$I^\omega$- and $R^\omega$-manifolds

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If $X$ is Polish, then $C(Y, X)$ has the Baire property.
When are maps close to homeomorphisms (near-homeomorphisms)?

Bing’s Shrinking Criterion: A map \( f: Y \to X \) between Polish spaces is a near-homeomorphism if and only if \( f(Y) \) is dense in \( X \) and the following condition is satisfied:

\[(⋆) \text{ For each } U \in \text{cov}(Y) \text{ and } V \in \text{cov}(X) \text{ there exist an open cover } W \in \text{cov}(X) \text{ and a homeomorphism } h: Y \to Y \text{ such that } f \circ h \in B(f, V) \text{ and } h^{-1}(W) \preceq U.\]

A closed surjection \( f: Y \to X \) between Polish spaces is a near-homeomorphism if and only if the following condition is satisfied:

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Limitation Topology

- When are maps close to homeomorphisms (near-homeomorphisms)?
- Bing’s Shrinking Criterion: A map $f: Y \to X$ between Polish spaces is a near-homeomorphism if and only if $f(Y)$ is dense in $X$ and the following condition is satisfied:
  
  \begin{itemize}
    \item[(\star)] For each $\mathcal{U} \in \text{cov}(Y)$ and $\mathcal{V} \in \text{cov}(X)$ there exist an open cover $\mathcal{W} \in \text{cov}(X)$ and a homeomorphism $h: Y \to Y$ such that $f \circ h \in B(f, \mathcal{V})$ and $hf^{-1}(\mathcal{W}) \prec \mathcal{U}$.
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$(\star\star)$ For each $\mathcal{U} \in \text{cov}(Y)$ and each $\mathcal{V} \in \text{cov}(X)$ there exists a homeomorphism $h : Y \to Y$ such that $f \circ h \in B(f, \mathcal{V})$ and the collection $\{hf^{-1}(x) : x \in X\}$ refines $\mathcal{U}$. 
A closed subset $A$ of a space $X$ is said to be a $Z$-set in $X$ if the set $\{ f \in C(X, X): f(X) \cap A = \emptyset \}$ is dense in the space $C(X, X)$. If the set $\{ f \in C(X, X): \text{cl}_X f(X) \cap A = \emptyset \}$ is dense in $C(X, X)$, then we say that $A$ is a strong $Z$-set in $X$. 

The concepts of the $Z$-set and the strong $Z$-set differ even for very simple spaces.
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The concepts of the Z-set and the strong Z-set differ even for very simple spaces.
Consider the subset

\[ X = ([0, 1] \times \{0\}) \cup \left( \bigcup_{n=1}^{\infty} \left\{ \left\{ \frac{1}{n} \right\} \times [0, 1] \right\} \right) \]

of the plane:
The point \((0, 0)\) is a \(Z\)-set, but not a strong \(Z\)-set.
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Every $Z$-set in a locally compact ANE-space is a strong $Z$-set.
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Compact subsets are \(Z\)-sets in the Hilbert space.
Z-set Unknotting Theorems

- Z-set Unknotting Theorem: Let $f : Z \rightarrow F$ be a homeomorphism between $Z$-sets of $\mathbb{I}^\omega$ (or $\mathbb{R}^\omega$). Then there is an autohomeomorphism $F : \mathbb{I}^\omega \rightarrow \mathbb{I}^\omega$ (respectively, $F : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$) which extends $f$. 

Several versions of this result exist. $\mathbb{I}^\omega$ can be replaced by any $\mathbb{I}^\omega$-manifold $X$ as long as $f$ is homotopic to the inclusion map $i_Z : Z \hookrightarrow X$. Same is true for $\mathbb{R}^\omega$-manifolds.
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Strong universality: A locally compact (Polish) space $X$ is strongly universal if for any compact (Polish) space $B$, its closed subset $A$, any $U \in \text{cov}(X)$ and any map $f : B \to X$ such that the restriction $f|A$ is a $Z$-embedding, there is a $Z$-embedding $g : B \to X$ which is $U$-close to $f$ and such that $g|A = f|A$. 

A space $X$ has DD$_n$P if the set 
$$\{f \in C(D^n_1 \oplus D^n_2, X) : f(D^n_1) \cap f(D^n_2) = \emptyset\}$$
is dense in $C(D^n_1 \oplus D^n_2, X)$.

A space $X$ has a Strong Discrete Approximation Property (SDAP) if the set 
$$\{f \in C(\bigoplus\{D^n : n \in \omega\}, X) : \{f(D^n) : n \in \omega\} \text{ is discrete}\}$$
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$\omega$- and $\mathbb{R}^\omega$-manifolds
Universality Properties; Disjoint (Discrete) Disks Properties

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- A space $X$ has $\text{DD}^nP$ if the set 
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A (locally) compact A(N)E is homeomorphic to $\mathbb{I}^\omega$ (to a $\mathbb{I}^\omega$-manifold) if and only if it is strongly universal for compact spaces (or equivalently, has the DD$^nP$ for each $n$).
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A Polish A(N)E is homeomorphic to $\mathbb{R}^\omega$ (to a $\mathbb{R}^\omega$-manifold) if and only if it is strongly universal for Polish spaces (equivalently, has the SDAP).
Properties of $\mathbb{I}^\omega$- and $\mathbb{R}^\omega$-manifolds

- $X \times \mathbb{I}^\omega$ is an $\mathbb{I}^\omega$-manifold iff $X$ is a locally compact ANE.
- $X \times \mathbb{I}^\omega \approx \mathbb{I}^\omega$ iff $X$ is a compact AE.

- Product of countably many non-degenerate compact AE's is $\mathbb{I}^\omega$.
- Product of countably many non-compact Polish AE's is $\mathbb{R}^\omega$.

- Homotopy equivalent $\mathbb{R}^\omega$-manifolds are homeomorphic.

- If $X$ and $Y$ are homotopy equivalent $\mathbb{I}^\omega$-manifolds, then $X \times \mathbb{I}^\omega \approx Y \times \mathbb{I}^\omega$. 
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- $X \times R^\omega$ is an $R^\omega$-manifold iff $X$ is a Polish ANE.
  $X \times R^\omega \approx R^\omega$ iff $X$ is a Polish AE.

▶ Product of countably many non-degenerate compact AE's is $I^\omega$.
▶ Product of countably many non-compact Polish AE's is $R^\omega$.
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▶ If $X$ and $Y$ are homotopy equivalent $I^\omega$-manifolds, then $X \times [0,1) \approx Y \times [0,1)$. 

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Properies of $\mathbb{I}^\omega$- and $\mathbb{R}^\omega$-manifolds

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