(A), K_0(A)^+$

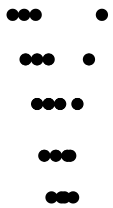
$\boxed{\mathbf{n}}$ $K_0(A; \mathbb{Z}/n)$

$\boxed{\mathbf{l}}$ $K_1(A)$

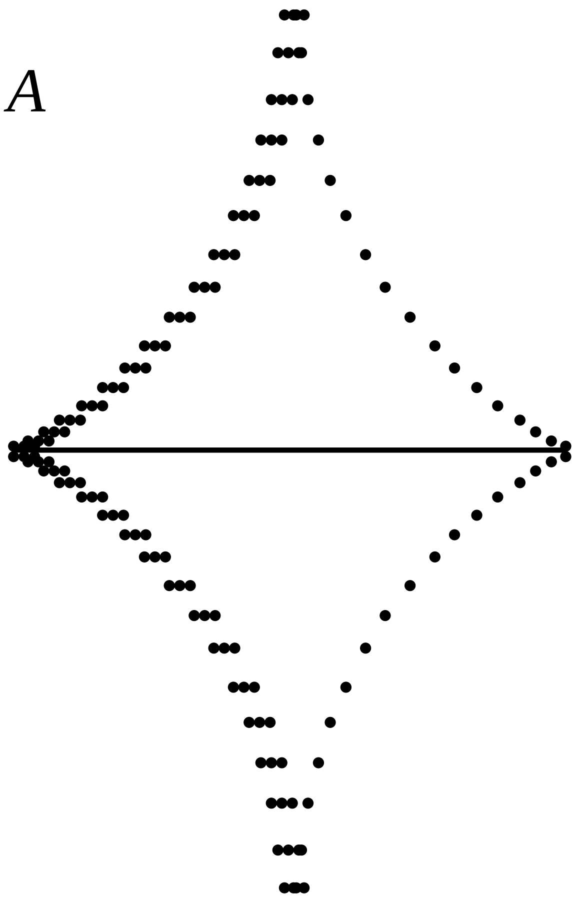
and:

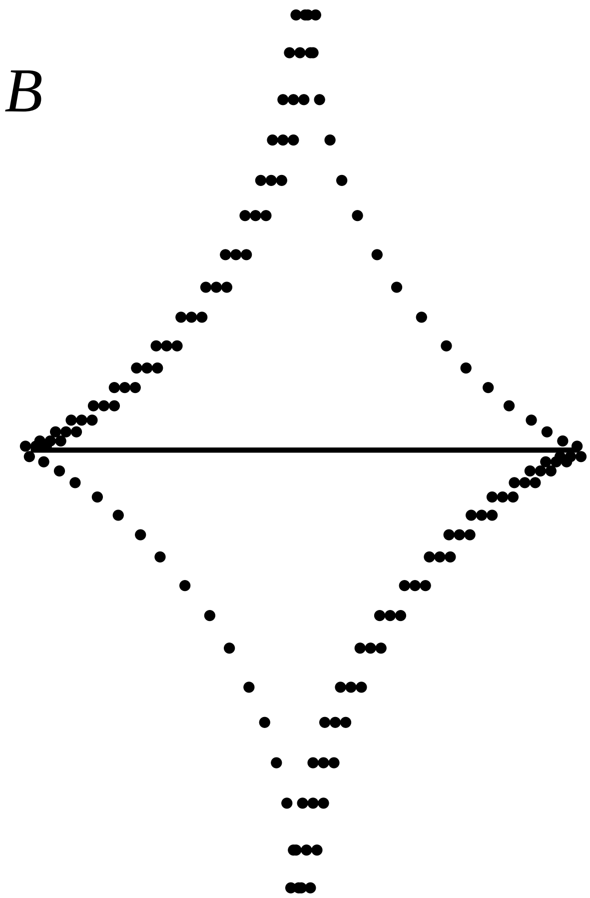
	$K_\star(-)^+$	$K_0(-;\mathbb{Z}/n)^+$	ρ	β
Simple	$-$	$-$	$-$	$-$
K_1 torsion free	$+$	$-$	$-$	$-$
K_1 torsion	$-$	$+$	$+$	$-$
K_1 mixed	$+$	$+$	$+$	?





$$A_5 = \mathbb{I}_4^{\sim} \oplus \bigoplus_{1}^{5} M_4 \oplus \bigoplus_{-5}^{-1} M_4$$





$$
\begin{tikzcd}
K_0(A) \arrow[d] & K_0(A;\mathbb{Z}/4) \arrow[r] \arrow[d] & K_1(A) \arrow[d] \\
K_0(B) & K_0(B;\mathbb{Z}/4) \arrow[r] & K_1(B)
\end{tikzcd}
$$

yet

$$A \not\cong B$$

$\boxed{\mathbf{IV}}$

$\boxed{\textbf{Corollary}}$ Let A be a unital AD algebra of real rank zero. Then

$$1 \longrightarrow \overline{\mathrm{Inn}(A)} \longrightarrow \mathrm{Aut}(A) \longrightarrow \mathbf{Aut}(\underline{\mathbf{K}}(A))_{1,+} \longrightarrow 1$$

is exact.

$\boxed{\textbf{Proposition}}$ [Dădărlat-Loring] A simple:

$$\mathbf{Aut}(\underline{\mathbf{K}}(A))_{1,+} \simeq \mathrm{Aut}(K_0(A))_+ \times \mathrm{Aut}(K_1(A)) \times \mathrm{Ext}(K_1(A), K_0(A)) / \mathrm{Pext}(K_1(A), K_0(A))$$

Fix φ_0, φ_1.

$$\begin{tikzcd}
K_0(A) \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\tilde{\varphi}_0"] & K_0(A; \mathbb{Q}/\mathbb{Z}) \arrow[r, shift left] \arrow[d, dashed, "\varphi"] & \operatorname{tor} K_1(A) \arrow[l, dotted, shift left] \arrow[d, "\tilde{\varphi}_1"] \\
K_0(A) \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] & K_0(A; \mathbb{Q}/\mathbb{Z}) \arrow[r, shift left] & \operatorname{tor} K_1(A) \arrow[l, dotted, shift left]
\end{tikzcd}$$

$$\varphi = \begin{bmatrix} \tilde{\varphi}_0 & 0 \\ * & \tilde{\varphi}_1 \end{bmatrix}$$

$$\begin{aligned}
\mathbf{Aut}(\overline{\mathbf{K}}(A))_{1,+} \ \simeq \ & \mathrm{Aut}(K_0(A))_+ \times \mathrm{Aut}(K_1(A)) \times \\
& \mathrm{Hom}(\operatorname{tor} K_1(A), K_0(A) \otimes \mathbb{Q}/\mathbb{Z})
\end{aligned}$$

$$\mathrm{Ext}(G_1, G_0) / \mathrm{Pext}(G_1, G_0)$$

$$\simeq$$

$$\mathrm{Hom}(\operatorname{tor} G_1, G_0 \otimes \mathbb{Q}/\mathbb{Z})$$

?