Overview of Shape Theory

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We now consider only metrizable compact spaces. For each such $X$ there exists an ANE-sequence associated with it. In other words, there exists an inverse sequence $S_X = \{X_n, p_n^{n+1}, \omega\}$ such that $X = \lim S_X$.

More sophisticated observation (Freudenthal): Any ($n$-dimensional) metrizable compactum is the limit of an inverse sequence consisting of ($n$-dimensional) ANE-compacta (even polyhedra) with surjective projections.
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To see this embed $X$ into the Hilbert cube $\mathbb{I}^\omega$, fix a metric $d$ on it and for each $n$ choose a closed ANE-neighborhood $X_n$ of $X$ such that $d(x, X) < \frac{1}{n}$ for every point $x \in X_n$. We may assume that the sequence of $X_n$’s is decreasing. Then this sequence together with inclusion maps forms an ANE-sequence.
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More sophisticated observation (Freudenthal): Any ($n$-dimensional) metrizable compactum is the limit of an inverse sequence consisting of ($n$-dimensional) ANE-compacta (even polyhedra) with surjective projections.
Freudenthal’s theorem is not valid for non-metrizable compact spaces. There exists a 1-dimensional compact space that cannot be represented as the limit space of an inverse spectrum consisting of metrizable ANE-compacta and surjective limit projections. Any compactum with $1 = \dim X < \text{ind } X$ will serve as an example.
Shape in terms of ANE-sequences

By a morphism

\[ \alpha: S_X = \{X_n, p_n^{n+1}, \omega\} \to S_Y = \{Y_n, q_n^{n+1}, \omega\} \]

between two ANE-sequences we understand a system 
\((\varphi, \{f_n\})\), where \(\varphi: \omega \to \omega\) is an increasing function and maps 
\(f_n: X_{\varphi(n)} \to Y_n\) are such that 
\(f_m \circ p_{\varphi(m)}^{\varphi(n)} \simeq q^n_m \circ f_n\) for every 
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Here is the diagram (commutes homotopically!)

\[
\begin{array}{ccc}
X_{\varphi(n)} & \xrightarrow{f_n} & Y_n \\
p^{\varphi(n)}_{\varphi(m)} & & q^n_m \\
X_{\varphi(m)} & \xrightarrow{f_m} & Y_m
\end{array}
\]
Suppose that we have two morphisms

\[ \alpha = (\varphi, \{f_n\}): \mathcal{S}_X = \{X_n, p_n^{n+1}, \omega\} \to \mathcal{S}_Y = \{Y_n, q_n^{n+1}, \omega\} \]

and

\[ \beta = (\psi, \{g_n\}): \mathcal{S}_X = \{X_n, p_n^{n+1}, \omega\} \to \mathcal{S}_Y = \{Y_n, q_n^{n+1}, \omega\} \]

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between two ANE-sequences.

Let us introduce a homotopy relation between \( \alpha \) and \( \beta \). We say that \( \alpha \simeq \beta \) if for each \( n \) there exists \( m \) such that \( m \geq \varphi(n), \psi(n) \) and \( f_n \circ p_{\varphi(n)}^m \simeq g_n \circ p_{\psi(n)}^m \).
Here is the diagram (commutes homotopically!)

\[
\begin{array}{ccc}
X_m & \xrightarrow{p^m_{\varphi(n)}} & X_{\varphi(n)} \\
\downarrow \quad p^m_{\psi(n)} & & \downarrow \quad f_n \\
X_{\psi(n)} & \xrightarrow{g_n} & Y_n
\end{array}
\]
Suppose we have two morphisms \( \alpha : S_X \to S_Y \) and \( \beta : S_Y \to S_X \). We say that \( S_X \simeq S_Y \) if \( \alpha \circ \beta \simeq \text{id}_{S_Y} \) and \( \beta \circ \alpha \simeq \text{id}_{S_X} \).
Suppose we have two morphisms $\alpha : S_X \to S_Y$ and $\beta : S_Y \to S_X$. We say that $S_X \simeq S_Y$ if $\alpha \circ \beta \simeq \text{id}_{S_Y}$ and $\beta \circ \alpha \simeq \text{id}_{S_X}$.

Turns out that the relation $\simeq$ is an equivalence relation for ANE-sequences.
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Two ANE-sequences associated with the same compactum are homotopy equivalent.
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Two ANE-sequences associated with the same compactum are homotopy equivalent.

We say that two compact spaces $X$ and $Y$ have the same shape (notation: $\text{Sh}(X) = \text{Sh}(Y)$) if $X$ and $Y$ have homotopy equivalent ANE-sequences associated with them.
Suppose we have two morphisms $\alpha : S_X \to S_Y$ and $\beta : S_Y \to S_X$. We say that $S_X \cong S_Y$ if $\alpha \circ \beta \cong \text{id}_{S_Y}$ and $\beta \circ \alpha \cong \text{id}_{S_X}$.

Turns out that the relation $\cong$ is an equivalence relation for ANE-sequences.

Two ANE-sequences associated with the same compactum are homotopy equivalent.

We say that two compact spaces $X$ and $Y$ have the same shape (notation: $\text{Sh}(X) = \text{Sh}(Y)$) if $X$ and $Y$ have homotopy equivalent ANE-sequences associated with them.

Note that choice of ANE-sequences is irrelevant.
Suppose $A$ and $B$ are compact subsets of the Hilbert cube $\mathbb{I}^\omega$. Translating above definitions to this situation by a shape morphism from $A$ to $B$ we mean a sequence of maps $f = \{f_n : \mathbb{I}^\omega \to \mathbb{I}^\omega\}$ with the following properties:

- For each neighborhood $V$ of $B$ there exists a neighborhood $U$ of $A$ and an integer $N$ such that for each $n \geq N$, $f_n(U) \subset V$ and $f_n|U \simeq f_{n+1}|U$ in $V$. 

If $f = \{f_n\}$ and $g = \{g_n\}$ are two shape morphisms from $A$ to $B$ then we say that $f$ and $g$ are homotopic if for any neighborhood $V$ of $B$ there exist a neighborhood $U$ of $A$ and an integer $N$ such that for each $n \geq N$ we have $f_n|U \simeq g_n|U$ in $V$. 
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If $f = \{f_n\}$ and $g = \{g_n\}$ are two shape morphisms from $A$ to $B$ then we say that $f$ and $g$ are homotopic if for any neighborhood $V$ of $B$ there exist a neighborhood $U$ of $A$ and an integer $N$ such that for each $n \geq N$ we have $f_n|U \simeq g_n|U$ in $V$. 
Let us note that if we have two morphisms $f : A \to B$ and $g : B \to A$ such that $gf \simeq \text{id}_A$ and $fg \simeq \text{id}_B$, then $Sh(A) = Sh(B)$ as defined above (routine verification - pen and paper).
Borsuk’s (original) Definition of Shape

Let us note that if we have two morphisms $f : A \rightarrow B$ and $g : B \rightarrow A$ such that $gf \simeq \text{id}_A$ and $fg \simeq \text{id}_B$, then $\text{Sh}(A) = \text{Sh}(B)$ as defined above (routine verification - pen and paper).

This definition does not depend on the given embedding of $A$ (or $B$) into $I^\omega$. 
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This definition does not depend on the given embedding of $A$ (or $B$) into $\mathbb{I}^\omega$.

A good and simple exercise: if $A$ and $B$ are ANE's then $Sh(A) = Sh(B)$ if and only if $A \simeq B$. 