Notes on End Theory

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Abstract

A short exposition of Freudenthal's end theory in certain topological spaces X and a description of the Freudenthal compactification φX is given. It is pointed out that the end theory describes the component structure of the corona $\beta X \backslash X$.

1 Introduction

The work by Freudenthal ([4],[5]) introduced and investigated the notion of *ends* for certain topological spaces. End theory is a way to describe the component structure of the topological space at infinity, and Freudenthal arrived at a beautiful theory centered around what is now called the Freudenthal compactification.

In this largely expository work, it is our aim to describe Freudenthal's end theory. Although Freudenthal's own exposition is very clear and self-contained, time has not passed by it without leaving scars, and we felt a need to rephrase the proofs using more modern tools when we first tried to understand [4]. Some readers may also prefer the choice of language here.

One observation in the present paper may be original, namely that the end theory completely describes the component structure of the *corona space* $\beta X \backslash X$. This result does not seem to appear in the literature, but as its proof is not hard, we suspect that it may very well be known to experts. The fact that end theory offers information about corona spaces is the starting point for joint work with C.A. Akemann ([1]) in which we lay down a theory of ends in noncommutative C^*-algebras in order to describe the ideal structure of the *corona algebra* $M(\mathfrak{A})/\mathfrak{A}$. To carry end theory through to a noncommutative setting, we employ generalized notions of connectivity defined and explored in [3].

It is the work in [1] which has sparked our interest in the subject, and the interested reader is invited to continue his reading here. Although completely self-contained, these notes are meant as a companion issue to [1], and we shall point out briefly at the end of each section how the classical theory compares to the one developed in [1].

We will confine ourselves to the following class of topological spaces.

Definition 1.1 A Hausdorff space X is called a *Raum*[1] if X is

(i) connected

(ii) locally connected

(iii) locally compact

(iv) σ-compact.

This is not be the largest class in which the arguments go through. Indeed, Freudenthal himself in [5] took the theory through to a far more general class of spaces, the essential property of which were *semicompactness* – every point possesses a neighborhood with compact boundary. Being interested in C^*-algebras, we have a preference for locally compact spaces, but that aside we also think that the basic idea of end theory comes out more clearly in the locally connected case. This condition ensures a certain finiteness condition which is crucial in [1], and simplifies matters greatly in the present exposition. To prove our point, we invite the reader to compare [4] and [5].

The terminology was inpired by a similar definition by Hopf in [6], although he used second countability instead of our (iv) to rule out certain pathological behavior. Note that a Raum may be compact, even though we are by far most interested in the noncompact case.

These notes are a revised version of sections of [2]. I would like to thank Chuck Akemann and Kurt Friedman on valuable comments on previous versions.

2 Sequences determining ends

Definition 2.1 Let X be a Raum. We say that a decreasing sequence (G_k) of nonempty subsets of X *determines an end* if

[1]Plural: Räume

(i) G_k is open.

(ii) G_k is connected.

(iii) ∂G_k is compact.

and

(iv) $\bigcap\limits_{n=1}^{\infty} \overline{G_k} = \emptyset$.

Note that (iv) above implies that no $\overline{G_k}$ can be compact, for by the finite intersection property, this would imply that some $\overline{G_k}$ were empty. The condition (iii) should be thought of as a condition of *connectivity at infinity.* In this sense, a half-line is connected at infinity, whereas a half-plane is not. The reader is asked to keep this dichotomy in mind while reading the paper, as it is a very instructive example.

We shall bring condition (iii) into play using the following trivial consequence of the definition of connected topological spaces.

Lemma 2.2 *If C is a connected set in X and G is open, then*

$$\left.\begin{array}{l} C \cap G \neq \emptyset \\ C \cap (X \backslash G) \neq \emptyset \end{array}\right\} \Longrightarrow C \cap \partial G \neq \emptyset.$$

Definition 2.3 Let (G_k) be a sequence determining an end of X. When H is a subset of X we say that $(G_k) \subseteq H$ if there exists a $k \in \mathbb{N}$ such that $G_k \subseteq H$.

Lemma 2.4 *Let (G_k) and (H_k) be sequences determining ends of the Raum X. The following conditions are equivalent.*

(i) $\forall l \in \mathbb{N} : G_l \cap H_l \neq \emptyset$.

(ii) $\forall l \in \mathbb{N} : (G_k) \subseteq H_l$.

(iii) $\forall l \in \mathbb{N} : (H_k) \subseteq G_l$.

Proof: By symmetry, it suffices to show (i)$\Longleftrightarrow$(ii). Given l, if $(G_k) \subseteq H_l$, then for some $k \in \mathbb{N}$, $G_l \cap H_l \supseteq G_k \cap H_l = G_k \neq \emptyset$. On the other hand, if (i) holds true and l is given, then since ∂H_l is compact,

$$\bigcap_{k=1}^{\infty} (\partial H_l \cap \overline{G_k}) \subseteq \bigcap_{k=1}^{\infty} \overline{G_k} = \emptyset$$

yields that for some $k \in \mathbb{N}$, $\partial H_l \cap \overline{G_k} = \emptyset$. *A fortiori*, $\partial H_l \cap G_k = \emptyset$, and as with $m = \max(k, l)$,

$$H_l \cap G_k \supseteq H_m \cap G_m \neq \emptyset,$$

we conclude from Lemma 2.2, that $G_k \subseteq H_l$. ∎

Definition 2.5 Let X be a Raum and $(G_k), (H_k)$ a pair of sequences determining ends.

1°: $(H_k) \approx (G_k)$ when (G_k) and (H_k) satisfy (i)–(iii) above.

2°: $[(G_k)]$ denotes the equivalence class represented by (G_k).

3°: $[(G_k)] \subseteq H$ when $(G_k) \subseteq H$.

4°: $\mathsf{E} = \{[(G_k)] | (G_k)$ determines an end of $X\}$.

5°: $\#_{\mathsf{E}} X$ denotes the number of elements of E, taking on values in $\mathbb{N} \cup \{0, \infty\}$.

We need to employ Lemma 2.4 to make sense out of these definitions. First use (i) for reflexivity and symmetry and (ii) for transitivity to see that "$\approx$" defined in 1° is an equivalence relation. Then note that "$\subseteq$" is defined on equivalence classes as in 3° because of (ii).

We call an equivalence class of sequences determining end for an *end*. We use e as a generic symbol for an end.

Example 2.6 It is easy to see that $\mathbb{R}^n$ is a Raum for all $n \in \mathbb{N}$. The sequence $(\{x \in \mathbb{R}^n | \|x\| > k\})_{k=1}^{\infty}$ determines an end for all $n > 1$, but not in $\mathbb{R}$. Here, $((k, \infty))_{k=1}^{\infty}$ and $((-\infty, -k))_{k=1}^{\infty}$ determine inequivalent sequences. It will be obvious from the following that in fact $\#_{\mathsf{E}}\mathbb{R} = 2$, $\#_{\mathsf{E}}\mathbb{R}^n = 1, n > 1$.

It is our goal to prove that when X is not compact, there exists at least one end of X, and to devise a way to find all there might be. The following lemma contains an essential consequence of our assumption that the X is a Raum, in particular, that they are locally connected.

Lemma 2.7 *Let X be a Raum, and assume that K is compact, G is open such that $K \subseteq G \subseteq X$. Writing*

$$X \backslash K = \bigcup_{i \in I} C_i,$$

where the C_i are the components of $X \backslash K$, we have that

$$I_0 = \{i \in I | C_i \not\subseteq G\}$$

is finite.

Figure 1: The set X of Example 2.8

Proof: As X is locally compact, there is an open set H containing K, such that $\overline{H}$ is compact, and because it clearly suffices to prove the lemma with $G \cap H$ instead of G, we may assume without loss of generality that G itself has compact closure.

If $K = \emptyset$, I has only one element, so assume further that $K \neq \emptyset$. From the local connectivity of X, each C_i is open in $X \backslash K$ and X. As they are components, they are closed in $X \backslash K$, but since X is connected, they are not closed here. We conclude that with closures taken in X, $\overline{C_i} \cap K \neq \emptyset$ for all $i \in I$, from which we further deduce that

$$C_i \cap G \neq \emptyset, \qquad \forall i \in I.$$

When $i \in I_0$, $C_i \cap (X \backslash G) \neq \emptyset$ follows by definition, and by the fact that C_i is connected,

$$C_i \cap \partial G \neq \emptyset, \qquad \forall i \in I_0. \tag{1}$$

As ∂G is compact by assumption, there is a finite set $I_1 \subseteq I$ such that $\partial G \subseteq \bigcup_{I_1} C_i$. The C_i are disjoint, whence from (1), $I_0 \subseteq I_1$. ■

Example 2.8 The subset X of $\mathbb{R}^2$ given by

$$\{0\} \times [0,1] \cup [0,\infty) \times \{0\} \cup \left(\bigcup_{n=1}^{\infty} \{n^{-1}\} \times [0,n] \right)$$

(see Figure 1) is not locally connected, for instance because the complement of the vertical line has a non-open component. It is hence not a Raum, and it does not have the property of the above lemma; indeed, when K is the vertical line and G is chosen with $\overline{G}$ compact, then an infinite number of components of $X \backslash K$ are not contained in G.

From the σ-compactness combined with the local compactness of a Raum X, we can choose compact sets L_k such that

$$(\dagger) \qquad L_1^\circ \subseteq L_1 \subseteq L_2^\circ \subseteq \cdots \subseteq L_k \subseteq L_{k+1}^\circ \subseteq L_{k+1} \subseteq \cdots, \qquad \bigcup_{k=1}^\infty L_k = X.$$

We call such a sequence a *compact nest* for X, and will assume in the following that one is fixed, using the dagger ($\dagger$) to refer to the chosen nest. We will also, for brevity, classify sets as *bounded* or *unbounded*, according to whether or not their closures are compact.

Proposition 2.9 *Let X be a noncompact Raum with compact nest ($\dagger$). Then*

(i) There is at least one, and at most a finite number of unbounded components of $X \backslash L_k$.

(ii) Let C be an unbounded component of $X \backslash L_k$. At least one unbounded component of $X \backslash L_{k+1}$ is contained in C.

(iii) Every unbounded component of $X \backslash L_{k+1}$ is contained in one of $X \backslash L_k$.

(iv) With W_k the union of all unbounded components of $X \backslash L_k$, $X \backslash W_k$ is compact.

Proof: Write $X \backslash L_k$ out in components $(C_i)_I$. By Lemma 2.7, the subset

$$I_0 = \{i \in I | C_i \not\subseteq L_{k+1}^\circ\}$$

is finite, and as C_i for $i \in I \backslash I_0$ is contained in the compact set L_{k+1}, there is a finite number of unbounded components. On the other hand, as the unbounded set $X \backslash L_{k+1}$ is contained in $\bigcup_{I_0} C_i$, at least one C_i, $i \in I_0$ must itself be unbounded, proving (i).

Letting

$$I_1 = \{i \in I | C_i \text{ is unbounded}\}$$

we have seen that $I_1 \subseteq I_0$, whence in the notation of (iv),

$$X \backslash W_k = X \backslash \bigcup_{i \in I_1} C_i = L_k \cup \bigcup_{i \notin I_1} C_i \subseteq L_{k+1}^\circ \cup \bigcup_{i \in I_0 \backslash I_1} C_i.$$

As $X \backslash W_k$ is closed and contained in a finite union of bounded sets, it is compact, proving (iv).

For (ii), let C be an unbounded component of $X \backslash L_k$ and note that we can write

$$C \cap (X \backslash L_{k+1}) = C \backslash L_{k+1} = \bigcup_{i \in I} D_i,$$

where the D_i are components of $X \backslash L_{k+1}$. Indeed, a component of $C \backslash L_{k+1}$ will be maximally connected in $X \backslash L_{k+1}$ as well, and we may take as the D_i the component of the former set. As above, all but a finite number of the D_i are contained in L°_{k+2}, and at least one of these must be unbounded, as $C \backslash L_{k+1}$ is.

Finally, every component D of $X \backslash L_{k+1}$ must be contained in one of $X \backslash L_k$, say C, and if D is unbounded, so is C. This proves (iii). ■

Remark 2.10 We have from (ii) above, that the number of unbounded components of $X \backslash L_k$ *increases* with k. Considering again the set X of Example 2.8, if we let $L_1 = \{(0,0)\}$ and $L_2 = \{0\} \times [0,1]$, then $X \backslash L_1$ has two unbounded components while $X \backslash L_2$ has only one.

Remark 2.11 There exists a complete analog to Lemma 2.4 in the general case ([1, 3.1.4]). However, Lemma 2.7 has no counterpart in the general case, and it turns out that the properties found in Proposition 2.9 may fail in the natural class of C^*-algebras generalizing Raum spaces. This forces us to consider, separately, two subclasses of Räume – one where analogs of Proposition 2.9 may be found, and one where they may not. This is discussed in [1, 2.1].

3 The Freudenthal compactification

Let X be a locally compact Hausdorff space and recall the definition of a *compactification* of X from, say, [8, 1.7.2]. We will assume all compactifications to be Hausdorff, and define a relation on these by saying that $(X \stackrel{\iota_1}{\to} Y_1) \geq (X \stackrel{\iota_2}{\to} Y_2)$ when there exists a continuous mapping $\kappa : Y_1 \to Y_2$ respecting the embeddings, i.e. with $\iota_2 = \kappa \circ \iota_1$. Note that κ is then automatically onto, and that if $\lambda : Y_2 \to Y_1$ exists with $\iota_1 = \lambda \circ \iota_2$, then κ is a homeomorphism with $\kappa^{-1} = \lambda$. The defined relation can thus be seen to be an order on the set of compactifications identified under embedding-preserving homeomorphism. The (homeomorphism class of the) one-point compactification, αX, is clearly minimal in this respect; the Stone-Čech compactification βX is maximal according to [8, 4.3.19]. We will use γX as a generic symbol for a compactification of X, and write $(X \hookrightarrow \gamma X)$ to indicate that X is identified with its image here. In this case, we write $\overline{Y}^{\gamma}$ to specify closure within γX. We write $\partial_\gamma Y$ for the γ-boundary of Y.

The purpose of this section is to define another compactification of the Raum X, formally adjoining the ends to the space. The following concept is crucial in doing this.

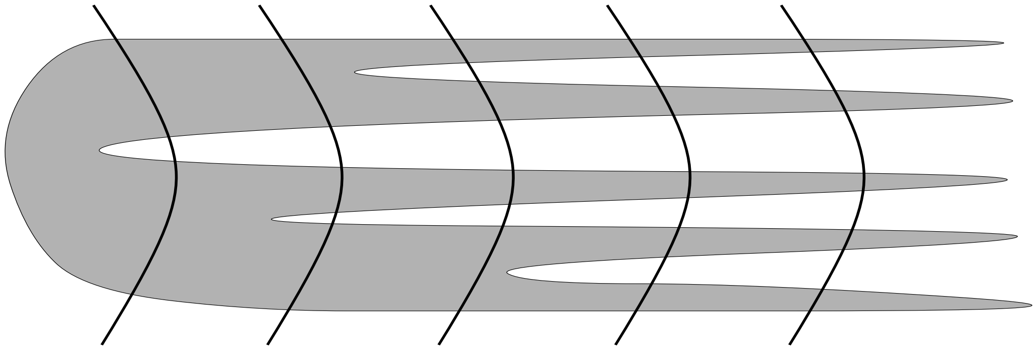

Figure 2: The set X of Example 3.2

Definition 3.1 Let a compact nest $(\dagger)$ be given. A $(\dagger)$-sequence is a decreasing sequence (G_k) where G_k is an unbounded component of $X \backslash L_k$.

By Proposition 2.9, $(\dagger)$-sequences exist and may be arranged notationally using multi-indices. The unbounded components of $X \backslash L_1$ are denoted by $H_1, \ldots, H_m$, and we then group the unbounded components of $X \backslash L_2$ according to which of the H_i, $i \in \{1, \ldots, m\}$, they are contained in; thus those lying inside H_1 are given indices $H_{11}, \ldots, H_{1j_1}$. A bit more formally, we may define a subset $T_\dagger$ of $\mathbb{N}^{\mathbb{N}}$ and sets $H_{i_1 \cdots i_k}$ for all $\underline{i} = (i_1, i_2, \ldots) \in T_\dagger$, in accordance with

(i) $H_{i_1 \cdots i_k}$ is an unbounded component of $X \backslash L_k$ when $\underline{i} \in T_\dagger$.

(ii) All unbounded components of $X \backslash L_k$ occur as $H_{i_1 \cdots i_k}$ for exactly one k-tuple $(i_1, \ldots, i_k)$ extending to $\underline{i} \in T_\dagger$.

(iii) $H_{i_1 \cdots i_k} \supseteq H_{i_1 \cdots i_k i_{k+1}}$ for $\underline{i} \in T_\dagger$.

because of (i)–(iii) in Proposition 2.9. It is clear that there are at most $\mathbb{N}^{\mathbb{N}} = \aleph$ $(\dagger)$-sequences. On the other hand, when X is not compact, there is at least one according to (i) and (ii) from Proposition 2.9.

Example 3.2 Consider the set X drawn in Figure 2 endowed with a compact nest as indicated by the solid lines – L_k is the subset to the left of the kth line. In this case, a systematic enumeration from top to bottom would give

$$T_\dagger = \{(1,1,1,1,\ldots), (1,1,2,1,\ldots), (2,1,1,1,\ldots), (2,2,1,1,\ldots), (2,2,1,2,\ldots)\}.$$

In the case of a 2-tree Y (Figure 3) we would have

$$T_\dagger = \{(i_1, i_2, \ldots) | i_k \in \{1, 2\}\},$$

demonstrating that the number of elements in $T_\dagger$ may be uncountable.

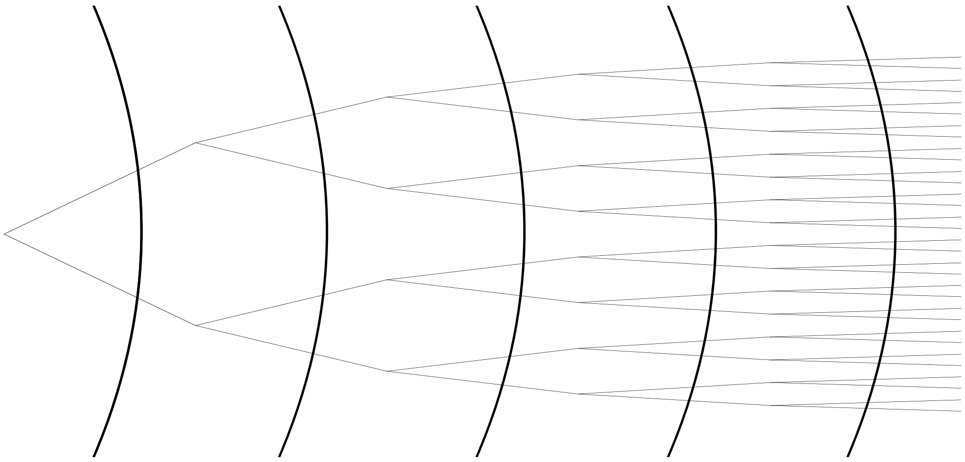

Figure 3: The set Y of Example 3.2

Lemma 3.3 *Let $(H_{i_1\cdots i_k})$ and $(H_{j_1\cdots j_k})$ be $(\dagger)$-sequences of the Raum X.*

(i) The sequence $(H_{i_1\cdots i_k})$ determines an end of X.

(ii) $(H_{i_1\cdots i_k}) \subseteq H_{j_1\cdots j_{k_0}}$ if and only if $i_k = j_k$ for all $k \in \{1, \ldots, k_0\}$.

(iii) $(H_{i_1\cdots i_k}) \approx (H_{j_1\cdots j_k})$ if and only if $i_k = j_k$ for each $k \in \mathbb{N}$.

Proof: Because each $H_{i_1\cdots i_k}$ is clopen relative to $X \backslash L_k$, $\partial H_{i_1\cdots i_k} \subseteq L_k$ and is thus compact. Consequently, $(H_{i_1\cdots i_k})$ determines an end, since

$$\bigcap_{k=1}^{\infty} \overline{H_{i_1\cdots i_k}} \subseteq \bigcap_{k=1}^{\infty} X \backslash L_k^\circ \subseteq X \backslash (\bigcup_{k=1}^{\infty} L_k^\circ) = \emptyset.$$

The second assertion follows directly from the fact that by definition, $(i_1, \ldots, i_k) \neq (j_1, \ldots, j_k)$ yields that $H_{i_1\cdots i_k}$ and $H_{j_1\cdots j_k}$ are disjoint, and clearly (iii) follows from (ii). ∎

Definition 3.4 With definitions as above,

$$\mathsf{E}_\dagger = \{(H_{i_1\cdots i_k}) | \underline{i} \in T_\dagger\}.$$

Note that by (iii) above, every $\mathsf{e} \in \mathsf{E}_\dagger$ has exactly one representative on the form $(H_{i_1\cdots i_k})$. We are ready to prove the existence of the Freudenthal compactification for noncompact Räume.

Proposition 3.5 *Let X be a Raum with compact nest $(\dagger)$. There is a topology making $X \cup \mathsf{E}_\dagger$ a Hausdorff compactification of X.*

Proof: We specify the new topology by giving bases of the neighborhood systems for each point in $X \cup \mathsf{E}_\dagger$. If $x \in X$, the basis is given by the set of neighborhoods of x in X. If $\mathsf{e} = [(H_{i_1 \cdots i_k})] \in \mathsf{E}_\dagger$, the basis is

$$V_{i_1 \cdots i_k} = H_{i_1 \cdots i_k} \cup \{\mathsf{e} \in \mathsf{E}_\dagger | \mathsf{e} \subseteq H_{i_1 \cdots i_k}\}. \tag{2}$$

Since these sets are nested, this does indeed define a neighborhood basis. That $X \cup \mathsf{E}_\dagger$ is Hausdorff follows by the fact that X is Hausdorff when the two points to be separated are both in X. When both points are in $\mathsf{E}_\dagger$, we note that if $\mathsf{e}_0 = [(H_{i_1 \cdots i_k})]$ and $\mathsf{e}_1 = [(H_{j_1 \cdots j_k})]$ are given with $\mathsf{e}_0 \neq \mathsf{e}_1$, then by Lemma 3.3(iii), there is a k with $i_k \neq j_k$ so that $H_{i_1 \cdots i_k}$ and $H_{j_1 \cdots j_k}$ are disjoint. Then so are the corresponding basic neighborhoods $V_{i_1 \cdots i_k}$ and $V_{j_1 \cdots j_k}$ by Lemma 3.3(ii). When $x \in X$ and $\mathsf{e} = [(H_{i_1 \cdots i_k})] \in \mathsf{E}_\dagger$ are given, we can pick a $k \in \mathbb{N}$ such that $x \in L_k^\circ$. Then there is a neighborhood V_x of x in X such that $V_x \subseteq L_k$, and then $V_x \cap V_{i_1 \cdots i_k} = V_x \cap H_{i_1 \cdots i_k} \subseteq V_x \cap (X \backslash L_k) = \emptyset$.

To see that the space is compact, let $(z_\lambda)_\Lambda$ be a universal net, cf. [8, 1.3.3,1.6.2]. If (z_λ) is eventually in X, we proceed as follows: If (z_λ) is eventually in some compact set of X, (z_λ) converges here, and hence in $X \cup \mathsf{E}_\dagger$. We may hence assume that when K is compact, (z_λ) is eventually in $X \backslash K$. By Proposition 2.9(iv), (z_λ) is eventually in the union of the unbounded components of $X \backslash L_1$, and since there is only a finite number of those by Proposition 2.9(i), it is eventually in one of them. We can thus pick $i_1 \in \mathbb{N}$ and $\lambda_1 \in \Lambda$ such that when $\lambda \geq \lambda_1$, $z_\lambda \in H_{i_1}$. Applying this argument again, (z_λ) is eventually in $H_{i_1 i_2}$ for some i_2, and we can choose a $(\dagger)$-sequence $(H_{i_1 \cdots i_k})$ and $\lambda_k \in \Lambda$ such that

$$\lambda \geq \lambda_k \Longrightarrow z_\lambda \in H_{i_1 \cdots i_k} \subseteq V_{i_1 \cdots i_k}.$$

By definition of the neighborhood system of $\mathsf{e} = [(H_{i_1 \cdots i_k})]$, $z_\lambda \to \mathsf{e}$. Now assume (z_λ) is eventually in $\mathsf{E}_\dagger$. We use the fact that for fixed k, the finitely many sets $\mathsf{E}_\dagger \cap V_{i_1 \cdots i_k}$ form a disjoint cover of $\mathsf{E}_\dagger$, and get a $(\dagger)$-sequence $(H_{i_1 \cdots i_k})$ and $\lambda_1 \leq \lambda_2 \leq \cdots$ such that

$$\lambda \geq \lambda_k \Longrightarrow z_\lambda \in \mathsf{E}_\dagger \cap V_{i_1 \cdots i_k} \subseteq V_{i_1 \cdots i_k}.$$

When $\mathsf{e} = [(H_{i_1 \cdots i_k})]$, $z_\lambda \to \mathsf{e}$. As X is clearly dense in $X \cup \mathsf{E}_\dagger$, we get a Hausdorff compactification of X this way. ∎

We denote the closure of a set Y in $X \cup \mathsf{E}_\dagger$ by $\overline{Y}^\dagger$.

Proposition 3.6 *Let G be an open set with compact boundary in the noncompact Raum X. When $\mathsf{e} \in \mathsf{E}_\dagger$, we have*

$$\mathsf{e} \subseteq G \Longleftrightarrow \mathsf{e} \in \overline{G}^\dagger.$$

Proof: We write $\mathsf{e} = [(H_{i_1 \cdots i_k})]$. If $H_{i_1 \cdots i_k} \subseteq G$ for some k, G meets all neighborhoods $V_{i_1 \cdots i_k}$ of e, and $\mathsf{e} \in \overline{G}^\dagger$. If $\mathsf{e} \in \overline{G}^\dagger$, every set $H_{i_1 \cdots i_k} = V_{i_1 \cdots i_k} \cap X$ meets G. As

$(H_{i_1\cdots i_k})$ is a sequence determining an end,

$$\bigcap_{k=1}^{\infty} (\partial G \cap \overline{H_{i_1\cdots i_k}}) = \emptyset,$$

and by compactness of ∂G, $\partial G \cap \overline{H_{i_1\cdots i_k}} = \emptyset$ for some $k \in \mathbb{N}$. By Lemma 2.2, we conclude from the above that $X\backslash G \cap \overline{H_{i_1\cdots i_k}} = \emptyset$, i.e. $H_{i_1\cdots i_k} \subseteq \overline{H_{i_1\cdots i_k}} \subseteq G$. ∎

Corollary 3.7 $\mathsf{E} = \mathsf{E}_\dagger$.

Proof: Let (G_l) be any sequence determining an end of X. The sets $(\overline{G_l}^\dagger)$ form a decreasing sequence of nonempty sets, and since $X \cup \mathsf{E}_\dagger$ is compact, their intersection is not empty. Since, by (iv) of the definition of sequences determining ends,

$$X \cap \bigcap_{l=1}^{\infty} \overline{G_l}^\dagger = \bigcap_{l=1}^{\infty} \overline{G_l} = \emptyset,$$

there is a point e of $\mathsf{E}_\dagger$ in $\bigcap_{l=1}^{\infty} \overline{G_l}^\dagger$, and hence in $\overline{G_l}^\dagger$ for all $l \in \mathbb{N}$. By the above proposition, $\mathsf{e} \subseteq G_l$ for all l, and $\mathsf{e} = [(G_l)]$ by Lemma 2.4. ∎

We have now seen that the above construction does not depend on the choice of nest, and may restate Proposition 3.5 as Theorem 3.8 below. To avoid confusion, let us explicitly define that a space Y is *totally disconnected* when every connected set is a singleton. However, due to [7, 1.3], any space on the form $\gamma X\backslash X$ will be totally disconnected precisely when it has (Čech) covering dimension zero.

Theorem 3.8 *Let X be a Raum. There is a topology making $X \cup \mathsf{E}$ a Hausdorff compactification of X such that E is totally disconnected and second countable in the relative topology. Furthermore, every neighborhood V of a point $\mathsf{e} \in \mathsf{E}$ contains an open neighborhood U of e such that $U \cap X$ is connected.*

Proof: We choose a nest (†) and do the construction of Proposition 3.5. Recall that a basic neighborhood of an end $\mathsf{e} = [(H_{i_1\cdots i_k})]$ is denoted by $V_{i_1\cdots i_k}$ and has the form given in (2). To see that E is totally disconnected, let $\mathsf{e}_0, \mathsf{e}_1 \in \mathsf{E}$ be given by $(H_{i_1^0\cdots i_k^0})$ and $(H_{i_1^1\cdots i_k^1})$, respectively. If $\mathsf{e}_0 \neq \mathsf{e}_1$, then there is a smallest $k \in \mathbb{N}$ such that $i_k^0 \neq i_k^1$. By definition of the neighborhood systems of points in E, the sets $\mathsf{E} \cap V_{i_1\cdots i_k}$ are open relative to E. Since they form a disjoint separation of E, and there is only a finite number of them, they are also relatively closed. As e_0 and e_1 lie in two different sets, they must lie in different components of E. We conclude that E is totally disconnected. For the second countability of E, note that as there are only a finite number of components of $X\backslash L_k$, the set of basic neighborhoods $V_{i_1\cdots i_k}$ is countable.

The last statement follows directly from the definitions; when $\mathsf{e} = [(H_{i_1\cdots i_k})]$, V contains $V_{i_1\cdots i_k}$ for some $k \in \mathbb{N}$. By definition, $V_{i_1\cdots i_k} \cap X = H_{i_1\cdots i_k}$ is connected in X as well as in $X \cup \mathsf{E}$. ■

We will refer to this compactification as *the Freudenthal compactification* or φX in the future, and we shall see shortly that φX is uniquely determined as the maximal compactification with a totally disconnected adjoined set.

Corollary 3.9 *Let X be a Raum equipped with a nest (†). We have*

$$\begin{aligned} \#_{\mathsf{E}} X &= \sup_{k\in\mathbb{N}} \#\{\textit{unbounded components of } X\backslash L_k\} \\ &= \lim_{k\in\mathbb{N}} \#\{\textit{unbounded components of } X\backslash L_k\} \end{aligned}$$

Proof: By Proposition 2.9, the number of unbounded components of $X\backslash L_k$ increases with k, so the limit exists and coincides with the supremum. As the number of unbounded components of $X\backslash L_k$ is the same as the number of indices $(i_1, \ldots, i_k)$ with $H_{i_1\cdots i_k} \neq \emptyset$, the result follows from Corollary 3.7. ■

Remark 3.10 As X is dense in φX, we get for an open set U, that $U \subseteq \overline{U \cap X}^{\varphi}$. Applying this to the set U mentioned in Theorem 3.8, we get that also U itself is connected, and φX is locally connected, hence a Raum. The condition given is stronger that requiring a compactification to be locally connected; for instance $\alpha\mathbb{R}$ is locally connected, but it does not have this property.

Remark 3.11 Notions of compact nests exist also in general C^*-algebras [1, 2.1.8], and they may be employed to define sequences determining ends just as in Definition 3.1 above. However, as we know of no generalization of the Freudenthal compactification, we must take a different approach in [1] to prove that every end has a representative subordinate to a given compact nest. This turns out not to be hard, see [1, 3.2.3]. With an algorithmic approach to ends in hand, the end structure is computable, and the number of ends can be found as in Corollary 3.9.

4 A characterization of φX

We now compare the Freudenthal compactification to other compactifications.

Lemma 4.1 *Let X be a Raum with compact nest (†). When $(X \hookrightarrow \gamma X)$ is a compactification, we have*

$$\bigcup_{\underline{i}\in T_\dagger} \bigcap_{k=1}^{\infty} \overline{H_{i_1\cdots i_k}}^{\gamma} = \gamma X\backslash X$$

where all sets $\bigcap_{k=1}^{\infty} \overline{H_{i_1\cdots i_k}}^{\gamma}$ *are connected and nonempty.*

Proof: We always have

$$\bigcap_{k=1}^{\infty} \overline{H_{i_1\cdots i_k}}^{\gamma} \cap X = \bigcap_{k=1}^{\infty} \overline{H_{i_1\cdots i_k}} = \emptyset,$$

so that every set $\bigcap_{k=1}^{\infty} \overline{H_{i_1\cdots i_k}}^{\gamma}$ is contained in $\gamma X \backslash X$. The intersection is nonempty and connected by compactness of γX, cf. [9, 28.2].

Let $\mathsf{g} \in \gamma X \backslash X$ be given. We shall work in φX to find a sequence $\underline{i} \in T_{\dagger}$ for which g is contained in $\bigcap_{k=1}^{\infty} \overline{H_{i_1\cdots i_k}}^{\gamma}$. Letting $\mathcal{N}$ be the set of open neighborhoods of g in γX, we have that

$$\mathcal{N}_X = \{U \cap X | U \in \mathcal{N}\}$$

has the finite intersection property since X is dense. From compactness of φX, $\bigcap_{V \in \mathcal{N}_X} \overline{V}^{\varphi} \neq \emptyset$, whereas $\bigcap_{\mathcal{N}_X} \overline{V} = \emptyset$ by the fact that γX is Hausdorff. This means that $\underline{i} \in T_{\dagger}$ exists with

$$\mathsf{e} = [(H_{i_1\cdots i_k})] \in \bigcap_{V \in \mathcal{N}_X} \overline{V}^{\varphi} \backslash X.$$

By definition of the topology on φX, every $V \in \mathcal{N}_X$ meets every $H_{i_1\cdots i_k}$, proving that $\mathsf{g} \in \overline{H_{i_1\cdots i_k}}^{\gamma}$ for every $k \in \mathbb{N}$. ■

The point of the above lemma is that we may reach all components of any compactification of X by means of our sets $H_{i_1\cdots i_k}$. We use this result to give a characterization of φX.

Proposition 4.2 *Let X be a Raum with compact nest ($\dagger$). When $(X \hookrightarrow \gamma X)$ is a compactification such that $\gamma X \backslash X$ is totally disconnected, then $(X \hookrightarrow \varphi X) \geq (X \hookrightarrow \gamma X)$.*

Proof: We define $\kappa : \varphi X \to \gamma X$. When $x \in X$, $\kappa(x) = x$. When $\mathsf{e} = [(H_{i_1\cdots i_k})]$ is given, we note that by Lemma 4.1 above, the intersection of all γ-closures is a connected set. It is hence a singleton, say

$$\bigcap_{k=1}^{\infty} \overline{H_{i_1\cdots i_k}}^{\gamma} = \{\mathsf{g}\}$$

where $\mathsf{g} \in \gamma X \backslash X$. We put $\kappa(\mathsf{e}) = \mathsf{g}$.

To see that κ is continuous, let U be an open neighborhood of $\mathsf{g} = \kappa([(H_{i_1\cdots i_k})])$. We have that $\mathsf{g} \notin \partial_{\gamma} U$, and conclude from

$$\bigcap_{k=1}^{\infty} (\partial_{\gamma} U \cap \overline{H_{i_1\cdots i_k}}^{\gamma}) = \partial_{\gamma} U \cap \{\mathsf{g}\} = \emptyset$$

by compactness that $\partial_\gamma U \cap \overline{H_{i_1\cdots i_k}}^\gamma = \emptyset$ for some $k \in \mathbb{N}$. As $\mathsf{g} \in U \cap \overline{H_{i_1\cdots i_k}}^\gamma$, we deduce from connectivity of $\overline{H_{i_1\cdots i_k}}^\gamma$ that $H_{i_1\cdots i_k} \subseteq \overline{H_{i_1\cdots i_k}}^\gamma \subseteq U$ according to Lemma 2.2. When $\mathsf{e} = [(H_{j_1\cdots j_k})] \in V_{i_1\cdots i_k}$, we have $j_k = i_k$ for $i \in \{1, \dots k\}$ by Lemma 3.3(ii), whence $\kappa(\mathsf{e}) \in \overline{H_{i_1\cdots i_k}}^\gamma$ by definition. We conclude that $\kappa(V_{i_1\cdots i_k}) \subseteq U$, and κ is continuous. ∎

Corollary 4.3 *Let X be a Raum. The Freudenthal compactification $(X \hookrightarrow \varphi X)$ is the largest compactification such that $\varphi X \backslash X$ is totally disconnected.*

Remark 4.4 In Proposition 4.2, assume further that γX has the same property as φX; that for every neighborhood V of $\mathsf{g} \in \gamma X \backslash X$, there is an open neighborhood U of g with $U \cap X$ connected. Then κ is 1–1, for with W_k the union of all $H_{i_1\cdots i_k}$, $X \backslash W_k$ is compact by Proposition 2.9(iv) and $V_k = \gamma X \backslash (X \backslash W_k) = (\gamma X \backslash X) \cup W_k$ is open. Fixing $\mathsf{g} \in \gamma X \backslash X$ and taking a neighborhood $U_k \subseteq V_k$ of g as above, $U_k \cap X$ must be contained in $H_{i_1\cdots i_k}$ for some index $(i_1, \dots, i_k)$, and $\mathsf{g} \in \overline{H_{i_1\cdots i_k}}^\gamma$ for only one tuple $\underline{n} \in \mathbb{N}^{\mathbb{N}}$. We conclude that $(X \hookrightarrow \varphi X) = (X \hookrightarrow \gamma X)$, giving a different characterization of φX.

Example 4.5

1°: $\varphi\mathbb{R} = [0, 1]$. $\varphi\mathbb{R} \backslash \mathbb{R} = \{0, 1\}$.

2°: $\varphi\mathbb{R}^2 = \alpha\mathbb{R}^2 = S^2$. $\varphi\mathbb{R}^2 \backslash \mathbb{R}^2 = \{pt\}$.

Consider now the spaces in Example 3.2.

3°: $\varphi X \backslash X$ is the discrete set with 5 points.

4°: $\varphi Y \backslash Y$ is the Cantor set.

Remark 4.6 We have no generalized notion of Freudenthal compactifications in [1]. A result parallel to Lemma 4.1 may be found, but its proof is, consequently, very different, and it is only good when $\gamma = \beta$. To see that the intersections of sequences determining ends cover the corona, one uses (a noncommutative version of) the following reasoning. Fix $\mathsf{g} \in \gamma X \backslash X$. It follows from Proposition 2.9 that at fixed levels $k < l$, g lies in $\overline{H_{i_1\cdots i_k}}^\gamma$ and $\overline{H_{j_1\cdots j_l}}^\gamma$, and by the strong separation properties of βX, $i_n = j_n$ for every $n \leq k$.

A further complication is that in the noncommutative setting, the analog of [9, 28.2] fails. One must hence use the special properties of a sequence determining an end to get that the intersection of the closures of a sequence determining an end is connected. This is the major technical difficulty in the proof of [1, 4.1.7].

5 Ends and corona components

The $H_{i_1\cdots i_k}$ even capture the component structure of the maximal compactification βX.

Proposition 5.1 *Let X be a Raum with compact nest (†). Every component of $\beta X \backslash X$ has the form*

$$\bigcap_{k=1}^{\infty} \overline{H_{i_1\cdots i_k}}^{\beta}$$

for one and only one $\underline{i} \in T_\dagger$.

Proof: Fix $\underline{i} \in T_\dagger$. We shall prove that $\bigcap_{k=1}^{\infty} H_{i_1\cdots i_k}$ is a component. It is connected by Lemma 4.1, so assume that it is nontrivially contained in a component C. Then C meets $\bigcap_{k=1}^{\infty} H_{j_1\cdots j_k}$ nontrivially for some $\underline{j} \in T_\dagger$. For some fixed k, $H_{i_1\cdots i_k}$ and $H_{j_1\cdots j_k}$ are different components of $X \backslash L_k$. In $X \backslash L_{k+1}^\circ$ the set $G = H_{j_1\cdots j_k} \backslash L_{k+1}^\circ$ is clopen, so that $1_G \in C^b(X \backslash L_{k+1}^\circ)$. We may extend 1_G to $f \in C^b(X)$ by Tietze's extension theorem, and note that

$$f(H_{i_1\cdots i_{k+1}}) = \{0\}, \qquad f(H_{j_1\cdots j_{k+1}}) = \{1\}.$$

Letting f denote the extension of f to βX as well, $\overline{H_{i_1\cdots i_{k+1}}}^{\beta} \subseteq f^{-1}(\{0\})$ and $\overline{H_{j_1\cdots j_{k+1}}}^{\beta} \subseteq f^{-1}(\{1\})$, so that

$$\overline{H_{i_1\cdots i_{k+1}}}^{\beta} \cap \overline{H_{j_1\cdots j_{k+1}}}^{\beta} = \emptyset,$$

a contradiction.

We conclude that $C = \bigcap_{k=1}^{\infty} H_{i_1\cdots i_k}$. Given an arbitrary component D, take $\mathsf{g} \in D$ and, employing Proposition 4.1 again, $\underline{i} \in T_\dagger$ with $\mathsf{g} \in \bigcap_{k=1}^{\infty} H_{i_1\cdots i_k}$. When two components are not disjoint, they are identical, and $D = \bigcap_{k=1}^{\infty} H_{i_1\cdots i_k}$. ∎

The above proposition relates the component structure of $\beta X \backslash X$ to that of $\varphi X \backslash X$. As the components of $\varphi X \backslash X$ are just the points of $\varphi X \backslash X$, we have established a 1–1 correspondence between the points of $\varphi X \backslash X$ and the components of $\beta X \backslash X$.

Remark 5.2 The main theorem of [1], [1, 4.1.7], is an analog of Proposition 5.1 above.

6 End theory for product spaces

Proposition 6.1 *Let X, Y be Räume with Y noncompact. Letting $Z = X \times Y$, Z is a Raum, and* *(i) If X is compact, then $\varphi Z\backslash Z$ and $\varphi X\backslash X$ are homeomorphic.*

(ii) If X is not compact, then Z has exactly one end.

Proof: That Z is a Raum is obvious as products of connected, resp. locally connected spaces are again connected, resp. locally connected. In (i), let (†) be a nest for Y; $X \times L_n$ forms a nest for Z, and we may arrange the unbounded components as

$$\tilde{H}_{i_1\cdots i_k} = X \times H_{i_1\cdots i_k},$$

where the axioms (i)–(iv) from Definition 2.1 are satisfied for the $\tilde{H}_{i_1\cdots i_k}$, because they are satisfied for the $H_{i_1\cdots i_k}$. According to Lemma 3.3(iii), the mapping $\kappa : \varphi X\backslash X \to \varphi Z\backslash Z$ given by

$$\kappa([(H_{i_1\cdots i_k})]) = [(\tilde{H}_{i_1\cdots i_k})]$$

is a well-defined bijection. By compactness, we need only prove it is continuous, and that follows from

$$\kappa^{-1}(\tilde{V}_{j_1\cdots j_k}\backslash Z) = \{[(H_{i_1\cdots i_k})] | i_k = j_k, 1 \leq i \leq k\} = V_{i_1\cdots i_k}\backslash X.$$

For (ii), let K_k be a compact nest for X; clearly $K_k \times L_k$ is a compact nest for Z, and we need only prove that $Z\backslash(K_k \times L_k)$ is connected for every $k \in \mathbb{N}$ according to Corollary 3.9. As X and Y are not compact, we can choose $x_k \in X\backslash K_k$ and $y_k \in Y\backslash L_k$ and write

$$\begin{aligned} Z\backslash(K_k \times L_k) &= \left(\bigcup_{x\notin K_k} \{x\} \times Y\right) \cup \left(\bigcup_{y\notin L_k} X \times \{y\}\right) \\ &= \left(\bigcup_{x\notin K_k} [\{x\} \times Y \cup X \times \{y_k\}]\right) \cup \left(\bigcup_{y\notin L_k} [Y \times \{y\} \cup \{x_k\} \times Y]\right). \end{aligned}$$

This set is connected by five applications of the fact that the union of connected sets with nonempty intersection is connected, cf. [9, 26.7(a)]. ■

Remark 6.2 There is a complete analog to Proposition 6.1(i) above in [1, 5.1.6] and one to Proposition 6.1(ii) in [1, 5.1.10].

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