Warped crossed products

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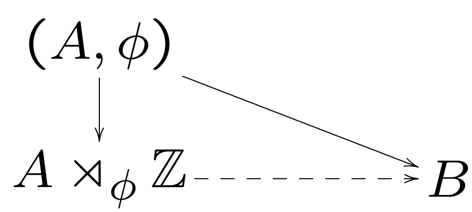
We consider a C^*-dynamical system (A, ϕ) with $\phi : A \to A$ a $*$-isomorphism. A **covariant representation** of (A, ϕ) into a C^*-algebra is a pair (π, U) such that

$$\pi(\phi(a)) = U\pi(a)U^*.$$

The (full) crossed product $A \rtimes_\phi \mathbb{Z}$ is the universal object factorizing all such covariant representations:



A C^*-**bimodule** X over A (to us) is an $A-A$ bimodule equipped with two inner products

$$\langle \cdot, \cdot \rangle \qquad (\cdot, \cdot)$$

satisfying axioms such as

$$\begin{array}{rcl} a\langle \xi, \eta \rangle & = & \langle a\xi, \eta \rangle \qquad a \in A, \xi, \eta \in X \\ (\xi, \eta)a & = & (\xi, \eta a) \qquad a \in A, \xi, \eta \in X \\ \langle \xi, \eta \rangle \zeta & = & \xi(\eta, \zeta) \qquad \xi, \eta, \zeta \in X \end{array}$$

If $\langle X, X \rangle$ and (X, X) are both total subsets in A se say that X is an **imprimitivity** bimodule (cf. Morita equivalence).

A *covariant representation* of (A, X) is a pair (π_A, π_X) of representations into some C^*-algebra B such that *both* module actions and *both* inner products become the ones inherited from B, i.e.

$$\pi_X(a\xi) = \pi_A(a)\pi_X(\xi) \qquad \pi_X(\xi a) = \pi_X(\xi)\pi_A(a)$$
$$\langle \xi, \eta \rangle = \pi_X(\xi)\pi_X(\eta)^* \qquad (\xi, \eta) = \pi_X(\xi)^*\pi_X(\eta)$$

where $a \in A$, $\xi, \eta \in X$.

Theorem [Abadie, E & Exel 99]

There is a universal object $A \rtimes_X \mathbb{Z}$ for covariant representations of bimodules

$$\begin{array}{ccc} (A, X) & & \\ \downarrow & \searrow & \\ A \rtimes_X \mathbb{Z} & \dashrightarrow & B, \end{array}$$

and A injects into $A \rtimes_X \mathbb{Z}$. Note that with A_ϕ the $A - A$ bimodule given as the set A equipped with

$$a \cdot \alpha = a\alpha \qquad \alpha \cdot a = \alpha\phi(a)$$

$$\langle a, b \rangle = ab^* \qquad (a, b) = \phi(a^* b)$$

we have $A \rtimes_\phi \mathbb{Z} = A \rtimes_{A_\phi} \mathbb{Z}$. Thus we may consider these universal objects as **generalized crossed products**.

Our motivation for this definition was the following result

Theorem [Abadie, E & Exel 99]

If a C^*-algebra B with a circle action χ_z is generated by its zeroth and first spectral subspaces

$$B_0 = \{b \in B \mid \chi_z(b) = b\} \qquad B_1 = \{b \in B \mid \chi_z(b) = zb\}$$

then $B = B_0 \rtimes_{B_1} \mathbb{Z}$.

which we then applied to quantum Heisenberg manifolds.

Observation

If B is generated by the first spectral subspace alone, then B_1 is a $B_0 - B_0$ imprimitivity bimodule.

Let us now focus our attention at $A \rtimes_X \mathbb{Z}$ in the case

- A is abelian, say $A = C(\Omega)$,

- X is an $A - A$ imprimitivity bimodule.

This means that $X \in \mathrm{Pic}(C(\Omega))$, the Picard group of $C(\Omega)$.

Theorem [see Abadie & Exel]
For any $C(\Omega) - C(\Omega)$ imprimivity bimodule X there exist

- A homeomorphism $\sigma : \Omega \to \Omega$

- A Hermitian line bundle $\mathcal{L}$ over Ω

such that $X \simeq \Gamma(\mathcal{L})_\sigma$ given as the set of continuous sections

$$\xi : \Omega \to \mathcal{L} \qquad p(\xi(\omega)) = \omega$$

of $\mathcal{L}$ with

$$f \cdot \xi = f\xi \qquad \xi \cdot f = \xi(f \circ \sigma)$$
$$\langle \xi, \eta \rangle(\omega) = \xi(\omega)\overline{\eta(\omega)} \qquad (\xi, \eta)(\omega) = \xi(\sigma(\omega))\overline{\eta(\sigma(\omega))}$$

Hence the generalization obtained is encoded by the Hermitian line bundle.

Definition

Given a triple $(\Omega, \sigma, \mathcal{L})$ as above we call

$$C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z} = C(\Omega) \rtimes_{\Gamma(\mathcal{L})_\sigma} \mathbb{Z}$$

a **warped** crossed product.

Become or cause to become bent or twisted out of shape, typically as a result of the effects of heat or dampness.

Motivating example [Connes & Dubois-Violette]

To any *quadratic algebra* $\mathcal{A}$ one may associate **geometric data** $(\Omega, \sigma, \mathcal{L})$ in a way relating $\mathcal{A}$ to $C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$.

The geometric data $(F_{\mathbf{u}}, \sigma, \mathcal{L})$ associated to the non-commutative sphere with generic parameter $\mathbf{u} \in \mathbb{T}^3$ yields a warped crossed product $C(F_{\mathbf{u}}) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ which is isomorphic to the mapping cone

$$\{f \in C[0,1] \otimes A_\eta \mid f(0) = \beta(f(1))\}$$

with A_η the irrational rotation algebra for $\eta \in \mathbb{R}\backslash\mathbb{Q}$ associated to $\mathbf{u}$ through theta functions, and

$$\beta(u) = u \qquad \beta(v) = u^4 v$$

Note that since A_η $*$-embeds into an AF-algebra so does $C[0,1] \otimes A_\eta$ and hence $C(F_{\mathbf{u}}) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$.

With Katsura: Undertake a systematic study of the structure of warped crossed products.

Mutatis mutandis

- Dual action and Takai duality [Abadie]
- Ideal structure
- Nuclearity and UCT
- Pimsner-Voiculescu sequence
- Classification **[95%]**

Theorem [Pimsner 85]

Consider a dynamical system (Ω, ϕ). The following are equivalent:

(i) No open set $U \subseteq \Omega$ has the property $\phi(\overline{U}) \subsetneq U$
[Chain recurrence]

(ii) $C(\Omega) \rtimes_\phi \mathbb{Z}$ is AF-embeddable

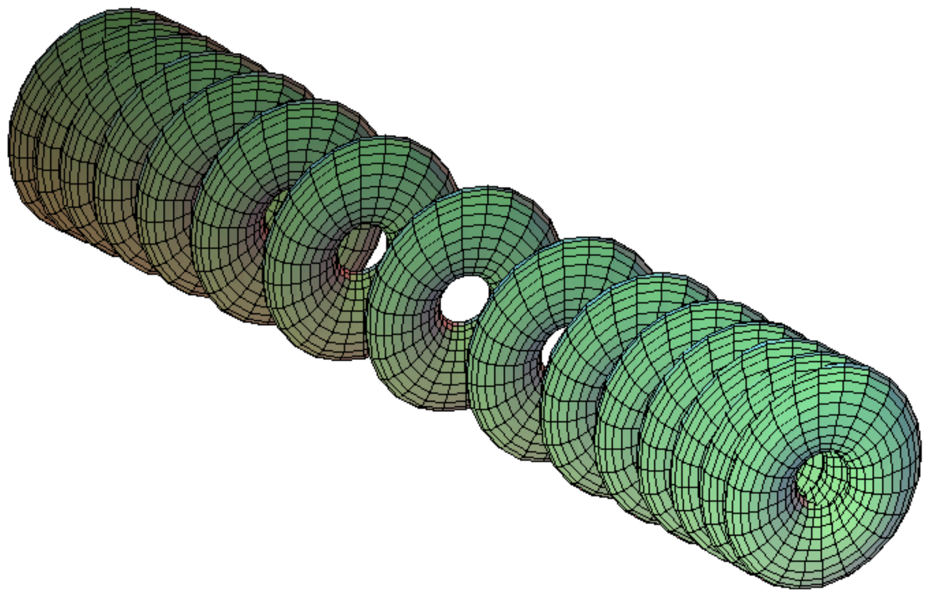
(iii) $C(\Omega) \rtimes_\phi \mathbb{Z}$ is quasidiagonal

(iv) $C(\Omega) \rtimes_\phi \mathbb{Z}$ is stably finite

Theorem [E & Katsura]
Any warped crossed product $C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ is AF-embeddable provided that (Ω, σ) is chain recurrent.

Idea of proof: Follow Pimsner but use the gauge invariance lemma to establish injectivity of the map produced by Berg's technique.

Theorem [E & Katsura]
There exists a warped crossed product $C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ which is stably finite even though (Ω, σ) is not chain recurrent.



(i) (Ω, σ) is chain recurrent

(ii) $C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ is AF-embeddable

(iii) $C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ is quasidiagonal

(iv) $C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ is stably finite

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\not\Rightarrow$ (i)

Theorem [E-Katsura, **95%**]
When

- σ is minimal

- $K_0(C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z})$ is dense in $\mathrm{Aff}(T(C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}))$

then $C(\Omega) \rtimes_{\sigma,\mathcal{L}} \mathbb{Z}$ has tracial rank zero and is hence classifiable by K-theory.