Connectivity and Ends in C^*-Algebra Theory

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0 Introduction

0.1 A historical account

As this project has evolved, goals have changed, viewpoints have shifted, and expositions have been rewritten several times, and as is no doubt very common for work of this kind, my focus has been on the "how"s rather than on the "why"s during most of the elapsed period of time. This is an attempt to explain to the reader – and to remind the author – which questions provoked the answers in this thesis.

The notion of connectivity for C^*-algebras originated in an attempt to find *Liapounoff theorems*, in the sense of [3], for the dual spaces of a C^*-algebra $\mathfrak{A}$. The question arose: for which C^*-algebras do we have

$$\hat{a}(S(\mathfrak{A})) = \hat{a}(P(\mathfrak{A})), \qquad \forall a \in \mathfrak{A}_{sa}, \tag{1}$$

where $S(\mathfrak{A})$ is the set of states of $\mathfrak{A}$, $P(\mathfrak{A})$ is the set of pure states; the extreme points of the convex set $S(\mathfrak{A})$, and $\hat{a}$ is the canonical image of $\mathfrak{A}$ in its double dual, i.e. $\hat{a}(\varphi) = \varphi(a)$.

The answer to this question, for $\mathfrak{A}$ σ-unital, can be found in Theorem 6.2.2, and states that (1) is true if and only if $\mathfrak{A}$ has no non-trivial decomposition as $\mathfrak{A} = \mathfrak{I}_0 \oplus \mathfrak{I}_1$, where $\mathfrak{I}_0, \mathfrak{I}_1$ are ideals of $\mathfrak{A}$. We were naturally led to consider the class of C^*-algebras with this Liapounoff property, and as time passed, the term "connected", which were originally a tentative working title for these C^*-algebras, changed into a label offering intuition as well as inspiration.

A further investigation made the list of equivalent statements longer, and the proof much shorter, and with these improvements, it seemed possible to engage in the problem of how connectivity behaves in relation to constructions of C^*-algebras, like "If $\mathfrak{A}$, $\mathfrak{B}$ are connected, when is $\mathfrak{A} \otimes_\gamma \mathfrak{B}$?". Questions along these lines have been a constant source of interesting problems, but for a long time the main motivating factor has been from the notion of *ends*.

The theory of ends in topological spaces tries to capture a global aspect; loosely speaking, it supplies a way of counting the essentially different paths to infinity in a connected, σ-compact, but not compact, topological space enjoying some additional properties. The approach is simple – one looks at the non-precompact components outside each element of an ascending sequence of compact sets, and as a notion of components should be easy to develop from a notion of connectivity, we set out to generalize the theory of ends to C^*-algebras, hoping to describe a global property of C^*-algebras and eventually relate it to the work in [5].

A bit of work went into finding an adequate definition of components and developing the notion, but the proofs needed were all rather easy to come by. However, it became increasingly clear that the definition of connectivity for $\mathfrak{A}$ is essentially commutative in the sense that it focuses on the structure of the center of $M(\mathfrak{A})$, the so-called *ideal center* of $\mathfrak{A}$, and while this fact seems to be essential for proving many of the results achieved, it also limits the applications of the theory, making it very difficult to come up with a reasonable definition of ends via the component theory constructed.

Somewhat discouraged, I started looking into the classical end theory as it is presented by Freudenthal in [14], and along the way, an idea for an alternative definition of ends evolved, generalizing the fact that for a not too "ugly" topological space X, each end of X is reflected by one and only one component of $\beta X \backslash X$. Thus, one way of counting ends of the connected, non-unital C^*-algebra $\mathfrak{A}$ could be counting components of the corona algebra $M(\mathfrak{A})/\mathfrak{A}$, following a long tradition of finding information about the global structure of a C^*-algebra in its corona.

Exited by the prospect of a simple, meaningful definition, we put a lot of work into trying to determine whether or not certain rules for counting ends of a product space would generalize. In particular, the commutative theory shows that if X, Y are connected and non-compact, then $X \times Y$ has only one end, although X, Y may have several each (consider $X = Y = \mathbb{R}$). The question of whether or not this holds in a more general setting is still unsolved, even in the case where the multiplier algebra can be calculated with the theory from [6].

Another hope was to count the number of ends in a different fashion and prove that for a reasonable class of C^*-algebras, the numbers of ends are the same. There seems to be quite a number of possible paths to this goal, but at this stage, exactly which one to follow does not seem clear. Partial results do, however, give an indication that a theorem of this kind might be within reach.

0.2 Prerequisites

As described in the previous section, most of the work presented in this thesis was done as part of a *tour* of the C^*-algebra theory from the point of view of a topological notion. The project served as a motivating factor for me to acquaint myself with even the hairiest details of different branches of this field, and as a result, many different parts are touched upon, in varying depth, in this thesis.

Writing down these pages, I have tried to base the proofs on a minimal number of standard textbooks, and to include self-contained descriptions of needed results, that I have not been able to find in those books. Part I thus primarily contains material from [2], [3], [4], [7], [12] and [15], presented in my own words with coherent notation. Also, reflecting the fact that I have spent a great amount of time learning to operate this theory, some pages are included with an account of the theory of the enveloping von Neumann algebra and supported subspaces of $\mathfrak{A}^*$, although this can be found in [22]. Advanced readers will probably want to skip this part entirely and return to it for reference purposes only.

I will assume that the reader is acquainted with the theories described in the textbook sections below, but will make explicit references whenever a main result is needed. For general functional analysis and basic C^*-algebra theory, including spectral theory, we use [23, 1–4]. For more advanced, but still basic, C^*-algebra theory, including states and representations, we refer to [22, 1,3.1–4.4], for von Neumann algebras we apply [17, 5.1,5.3–5.5,7.1–8.3], for direct sums and tensor products, [20, 6].

0.3 Notation

We will apply standard terms from functional analysis, topology and measure theory without further mention. Note that when M is a subset of some normed space E, we write M_1 for the intersection of M with the closed unit ball of E. We use "$\bar{\ }$" for closure, except when referring to norm closure in C^*-algebras or their duals, where we use "$\bar{\bar{\ }}$". When X is a topological space, $C^b(X)$ is the set of bounded, continuous functions on X, $C_0(X)$ those continuous functions that vanish at infinity, and $C_{00}(X)$ those continuous functions that have compact support. When X is compact, all these spaces coincide, and we just write $C(X)$. We write βX for the Stone-Čech compactification of a space, cf. [23, 4.3.18].

Let $\mathfrak{A}$ be a C^*-algebra, unital or not. We write $\mathfrak{A}^\sim$ for the unitization of $\mathfrak{A}$ when $\mathfrak{A}$ is non-unital – when $\mathfrak{A}$ had a unit to begin with, $\mathfrak{A}^\sim = \mathfrak{A}$. Spectra of elements $a \in \mathfrak{A}$ will be taken in $\mathfrak{A}^\sim$ and will be denoted by $\mathrm{sp}(a)$. We will use standard manipulation with elements $a \in \mathfrak{A}$, e.g. $\mathrm{Re}\ a, \mathrm{Im}\ a, a_+, a_-, a^{1/2}$. The center of $\mathfrak{A}$ is referred to by $\mathcal{Z}(\mathfrak{A})$, the unitary group (when $\mathfrak{A}$ is unital) by $\mathcal{U}(\mathfrak{A})$. Ideals of $\mathfrak{A}$ are assumed to be closed.

When $\mathfrak{H}$ is a Hilbert space, we denote the inner product by $(\cdot|\cdot)$. We write $\mathbb{B}(\mathfrak{H}, \mathfrak{K})$ for the set of bounded operators between $\mathfrak{H}$ and $\mathfrak{K}$; $\mathbb{B}(\mathfrak{H}) = \mathbb{B}(\mathfrak{H}, \mathfrak{H})$ and $\mathbb{K}(\mathfrak{H})$ is the set of compact operators here. The C^*-algebra of complex $n \times n$-matrices is denoted by $M_n(\mathbb{C})$, and by $M_n(\mathfrak{A})$ we mean the matrices over $\mathfrak{A}$ with norm stemming from a faithful representation of $\mathfrak{A}$.

When $\mathcal{M}$ is a von Neumann algebra, we write $\mathcal{M}'$ for its commutant and $\mathcal{M}_*$ for its predual. When $p, q \in \mathcal{M}$ are projections, we denote by $p \wedge q$ the infimum, by $p \vee q$ the supremum of the two. We shall use this terminology for arbitrary families of projections as well, using tacitly that the resulting projections lie in fact in $\mathcal{M}$, cf. [17, 5.1.8].

Whenever M is a subset of a space in which we have notions of positivity or self-adjointness, we denote by M_+ and M_{sa} the positive and self-adjoint part of M, respectively. In $\mathfrak{A}^*$ we denote the *quasi-states* $[((\mathfrak{A}^*)_+)_1]$ by $Q(\mathfrak{A})$, the *states* $[\{\varphi \in Q(\mathfrak{A}) | \|\varphi\| = 1\}]$ by $S(\mathfrak{A})$ and the *pure states* $[\operatorname{ext} S(\mathfrak{A})]$ by $P(\mathfrak{A})$. Where no confusion can arise, we abbreviate by Q, S, P. When $\varphi \in S(\mathfrak{A})$, we denote the GNS triple from φ by $(\pi_\varphi, \mathfrak{H}_\varphi, \xi_\varphi)$. We also write L_φ for the left kernel of φ, i.e.

$$L_\varphi = \{x \in \mathfrak{A} | \varphi(x^* x) = 0\}.$$

In the "miscellany" department, note that we use the hat to symbolize embedding into double duals in two different ways: $\hat{x}$ denotes the character $\hat{x} : C_0(X) \to \mathbb{C}$ defined by $\hat{x}(f) = f(x)$ and for $a \in \mathfrak{A}$, $\hat{a} : \mathfrak{A}^* \to \mathbb{C}$ is given by $\hat{a}(\varphi) = \varphi(a)$. Also, $\aleph_0$ is the cardinality of $\mathbb{N}$, $\aleph$ that of $\mathbb{R}$.

More notation will be introduced in the sections where it is first needed. Also, some terminology regarding convexity and connectivity will be introduced in the appendices, and further terms regarding representations and states will be introduced along with a brief explanation in Section 1. The reader will find a list of symbols, complete with references, starting on page 105.

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Part I

Preliminaries

1 Introduction to $\mathfrak{A}^{**}$

As mentioned in Section 0.2, we formally assume our reader to be acquainted with [22, 3], in which a thorough (albeit compact) exposition of the theories described in this section can be found. However, we are going to make full use of the tools that the enveloping von Neumann algebra of a C^*-algebra provides, so in the intent of assisting our reader, as well as establishing some notation, we include here a short description of the fundamental results.

1.1 Terminology and basic results

We define the *universal representation* $(\pi_u, \mathfrak{H}_\mathfrak{u})$ of $\mathfrak{A}$ as

$$\pi_u = \bigoplus_{\varphi \in S(\mathfrak{A})} \pi_\varphi, \qquad \mathfrak{H}_\mathfrak{u} = \bigoplus_{\varphi \in \mathfrak{S}(\mathfrak{A})} \mathfrak{H}_\varphi$$

and, similarly, the *reduced atomic representation* $(\pi_a, \mathfrak{H}_\mathfrak{a})$ as

$$\pi_u = \bigoplus_{\varphi \in P(\mathfrak{A})} \pi_\varphi, \qquad \mathfrak{H}_\mathfrak{u} = \bigoplus_{\varphi \in \mathfrak{P}(\mathfrak{A})} \mathfrak{H}_\varphi.$$

Both of these representations are *faithful*, i.e. 1–1. It turns out that the von Neumann algebra $\pi_u(\mathfrak{A})''$ is isomorphic as a Banach space to the second dual of $\mathfrak{A}$, and we write $\mathfrak{A}^{**}$ for the von Neumann algebra as well. The *enveloping von Neumann algebra* of $\mathfrak{A}$, as $\mathfrak{A}^{**}$ is called, contains a lot of information about $\mathfrak{A}$, and we shall identify $\mathfrak{A}$ with its image in $\mathfrak{A}^{**}$ and use this as the default framework for constructions involving $\mathfrak{A}$. We will thus tacitly consider the unitization $\mathfrak{A}^\sim$ as well as the multiplier algebra $M(\mathfrak{A})$ as subalgebras of $\mathfrak{A}^{**}$, and denote by $\mathfrak{A}'$ the commutant of $\mathfrak{A}$ in $\mathbb{B}(\mathfrak{H}_\mathfrak{u})$, by $\mathfrak{A}^m$ the monotone increasing closure[1] of $\mathfrak{A}$ in $\mathfrak{A}^{**}$ and so on.

Two representations $\pi : \mathfrak{A} \to \mathbb{B}(\mathfrak{H}), \rho : \mathfrak{A} \to \mathbb{B}(\mathfrak{K})$ are *equivalent* when $\rho = \psi \circ \pi$ for some isomorphism $\psi : \pi(\mathfrak{A})'' \to \rho(\mathfrak{A})''$. They are said to be *unitarily equivalent*, when ψ can be chosen as adjunction by a unitary, i.e. when

$$\rho(\cdot) = u\pi(\cdot)u^*$$

[1] Contained in $\mathfrak{A}^{**}$ because monotone nets converge strongly.

for some unitary $u \in \mathbb{B}(\mathfrak{H}, \mathfrak{K})$, cf. [22, 3.3.6]. One of the virtues of $\mathfrak{A}^{**}$ is that every non-degenerate representation $(\pi, \mathfrak{H})$ of $\mathfrak{A}$ is equivalent to one of the form

$$a \mapsto ap$$

where $p \in \mathfrak{A}^{**}$ is a central projection. Indeed, there is a unique extension of π to a normal representation $\pi^{**} : \mathfrak{A}^{**} \to \mathbb{B}(\mathfrak{H})$ such that $\pi^{**}(\mathfrak{A}^{**}) = \pi(\mathfrak{A})''$, and since $\ker \pi^{**}$ is a strongly closed ideal, we can find p as the projection satisfying $\ker \pi^{**} = \mathfrak{A}^{**}(1-p)$. For details, cf. [22, 3.8.1]. The unique projection p is called *the (central) cover* of π and there is a 1–1 correspondence between the set of non-zero central projections of $\mathfrak{A}^{**}$ and equivalence classes of representations of $\mathfrak{A}$.

When φ is a pure state of $\mathfrak{A}$, the GNS representation π_φ is *irreducible*, i.e. $\pi_\varphi(\mathfrak{A})'' = \mathbb{B}(\mathfrak{H}_\varphi)$. On the other hand, if $(\pi, \mathfrak{H})$ is irreducible, $a \mapsto (\pi(a)\xi|\xi)$ is a pure state for every unit vector $\xi \in \mathfrak{H}$. Equivalent irreducible representations are automatically unitarily equivalent. In fact, we have the following:

Proposition 1.1.1 *Let $\varphi, \psi \in P(\mathfrak{A})$. The following conditions are equivalent:*

(i) π_φ is equivalent to π_ψ.

(ii) π_φ is unitarily equivalent to π_ψ.

(iii) $\varphi(\cdot) = \psi(u^* \cdot u)$ for some $u \in \mathcal{U}(\mathfrak{A}^\sim)$.

Definition 1.1.2 We say that pure states φ, ψ are *equivalent*, and write $\varphi \sim \psi$, when they satisfy (i)-(iii) above.

Proof of Proposition 1.1.1: Combine [22, 3.13.3] and [22, 3.13.4]. ■

Note that the cover p of an irreducible representation is a minimal central projection, as $\mathfrak{A}^{**}p \simeq \mathbb{B}(\mathfrak{H})$ for some $\mathfrak{H}$. We infer that non-equivalent irreducible representations have orthogonal supports. We denote by $\hat{\mathfrak{A}}$ the set of equivalence classes of non-zero irreducible representations of $\mathfrak{A}$. A related set is $\mathrm{Prim}(\mathfrak{A})$, the set of primitive ideals of $\mathfrak{A}$, i.e. those that are kernels of irreducible representations. We endow this set with hull-kernel topology, and write Hull and Ker, capitalized, to denote the operations involved in defining this topology, cf. [22, 4.1].

As $\mathfrak{A}^{**}$ is a von Neumann algebra, we may talk about its predual, the set of normal functionals on $\mathfrak{A}^{**}$, cf. [22, 3.6.4]. It turns out that the normal functionals are exactly (the extensions of) the norm continuous functionals on $\mathfrak{A}$, and we shall identify $\mathfrak{A}^*$ with $(\mathfrak{A}^{**})_*$. We may thus apply Sakai's polar decomposition to elements of $\mathfrak{A}^{**}$ – in particular, $|\varphi|$ makes sense for all $\varphi \in \mathfrak{A}^*$ as in [22, 3.6.7]. We say that a projection $p \in \mathfrak{A}^{**}$ *supports* $\varphi \in S(\mathfrak{A})$, when $\varphi(p) = 1$. As φ is normal on $\mathfrak{A}^{**}$, it has *a support*, i.e. a minimal supporting projection, cf. [17, 7.2.5]. According to [22, 3.13.6], a state is pure if and only if it has a minimal support. We make the following definition.

Definition 1.1.3

$$z = \bigvee \{p | p \text{ minimal}\}$$

Let $u \in \mathcal{U}(\mathfrak{A}^{**})$. As u^*pu is minimal when p is, $u^*zu = z$, and as the subspace spanned by the unitaries is strongly dense in $\mathfrak{A}^{**}$, z is central. The projection z covers the part of $\mathfrak{A}^{**}$ on which the pure states "live". We shall see shortly, that z is in fact the cover of the reduced atomic representation π_a.

2 On projections in $\mathfrak{A}^{**}$

Being a von Neumann algebra, $\mathfrak{A}^{**}$ contains an abundance of projections – too many, for most purposes. In this section we look at different classes of projections that are close to the original C^*-algebra $\mathfrak{A}$ in different senses.

2.1 Some subspaces of $\mathfrak{A}^*$

The main purpose of this section is to introduce some notation and to state the central theorems of this part of the theory, as a base for the sections to follow. Most of the results can be found in [22, 3.6], and we will refer the reader to this source for proofs, whenever possible. We shall use a notation somewhat more explicit than that of [22, 3.6], inspired by Brown ([10]), stressing the importance of the projections. Another difference between this exposition and that of Pedersen is that we introduce the faces earlier, taking into account also norm closed faces of Q. Most of the proofs in this section have been included to fill the gap caused by this small variation.

Let $\mathfrak{A}$ be a C^*-algebra, and let M be a subspace of $\mathfrak{A}^*$. Let $\mathfrak{B}$ be a sub-C^*-algebra of $\mathfrak{A}^{**}$. We say that M is *left invariant* under $\mathfrak{B}$, if

$$\varphi(b\cdot) \in M, \qquad \forall \varphi \in M, b \in \mathfrak{B}.$$

Right invariance is defined similarly. Subspaces that are both left and right invariant are called *invariant*. When M is a cone sitting within $(\mathfrak{A}^*)_+$, i.e. when M satisfies $M + M \subseteq M$ and $\mathbb{R}_+ M \subseteq M$, we say that M is *hereditary*, when for $\varphi \in (\mathfrak{A}^*)_+$, $\psi \in M$,

$$0 \leq \varphi \leq \psi \Longrightarrow \varphi \in M.$$

Faces are introduced in Appendix A.

Definition 2.1.1 Let $p \in \mathfrak{A}^{**}$ be a projection. We say that p *supports*

(i) the left invariant subspace $L(p)$ of $\mathfrak{A}^*$,

$$L(p) = {}^{\perp}(\mathfrak{A}^{**}(1-p)) = \{\varphi \in \mathfrak{A}^* | \forall a \in \mathfrak{A}^{**} : \varphi(a(1-p)) = 0\}$$

(ii) the hereditary cone $C(p)$ of $\mathfrak{A}^*$,

$$C(p) = \{\varphi \in (\mathfrak{A}^*)_+ | \varphi(1-p) = 0\}$$

(iii) the face $F(p)$ of Q,

$$F(p) = \{\varphi \in Q | \varphi(1-p) = 0\}$$

(iv) the subset $P(p)$ of P,

$$P(p) = F(p) \cap P = \{p \in P | \varphi(p) = 1\}$$

and call these sets the left invariant subspace, the hereditary cone, the face and the set of pure states *supported* by p, respectively.

We will start out by concerning ourselves primarily with the first three types of sets, justifying the implicit assertions in their names along the way. First we need to make sure that a closed subspace is left invariant under $\mathfrak{A}$ and $\mathfrak{A}^{**}$ simultaneously.

Lemma 2.1.2 *Let M be a norm closed subspace of $\mathfrak{A}^*$. If M is left invariant under $\mathfrak{A}$, it is left invariant under $\mathfrak{A}^{**}$.*

Proof: Fix $\varphi \in M$ and $a \in \mathfrak{A}^{**}$. As $\mathfrak{A}$ is σ-weakly dense in $\mathfrak{A}^{**}$ we can take a net b_λ converging to a in this topology, and since it is induced by $\mathfrak{A}^*$ we have $\varphi(b_\lambda c) \to \varphi(ac)$ for all $c \in \mathfrak{A}^{**}$ since $\varphi(\cdot c) \in \mathfrak{A}^*$. Hence $\varphi(b_\lambda \cdot) \to \varphi(a \cdot)$ in the weak topology induced by $\mathfrak{A}^{**}$ on $\mathfrak{A}^*$, and since M is a norm closed convex subset of $\mathfrak{A}^*$, it is weakly closed. By assumption, each $\varphi(b_\lambda \cdot)$ lies in M, hence so does $\varphi(a \cdot)$. ■

Theorem 2.1.3 *The mappings L, C and F are order-preserving bijections from the class of projections in $\mathfrak{A}^{**}$ to the classes of*

(i) norm closed subspaces of $\mathfrak{A}^*$ that are left invariant under $\mathfrak{A}$,

(ii) norm closed hereditary cones of $\mathfrak{A}^*$,

(iii) norm closed faces of Q *containing* 0,

respectively. Furthermore, $L(p)_+ = C(p)$, $C(p) \cap Q = F(p)$ *and* $\operatorname{ext} F(p) = P(p) \cup \{0\}$.

Proof: Most of this can be found by applying [22, 3.6.11] to the von Neumann algebra $\mathfrak{A}^{**}$ with predual $\mathfrak{A}^*$. This way we get that $L(p)$ is a norm closed subspace of $\mathfrak{A}^*$ which is left invariant under $\mathfrak{A}^{**}$, and that all such spaces can be obtained this way in a 1–1 manner. But if M is a closed subspace that is left invariant under $\mathfrak{A}$, it is also left invariant under $\mathfrak{A}^{**}$ according to Lemma 2.1.2. We also get from [22, 3.6.11] that $L(p)_+ = C(p)$ and that C is a bijection.

It is clear that $F(p)$ is a norm closed face of Q containing 0. On the other hand, when F is such a face, let $C = \mathbb{R}_+ F$. Using convexity of F, we get that C is a cone, and it is hereditary because if $0 \le \varphi \le \lambda\psi$, with $\psi \in F$, $\lambda \in \mathbb{R}_+$, we can write

$$\begin{aligned} \psi &= \lambda^{-1}\|\varphi\|\|\varphi\|^{-1}\varphi + \left\|\psi - \lambda^{-1}\varphi\right\|\left\|\psi - \lambda^{-1}\varphi\right\|^{-1}(\psi - \lambda^{-1}\varphi) \\ &= \lambda^{-1}\|\varphi\|\chi_1 + \left\|\psi - \lambda^{-1}\varphi\right\|\chi_2 + (1 - \|\psi\|)0, \end{aligned}$$

where $\chi_1 = \|\varphi\|^{-1}\varphi$, $\chi_2 = \|\psi - \lambda^{-1}\varphi\|^{-1}(\psi - \lambda^{-1}\varphi)$. As

$$\begin{aligned} \lambda^{-1}\|\varphi\| + \left\|\psi - \lambda^{-1}\varphi\right\| + (1 - \|\psi\|) &= \\ \lambda^{-1}\varphi(1) + \psi(1) - \lambda^{-1}\varphi(1) + 1 - \psi(1) &= 1, \end{aligned}$$

and $\chi_1, \chi_2, 0 \in Q$(where $0 \le \chi_2$ since $\varphi \le \lambda\psi$), we get from the fact that F is a face that $\chi_1, \chi_2 \in F$. In particular, $\varphi = \|\varphi\|\chi_1 \in \mathbb{R}_+ F = C$.

C is also norm closed, for if $\varphi_n \in F$,$\alpha_n \in \mathbb{R}_+$ and $\alpha_n\varphi_n \to \varphi$ in norm then $\varphi \ge 0$ and we may assume $\varphi \ne 0$. Furthermore, we may assume that $\|\varphi_n\| \ne 0$, and, by replacing α_n and φ_n by $\alpha_n\|\varphi_n\|$ and $\|\varphi_n\|^{-1}\varphi_n$, that $\varphi_n \in S$, for since F is a face, we conclude from $\|\varphi_n\|(\|\varphi_n\|^{-1}\varphi_n) + (1 - \|\varphi_n\|)0 = \varphi_n \in F$ that also $\|\varphi_n\|^{-1}\varphi_n \in F$. As $\alpha_n = \|\alpha_n\varphi_n\| \to \|\varphi\|$ we can write $\varphi = \alpha\psi$, where $\alpha_n \to \alpha$ and $\psi \in S$. Finally,

$$\|\psi - \varphi_n\| \le \alpha^{-1}(\|\varphi - \alpha_n\varphi_n\| + |\alpha_n - \alpha|\|\varphi_n\|) \to 0,$$

so since F is closed, $\psi \in F$ and $\varphi \in \mathbb{R}_+ F$.

Now $\mathbb{R}_+ F = C(p)$ for some $p \in \mathfrak{A}^{**}$, and we establish that $F = \mathbb{R}_+ F \cap Q$ holds. The inclusion from left to right is obvious, so assume that $\lambda \in \mathbb{R}_+$, $\varphi \in F$ with $\|\lambda\varphi\| \le 1$. As above, we see that $\|\varphi\|^{-1}\varphi \in F$, so we may assume that $\varphi \in S$ and, consequently, that $\lambda \le 1$. Since $0 \in F$,

$$\lambda\varphi = \lambda\varphi + (1 - \lambda)0 \in F.$$

By definition, $F = C(p) \cap Q = F(p)$, and every face is supported by a p. Since also $C(p) = \mathbb{R}_+ F(p)$, F is injective because C is, and the second formula claimed holds true in general. For the last, we just note that since $F(p)$ is a face, $\operatorname{ext} F(p) = F(p) \cap \operatorname{ext} Q = F(p) \cap (P \cup \{0\})$.

If $p \leq q$, clearly $C(p) \subseteq C(q)$ and $F(p) \subseteq F(q)$ as $0 \leq \varphi(1-q) \leq \varphi(1-p)$ when $\varphi \in (\mathfrak{A}^*)_+$. That L is order-preserving is seen by invoking the polar decomposition theorem [22, 3.6.7] ∎

2.2 Open, closed and compact projections

The subspaces introduced in the previous sections were all norm closed, and, generally speaking, it is necessary to assume closedness to assure that the subspaces in question have anything to do with the internal structure of $\mathfrak{A}$. In some cases, one must go even further and demand that the subspaces are *weak** closed. It turns out that if any of $L(p), C(p), F(p)$ is *weak** closed, so are the other two. The following theorem gives a close description of the p supporting *weak** closed subspaces.

By a left ideal $\mathfrak{L}$ in $\mathfrak{A}$ we mean a *closed* left ideal. A sub-C^*-algebra $\mathfrak{B}$ is called *hereditary*, when its positive cone $\mathfrak{B}_+$ is hereditary, i.e. when for $a \in \mathfrak{A}_+, b \in \mathfrak{B}_+$

$$a \leq b \Longrightarrow a \in \mathfrak{B}_+.$$

From [22, 1.5.2] we get that

$$\mathfrak{L} \mapsto \mathfrak{L} \cap \mathfrak{L}^*$$
$$\mathfrak{B} \mapsto \{a \in \mathfrak{A} | a^* a \in \mathfrak{B}\}$$

are corresponding, order-preserving bijections between the class of left ideals and the class of hereditary subalgebras of $\mathfrak{A}$.

Definition 2.2.1 Let $p \in \mathfrak{A}^{**}$ be a projection. The *hereditary subalgebra* supported by p is

$$H(p) = p\mathfrak{A}^{**}p \cap \mathfrak{A}.$$

Clearly, $H(p)$ is a closed sub-C^*-algebra of $\mathfrak{A}$, and since

$$H(p)_+ = \{a \in \mathfrak{A}_+ | a \leq p\},$$

it is also hereditary.

Theorem 2.2.2 *Let p be a projection of $\mathfrak{A}^{**}$. The following conditions are equivalent* *(i)* $L(1-p)$ *is weak* closed.*

(ii) $C(1-p)$ *is weak* closed.*

(iii) $F(1-p)$ *is weak* closed.*

(iv) $p \in (\mathfrak{A}_+)^m$.

(v) $p \in ((\mathfrak{A}^{\sim}_{sa})^m)^=$.

(vi) $p \in H(p)^-$.

(vii) $H(p)^- = p\mathfrak{A}^{**}p$.

The closures in (vi) and (vii) are in the strong operator topology.

Definition 2.2.3 If $p \in \mathfrak{A}^{**}$ satisfies the conditions above, p is called *open*. If $1-p$ satisfies the conditions above, p is called *closed*.

Proof of Theorem 2.2.2: All the necessary arguments for this can be found in the proofs of [22, 3.10.7] and [22, 3.11.9]. We repeat the words that are not to be found in the theorems themselves.

That (i)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iii) follow by the formulae in Theorem 2.1.3, since $(\mathfrak{A}^*)_+$ and Q are *weak** closed, the latter since it is *weak** compact by Alaoglu's theorem. We get (iii)$\Longrightarrow$(i) from [22, 3.10.7], and [22, 3.11.9] shows equality of (iii), (iv), (v) and (vi), so we are done when (vi)$\Longrightarrow$(vii) is established, since clearly (vii)$\Longrightarrow$(vi). For that, note that by Kaplansky's density theorem, there is a net (y_λ) in $H(p)_1$ converging strongly to p, since $p \in (H(p)^-)_1$. When $a \in p\mathfrak{A}^{**}p$, there is a net (x_μ) in $\mathfrak{A}$ with

$$a = \lim_M px_\mu p = \lim_{\Lambda,M} y_\lambda^* x_\mu y_\lambda$$

because multiplication is strongly continuous on $\mathbb{B}(\mathfrak{H})_1 \times \mathbb{B}(\mathfrak{H})$. As $H(p)$ is hereditary, we conclude from

$$y_\lambda^* x_\mu y_\lambda \leq \|x_\mu\| y_\lambda^* y_\lambda \in H(p)$$

that $a \in H(p)^-$. ■

Corollary 2.2.4 *The mapping H is an order-preserving bijection from the set of open projections of $\mathfrak{A}^{**}$ to the set of hereditary subalgebras of $\mathfrak{A}$.*

Proof: When $\mathcal{L}$ is a left ideal of $\mathfrak{A}$, $\mathcal{L}^\perp$ is a *weak** closed left invariant subspace of $\mathfrak{A}^*$. When M is a left invariant subspace of $\mathfrak{A}^*$, ${}^\perp M$ is a left ideal. By the double polar theorem A.1.2 applied to the spaces $(\mathfrak{A}, \|\cdot\|)$, $(\mathfrak{A}^*, weak^*)$ in duality, the mappings

$\mathcal{L} \mapsto \mathcal{L}^\perp$ and $M \mapsto {}^\perp M$ are bijections between the class of left ideals of $\mathfrak{A}$ and the class of *weak** closed left invariant subsets of $\mathfrak{A}^*$. Combining this with Theorem 2.2.2 and the correspondence mentioned above, we get a bijection between the set of open projections and the hereditary subalgebras of $\mathfrak{A}$ via

$$\begin{aligned} {}^\perp L(p) \cap ({}^\perp L(p))^* &= \{a \in \mathfrak{A} | ap = a, pa = a\} \\ &= \{a \in \mathfrak{A} | pap = a\} \\ &= p\mathfrak{A}^{**}p \cap \mathfrak{A} = H(p) \end{aligned}$$

■

Remark 2.2.5 We have already noted, that $H(p)$ is a hereditary subalgebra whether p is open or not. The point of the last four statements of Theorem 2.2.2 is that $H(p)$ is big enough to reach p only if p is open. Indeed, $H(p) = \{0\}$ is quite conceivable for non-open $p \neq 0$. Another purpose for open projections is that they are in a 1-1 correspondence with the hereditary subalgebras from Corollary 2.2.4. Shedding light on the choice of terminology, the exact analogy to keep in mind for the commutative case $\mathfrak{A} = C_0(X)$ is that every ideal can be written uniquely as

$$\begin{aligned} \mathfrak{I}_G &= \{f \in \mathfrak{A} | f(X \backslash G) \subseteq \{0\}\} \\ &= \{f \in \mathfrak{A} | f = f1_G\}, \end{aligned}$$

where G is open. Again, $\mathfrak{I}_Y$ is a closed ideal for all $Y \subseteq X$, but if Y is not open, $\mathfrak{I}_Y$ might be very small. For instance, $\mathfrak{I}_Y = \{0\}$ if $X \backslash Y$ is dense.

Let us take a closer look at the commutative case, assuming that X is compact. Letting $M_p(X)$ denote the set of probability Radon measures on X, we have that

$$\mathfrak{H}_\mathfrak{u} = \bigoplus_{\mu \in \mathfrak{M}_\mathfrak{p}(\mathfrak{X})} \mathfrak{L}^2(\mathfrak{X}, \mu)$$

where $\pi_u : \mathfrak{A} \to \mathbb{B}(\mathfrak{H}_\mathfrak{u})$ is a direct sum of representations as multiplication operators. We get that

$$\mathfrak{A}^{**} = \pi_u(\mathfrak{A})'' = \bigoplus_{\mu \in M_p(X)} L^\infty(X, \mu),$$

cf. [23, 4.7.6] for details. We have $f \in \mathfrak{A}$ sitting in $\mathfrak{A}$ as a constant tuple (f), and want to prove that the open projections are the ones of the form (1_G), where $G \subseteq X$ is open. This is most easily seen by Theorem 2.2.2(iv). First note that the set

$$\Lambda = \{f \in \mathfrak{A} | 0 \leq f \leq 1_G\}$$

is upward directed, and that as a net $(f_\lambda)_\Lambda$ indexed by itself, we get $f_\lambda \nearrow 1_G$ pointwise since by Urysohn's lemma, for every $x \in G$ there is an $f \in \Lambda$ such that $f(x) = 1$. We get, for every $\mu \in M_p(X)$, $g \in L^2(\mu)$,

$$(f_\lambda g|g)_\mu = \int_X f_\lambda |g|^2 d\mu \nearrow \int_X 1_G |g|^2 d\mu = (1_G g|g)_\mu$$

by the monotone convergence theorem, showing that $f_\lambda \nearrow 1_G$. On the other hand, if $p = (1_{H_\mu})$ is open and $f_\lambda \in \mathfrak{A}_+$ is given with $f_\lambda \nearrow p$, let

$$G = \bigcup_{\lambda\in\Lambda} \{x \in X | f_\lambda(x) > 0\}.$$

Clearly G is open, and for every $x \in G$, $f_\lambda(x) \nearrow 1$, because when μ_x is the Dirac measure on x, $1 \in L^2(\mu_x)$ and $f_\lambda(x) = \int_X 1 f_\lambda d\mu \to 1_{H_{\mu_x}}(x)$, where $x \in H_{\mu_x}$ follows by the fact that $f_\lambda(x) > 0$ for some λ. As above, we get that $f_\lambda \nearrow 1_G$, whence $1_G = 1_{H_\mu}$ (a.e. $[\mu]$) for every $\mu \in M_p(X)$.

That the *weak** closure of a left invariant subspace of $\mathfrak{A}^*$ is itself left invariant under $\mathfrak{A}$ follows directly, and by virtue of Theorem 2.1.3 we can make the following definition.

Definition 2.2.6 Let p be a projection of $\mathfrak{A}^{**}$. The unique projection q satisfying

$$L(p)^{-} = L(q)$$

is called the *closure* of p. We write $\overline{p} = q$. Also, when p is open, we write $\partial p = \overline{p} - p$.

By the order-preserving property of L, $\overline{p}$ is the smallest closed projection dominating p.

The above theorem is the work of Effros, Akemann, and Pedersen. The rest of this section further explores the notion of open and closed projections, and most of the results can be found in [2, II], although we carry the results through for non-unital C^*-algebras as well and use some results that were not available when this paper were written. Note that as the usual duality between open and closed holds, all results about one class can be translated to a result about the other along well-known lines. But as we will see, caution must be exerted doing this.

Proposition 2.2.7 *If $p \in \mathfrak{A}^{**}$ is a minimal projection, p is closed.*

Proof: When $\varphi \in C(p)$,

$$\varphi(pap) = \varphi(a)$$

by the Cauchy-Schwarz inequality, and since $p\mathfrak{A}^{**}p$ is one-dimensional (a von Neumann algebra with no non-trivial projections), $C(p) = \mathbb{R}_+\varphi_0 \cup \{0\}$, where φ_0 is some non-zero element of $C(p)$. Clearly, then, $C(p)$ is closed. ∎

Proposition 2.2.8 *If $(p_i)_I$ is a family of closed projections, $\bigwedge_I p_i$ is closed.*

Proof: Note that $\bigcap_I C(p_i)$ is a *weak** closed hereditary cone. As C is an order-preserving mapping by Theorem 2.1.3, we get

$$C(\bigwedge_I p_i) = \bigcap_I C(p_i)$$

■

In the next proposition, note that when p is central, the norm condition is automatically met.

Proposition 2.2.9 *If $p, q \in \mathfrak{A}^{**}$ are closed projections with $\|pq - p \wedge q\| < 1$, then $p \vee q$ is closed.*

Proof: Let

$$M = L(p) + L(q)$$

and note that M is left invariant. If we can prove that M is *weak** closed we are done, for since M is then norm closed, we get $M = L(p \vee q)$ by the order-preserving properties of L stated in Theorem 2.1.3. Note that with $q' = q - p \wedge q$ we get $M = L(p) + L(q')$, since if $\varphi \in L(p), \psi \in L(q)$ we may write

$$\begin{aligned}(\varphi + \psi)(\cdot) &= [\varphi(\cdot) + \psi(\cdot(p \wedge q))] + [\psi(\cdot) - \psi(\cdot(p \wedge q))] \\ &= [\varphi(\cdot) + \psi(\cdot(p \wedge q))] + \psi(\cdot q').\end{aligned}$$

We may thus assume, without loss of generality, that $p \wedge q = 0$, whence $L(p) \cap L(q) = L(p \wedge q) = \{0\}$.

The mapping $\Psi : L(p) \oplus L(q) \to M$ given by $\Psi(\varphi, \psi) = \varphi + \psi$ is 1–1 by assumption, and is bounded away from zero, when $L(p) \oplus L(q)$ is equipped with maximum-norm. For assume that $(\varphi_n), (\psi_n)$ were sequences in $L(p), L(q)$ such that

$$\|(\varphi_n, \psi_n)\|_\infty = \max(\|\varphi_n\|, \|\psi_n\|) = 1, \qquad \|\varphi_n + \psi_n\| \leq 2^{-n}, \tag{2}$$

we learn that $|\|\varphi_n\| - \|\psi_n\|| \leq 2^{-n}$ and $\|\varphi_n\|, \|\psi_n\| \geq 1 - 2^{-n} \geq 1/2$. Write

$$\begin{aligned}\left\| \|\varphi_n\|^{-1}\varphi_n + \|\psi_n\|^{-1}\psi_n \right\| &= \|\varphi_n\|^{-1} \left\| \varphi_n + \frac{\|\varphi_n\|}{\|\psi_n\|}\psi_n \right\| \\ &\leq 2\|\varphi_n + \psi_n\| + 2\left|1 - \frac{\|\varphi_n\|}{\|\psi_n\|}\right| \|\psi_n\| \\ &\leq 4 \cdot 2^{-n} = 2^{-n+2}\end{aligned}$$

to see that we may assume (by normalizing and renumbering) that $\|\varphi_n\| = \|\psi_n\| = 1$ in (2). We have from the assumption that $\|pq - p \wedge q\| = \|pq\| < 1$ that

$$1 = \varphi_n(p) \leq \psi_n(p) + 2^{-n} = \psi_n(pq) + 2^{-n} \leq \|pq\| + 2^{-n} < 1$$

for large n, a contradiction. We may hence assume that when $\|\Psi(\varphi,\psi)\| \leq 1$, then $\|(\varphi,\psi)\|_\infty \leq K$ for some $K \in \mathbb{R}$.

By the Krein-Smulian theorem, as stated, for instance, in [23, 2.5.10], we need only prove that M_1 is *weak** closed. When $\varphi_\lambda \in L(p), \psi_\lambda \in L(q)$ with $\varphi_\lambda + \psi_\lambda \in M_1$ and

$$\varphi_\lambda + \psi_\lambda \to \chi \in \mathfrak{A}^*$$

we get that $\|\varphi_\lambda\|, \|\psi_\lambda\| \leq \|(\varphi_\lambda, \psi_\lambda)\|_\infty \leq K$. The K-balls of the *weak** closed sets $L(p), L(q)$ are compact, so by passing to subnets, we may assume that $\varphi_\lambda \to \varphi, \psi_\lambda \to \psi$, wherefore $\chi = \varphi + \psi \in M$ as desired. ■

Akemann ([2, II.6]) gives examples of closed p,q with $p \vee q$ not closed.

Open and closed projections behave more nicely under the normal extension π_a^{**} of the reduced atomic representation to $\mathfrak{A}^{**}$ than an arbitrary projection of $\mathfrak{A}^{**}$ does. In fact, π^{**} turns out to be 1–1 on the set of open or closed projections, showing that for such $p \neq 0$, $\pi^{**}(p) \neq 0$ for some $\pi \in \hat{\mathfrak{A}}$, since $\pi_a^{**} = \bigoplus_{P(\mathfrak{A})} \pi_\varphi^{**}$ by normality. We conclude that $\varphi(p) = 1$ for some $\varphi \in P(\mathfrak{A})$. For closed projections this last statement is obvious; when $F(p)$ is closed, $P(p) \neq \emptyset$ follows from the Krein-Milman theorem and Theorem 2.2.2, for because $F(p) = \overline{\mathrm{co}}(P(p) \cup \{0\})$, $P(p)$ can not be empty. But for the more general statements, we need to consult [22] again.

Definition 2.2.10 The element $c \in \mathfrak{A}_{sa}^{**}$ is said to be *universally measurable*, if for every $\varepsilon > 0$ and $\varphi \in S(\mathfrak{A})$ there exist $a, b \in (\mathfrak{A}_{sa})^m$ such that

$$-b \leq x \leq a, \qquad \varphi(a+b) < \varepsilon.$$

The set of universally measurable elements of $\mathfrak{A}_{sa}^{**}$ is denoted by $U(\mathfrak{A})$.

Proposition 2.2.11 *The set $U(\mathfrak{A})$ is a norm closed real vector space containing $((\mathfrak{A}_{sa}^{\sim})^m)^=$. In particular, every open and closed projection lies in $U(\mathfrak{A})$.*

Proof: The first statement is [22, 4.3.13]. To see from this that open projections are universally measurable, apply Theorem 2.2.2(v). As 1 is obviously open, we get the last statement from the real linearity of $U(\mathfrak{A})$. ■

Theorem 2.2.12 *The extension of the reduced atomic representation π_a^{**} is norm-preserving on $U(\mathfrak{A})$. Furthermore, $\pi_a^{**}(a) \geq 0$ implies $a \geq 0$ for $a \in U(\mathfrak{A})$.*

Proof: All this can be found in [22, 4.3.15]. ■

We are now able to prove the following fact, claimed in Section 1.1, quite easily.

Proposition 2.2.13 *The central cover of π_a is z.*

Proof: As outlined in Section 1.1, we need to show that when $\ker \pi_a^{**} = \mathfrak{A}^{**}(1-z')$, then $z = z'$. We get $z \geq z'$ from $\pi_a^{**}(z) = 1$, which follows by showing that for all irreducible $(\pi, \mathfrak{H})$, $\pi^{**}(z) = 1$. But for ξ a unit vector of $\mathfrak{H}$, $(\pi^{**}(\cdot)\xi|\xi)$ is a pure state, whence for some minimal projection p, $1 = (\pi^{**}(p)\xi|\xi) \leq (\pi^{**}(z)\xi|\xi) \leq \|z\| = 1$. Now assume that $z > z'$ and take a minimal projection $p \neq 0$ orthogonal to z'. As $p \leq 1 - z'$, $\pi_a^{**}(p) = 0$, contradicting Proposition 2.2.7 and the two results above. ■

We may now think of the extension π_a^{**} of the reduced atomic representation as $a \mapsto az$ and state our next results in this more practical language.

Corollary 2.2.14 *If p is an open or closed projection, $pz = 0$ yields $p = 0$.*

Proof: By Proposition 2.2.11, open and closed projections are universally measurable. By Theorem 2.2.12, $a \mapsto az$ is 1–1 on $U(\mathfrak{A})$. ■

Corollary 2.2.15 *If p and q are open or closed projections in $\mathfrak{A}^{**}$, and $pz \leq qz$, then $p \leq q$.*

Proof: As $U(\mathfrak{A})$ is real linear, $q - p \in U(\mathfrak{A})$. Apply Theorem 2.2.12. ■

For p closed, Corollary 2.2.15 also follows by the ensuing proposition, as $pz = 0$ yields $\overline{pz} = p = 0$.

Proposition 2.2.16 *When p is closed, $\overline{zp} = p$.*

Proof: By Theorem 2.2.2, $F(p)$ is *weak** closed, and by the last formula of Theorem 2.1.3 combined with the Krein-Milman theorem,

$$\begin{aligned} F(p) &= \overline{\mathrm{co}}(P(p) \cup \{0\}) \\ &\subseteq \overline{\mathrm{co}}(F(p) \cap F(z)) \\ &= \overline{\mathrm{co}}(F(pz)) \\ &= F(pz)^{-}, \end{aligned}$$

where we use the fact that $F(pz)$ is convex. Obviously, $pz \leq \overline{pz} \leq p$, so that

$$F(pz) \subseteq F(\overline{pz}) \subseteq F(p).$$

and $F(pz)^{-} = F(\overline{pz}) = F(p)$. By Theorem 2.1.3, $\overline{pz} = p$. ■

We end this section with a couple of results about *compact* projections. As we will only need the results in the rather sketchy last section of this thesis, we will allow ourselves to state the following result without a full proof.

Lemma 2.2.17 *For a projection $p \in \mathfrak{A}^{**}$, the following conditions are equivalent*

(i) $F(p) \cap S(\mathfrak{A})$ *is weak* closed.*

(ii) p *is closed and* $p = ap$ *for some* $a \in \mathfrak{A}_+$ *with* $\|a\| = 1$.

Definition 2.2.18 A projection that satisfies (i) and (ii) above is called *compact.*

Proof of Lemma 2.2.17: This is found in [5, 2.4]. ■

The reader should note that if $\mathfrak{A}$ is unital, every closed projection is compact. More generally, if $p \leq q$, p is closed and q is compact, then if $aq = q$, we have $ap = aqp = qp = p$ and p is also compact. Compact projections share the following property with their commutative counterparts.

Proposition 2.2.19 *Let q be a compact projection and $(p_i)_I$ a family of closed projections such that for all finite subsets I_0 of I, $(\bigwedge_{I_0} p_i) \wedge q \neq 0$. Then*

$$\left(\bigwedge_{i \in I} p_i\right) \wedge q \neq 0.$$

Proof: For every finite subset I_0 of I,

$$F((\bigwedge_{i \in I_0} p_i) \wedge q) \cap S = \left(\bigcap_{i \in I_0} F(p_i)\right) \cap (F(q) \cap S) \neq \emptyset,$$

and as $F(q) \cap S$ is a *weak** compact as a closed subset of $(\mathfrak{A}^*)_1$, we get via the finite intersection property for compact sets, that $F((\bigwedge_I p_i) \wedge q) \cap S \neq \emptyset$, whence $(\bigwedge_I p_i) \wedge q \neq 0$. ■

2.3 Regular projections

The theory of regular projections was founded by Tomita and brought out to the public by Effros in [12]. Unfortunately for us, this important paper dates back to the good old days where C^*-algebras always had a unit, and as pointed out by Brown in [10], some of Effros' methods do not readily extend to the non-unital case. The purpose of this section is to state and prove the basic facts of regular projections in the language of [22], making sure along the way that the results hold true for non-unital C^*-algebras as well.

As can be seen in Theorem 2.2.2, closedness of projections relates closely to *weak** closedness of the corresponding subsets of $\mathfrak{A}^*$. The theory of *regular* projections is concerned with investigating how the closure of projections relates to the *weak** closure of the sets supported by it. By definition, $L(\overline{p}) = L(p)^-$, but since there is no warrant that $C(p)^-$ is a hereditary cone or $F(p)^-$ a face, $C(\overline{p}) \neq C(p)^-$ and $F(\overline{p}) \neq F(p)^-$ is quite possible. The following theorem characterizes the class of projections where the sets supported by $\overline{p}$ relate naturally to the closures of the sets supported by p. We need the following preliminaries:

Let $p \in \mathfrak{A}^{**}$ be a projection. We define $\pi_p : \mathfrak{A} \mapsto \mathfrak{A}^{**}$ by

$$\pi_p(a) = ap \tag{3}$$

and note that $\ker \pi_p$ is a norm closed subspace of $\mathfrak{A}$.

Lemma 2.3.1 *With $p \in \mathfrak{A}^{**}$ a projection and π_p defined as in (3) we get for each $a \in \mathfrak{A}$*

$$\|a + \ker \pi_p\| = \sup_{\varphi \in (L(p)^-)_1} |\varphi(a)|$$

and

$$\|ap\| = \sup_{\varphi \in L(p)_1} |\varphi(a)|.$$

Proof: First note that $L(p) = \{\varphi(\cdot p) | \varphi \in \mathfrak{A}^*\}$ as a direct consequence of the definition. From the isometric isomorphism inducing $(\mathfrak{A}/\ker \pi_p)^* \simeq (\ker \pi_p)^\perp$ we get

$$\|a + \ker \pi_p\| = \sup_{\varphi \in (\ker \pi_p^\perp)_1} |\varphi(a)|.$$

But since $\ker \pi_p = \{a \in \mathfrak{A} | ap = 0\} = {}^\perp L(p)$, $(\ker \pi_p)^\perp = ({}^\perp L(p))^\perp = L(p)^-$. For the second equality, we note that $\|ap\| = \sup_{\varphi \in (\mathfrak{A}^*)_1} |\varphi(ap)|$ and write

$$\sup_{\varphi \in (\mathfrak{A}^*)_1} |\varphi(ap)| \leq \sup_{\varphi \in L(p)_1} |\varphi(a)| = \sup_{\varphi \in L(p)_1} |\varphi(ap)| \leq \sup_{\varphi \in (\mathfrak{A}^*)_1} |\varphi(ap)|,$$

where the first inequality holds since if $\varphi \in (\mathfrak{A}^*)_1$, then $\varphi(\cdot p) \in L(p)_1$. ■

Lemma 2.3.2 *Assume that $(\varphi_\lambda)_\Lambda$ is a net converging weak* to φ in $(\mathfrak{A}^*)_1$. If $\|\varphi\| = 1$, $\|\varphi_\lambda\| \to 1 = \|\varphi\|$.*

Proof: We have $\liminf_\Lambda \|\varphi_\lambda\| \geq 1$, for if $a \in \mathfrak{A}$, $\|a\| = 1$ is chosen such that $|\varphi(a)| > 1 - \varepsilon$ then $\|\varphi_\lambda\| \geq |\varphi_\lambda(a)| > 1 - \varepsilon$ when $\lambda \geq \lambda_0$ for some λ_0. Since $\|\varphi_\lambda\| \leq 1$, $\|\varphi_\lambda\| \to 1 = \|\varphi\|$. ■

Theorem 2.3.3 *Let $p \in \mathfrak{A}^{**}$ be a projection and let π_p be given by (3). Then the following conditions are equivalent:*

(i) $\forall a \in \mathfrak{A} : \|a\overline{p}\| = \|ap\|$.

(ii) $\forall a \in \mathfrak{A} : \|a + \ker \pi_p\| = \|ap\|$.

(iii) $(L(p)_1)^{-} = L(\overline{p})_1$.

(iv) $F(p)^{-} = F(\overline{p})$.

Definition 2.3.4 A projection $p \in \mathfrak{A}^{**}$ with these properties is called *regular*.

Proof of Theorem 2.3.3: First note that according to Theorem 2.2.2 and the fact that $(\mathfrak{A}^*)_1$ is *weak** compact, $L(\overline{p})_1$ is *weak** closed. This proves that $(L(p)_1)^{-} \subseteq L(\overline{p})_1$, and similarly $F(p)^{-} \subseteq F(\overline{p})$. By applying Lemma 2.3.1 to $\overline{p}$ and using the fact that $L(\overline{p})$ is *weak** closed, we get that

$$\|a\overline{p}\| = \sup_{\varphi \in L(\overline{p})_1} |\varphi(a)| = \|a + \ker \pi_{\overline{p}}\|,$$

so (i)$\Longrightarrow$(ii) follows by $\ker \pi_p = {}^{\perp}L(p) = {}^{\perp}L(\overline{p}) = \ker \pi_{\overline{p}}$, using the fact that $L(p)$ is dense in $L(\overline{p})$.

Assume that (iii) does not hold, i.e. that $\varphi_0 \in L(\overline{p})_1 \backslash (L(p)_1)^{-}$ exists. Since $(L(p)_1)^{-}$ is convex and *weak** compact, we can find $a \in \mathfrak{A}$ such that

$$\text{Re } \varphi_0(a) > \alpha > \beta > \text{Re } \varphi(a), \qquad \forall \varphi \in (L(p)_1)^{-}$$

for some $\alpha, \beta \in \mathbb{R}$. From Lemma 2.3.1, $\|a + \ker \pi_p\| \geq \alpha$ while $\beta \geq \|ap\|$ (as we may replace Re $\varphi(a)$ by $|\varphi(a)|$ in the above equation since $(L(p)_1)^{-}$ is closed under multiplication by scalars of argument 1), so that (ii) does not hold. When (iii) holds, we get (i) from the equation

$$\begin{aligned}
\|ap\| &= \sup_{\varphi \in L(p)_1} |\varphi(a)| \\
&= \sup_{\varphi \in (L(p)_1)^{-}} |\varphi(a)| \\
&= \sup_{\varphi \in L(\overline{p})_1} |\varphi(a)| \\
&= \|a\overline{p}\|.
\end{aligned}$$

To see that (iii)$\Longrightarrow$(iv) we need only prove that $F(\overline{p}) \subseteq F(p)^{-}$ when (iii) holds, so take $\varphi \in F(\overline{p}) \subseteq L(\overline{p})_1$. We may assume that $\|\varphi\| = 1$ and according to the hypothesis we can find a net (φ_λ) in $L(p)_1$ converging *weak** to φ. By Lemma 2.3.2, $\|\varphi_\lambda\| \to 1$. As $\||\varphi_\lambda|\| = \|\varphi_\lambda\| \le 1$, $(|\varphi_\lambda|)$ is a net in the *weak** compact set $(\mathfrak{A}^*)_1$ and we may assume that $|\varphi_\lambda| \to \psi$ for some $\psi \in (\mathfrak{A}^*)_1$. Clearly ψ is positive, and for all $a \in \mathfrak{A}$ and all λ

$$|\varphi_\lambda(a)|^2 \le \|\varphi_\lambda\| |\varphi_\lambda|(a^*a)$$

by [22, 3.6.7]. Taking this to the limit we get

$$|\varphi(a)|^2 \le \psi(a^*a) = \|\varphi\|\psi(a^*a)$$

which extends to all of $\mathfrak{A}^{**}$ since, by Kaplansky, $(\mathfrak{A})_1$ is strongly dense in $(\mathfrak{A}^{**})_1$ and φ, ψ are normal, hence strongly continuous on bounded sets of $\mathfrak{A}^{**}$. We also learn that $\|\psi\| = 1$ because

$$1 \ge \|\psi\| \ge \sup_{a\in\mathfrak{A}, \|a\|=1} \psi(a^*a) \ge \sup_{a\in\mathfrak{A}, \|a\|=1} |\varphi(a)|^2 = 1.$$

Now apply the uniqueness assertion of [22, 3.6.7] to get $\psi = |\varphi| = \varphi$. Since $|\varphi_\lambda| = \varphi_\lambda(u_\lambda \cdot)$ for some $u_\lambda \in \mathfrak{A}^{**}$ and $L(p)$ is left invariant, $|\varphi_\lambda| \in L(p) \cap Q = F(p)$. We conclude that $\varphi \in F(p)^-$.

Finally, we must prove (iv)$\Longrightarrow$(iii), so take $\varphi \in L(\overline{p})_1$. We may assume $\|\varphi\| = 1$, and since $\psi = |\varphi| \in F(\overline{p})$ as $\psi(\cdot) = \varphi(u^* \cdot)$ for some $u \in \mathfrak{A}^{**}$, we can employ (iv) to get a net (ψ_λ) in $F(p)$ converging *weak** to ψ. From Lemma 2.3.2, $\|\psi_\lambda\| \to 1$, and by replacing ψ_λ with $\|\psi_\lambda\|^{-1}\psi_\lambda \in F(p)$ if necessary, we can assume that $\psi_\lambda \in S$. Since $\mathfrak{A}$ is σ-weakly dense in $\mathfrak{A}^{**}$, we can find a net (u_λ) converging to u in this topology. The set

$$A = \{\psi \in \mathfrak{A}^* | |\varphi(a_i) - \psi(a_i)| < \varepsilon, i \in \{1, \dots, n\}\}$$

is a generic *weak** neighborhood of φ; to see that $\varphi \in (L(p)_1)^-$, we need only find $\psi \in A \cap L(p)_1$. And since

$$|\varphi(a_i) - \psi_\lambda(u_\mu a_i)| \le |\psi((u - u_\mu)a_i)| + |(\psi - \psi_\lambda)(u_\mu a_i)|,$$

we can, by choosing first μ, then λ, find $\psi_\lambda(u_\mu \cdot) \in A$. As $\psi_\lambda(u_\mu \cdot) \in L(p)_1$ by the left invariance, we are done. ■

Remark 2.3.5 Condition (i) is regularity according to Akemann, (ii) the classical definition by Tomita and Effros. In [12, 6.1], Effros proves that when $\mathfrak{A}$ is unital, regularity is equivalent to the condition $C(p)^- = (L(p)^-)_+$. This is, by the last statement of Theorem 2.2.2, equivalent to $C(p)^- = C(\overline{p})$. In fact, the assumption that the C^*-algebra has a unit enters Effros' proof several times (in the form that if φ_λ, φ are positive and $\varphi_\lambda \to \varphi$ *weak**, then $\|\varphi_\lambda\| = \varphi_\lambda(1) \to \varphi(1) = \|\varphi\|$), but except in the proof of the implication $C(p)^- = (L(p)_+)^- \Longrightarrow (L(p)_1)^- = L(\overline{p})_1$, the problem can easily be circumvented by Lemma 2.3.2. Whether the condition is strictly weaker than regularity is unknown[2].

[2]Well, at least Brown doesn't have it in [10]...

Corollary 2.3.6 *A central projection $p \in \mathfrak{A}^{**}$ is regular.*

Proof: When p is central, $\ker \pi_p$ is an ideal in $\mathfrak{A}$, and π_p lifts to $\tilde{\pi}_p : \mathfrak{A}/\ker \pi_p \to \mathfrak{A}^{**}$. This is a C^*-monomorphism, hence an isometry, and by (ii) in the theorem above, p is regular. ■

Recall the definition of P from Definition 2.1.1. The following lemma will prove very useful in the ensuing sections.

Lemma 2.3.7 *If $p \in \mathfrak{A}^{**}$ is regular and dominated by z, then*

$$\overline{P(p)} = P(\overline{p}),$$

where the closure on the left hand side is relative to $P(\mathfrak{A})$.

Proof: We first apply the fact that $p \leq z$ to prove that $\overline{\mathrm{co}}(P(p) \cup \{0\}) = F(p)^{-}$. Since $F(p)$ is a *weak** closed convex set containing $P(p)$, the inclusion from left to right is clear. By the double polar theorem[3] A.1.2 applied to the real spaces $(\mathfrak{A}_{sa}, \|\cdot\|)$ and $(\mathfrak{A}^*_{sa}, weak^*)$ in duality (cf. [22, 3.1.1]) and the fact that $0 \in F(p)$, the other inclusion is equivalent to $P(p)^{\circ\circ} \supseteq F(p)^{\circ\circ}$, which follows in turn from $P(p)^{\circ} \subseteq F(p)^{\circ}$. Take $a \in P(p)^{\circ}$, by definition $a \in \mathfrak{A}^{**}$ is self-adjoint and satisfies

$$\forall \varphi \in P(p) : \varphi(a) \geq -1.$$

We prove that $p(a+1)p \geq 0$. For this, we must show that for every unit vector $\xi \in \mathfrak{H}_{\mathrm{u}}$, $(p(a+1)p\xi|\xi) \geq 0$, but since $\mathfrak{A}^{**} = \pi_u(\mathfrak{A})'' \subseteq \bigoplus_{\varphi \in S} \mathbb{B}(\mathfrak{H}_\varphi)$, we may assume that ξ is a unit vector of $\mathbb{B}(\mathfrak{H}_\varphi)$ for some $\varphi \in S$. We may also assume that $p\xi \neq 0$, whence φ above must be pure, as $p \leq z$. Since

$$\psi(x) = \|p\xi\|^{-2}(pxp\xi|\xi)$$

is an element of $P(p)$, $\psi(a+1) \geq 0$ by assumption. With $\varphi \in F(p)$,

$$\varphi(a) = \varphi(pap) \geq -\varphi(p) = -\|\varphi\| \geq -1,$$

and $a \in F(p)^{\circ}$.

When $P(p)^{-}$ denotes the closure of $P(p)$ relative to P, it is clear that $P(p)^{-} \subseteq P(\overline{p})$. Using the fact that p is regular along with the above observation, we get

$$\overline{\mathrm{co}}(P(p) \cup \{0\}) = F(p)^{-} = F(\overline{p}).$$

By Proposition A.2.1 the extremal points of $F(\overline{p})$ are contained in $(P(p) \cup \{0\})^{-}$, and by Theorem 2.1.3, these are exactly $P(\overline{p}) \cup \{0\}$. We conclude that $P(\overline{p}) \subseteq P(p)^{-}$, as required. ■

[3] The reader should check appendix A for our definition of polars.

Remark 2.3.8 If p is not regular, $F(p)^{-} \subsetneq F(\overline{p})$, and $P(p)^{-} \subsetneq P(\overline{p})$. For an example showing that asserting $p \leq z$ is necessary in the above lemma, consider $p = 1 - z$. From Corollary 2.3.6, p is regular, but even though $F(p)^{-} = F(\overline{p})$, we have

$$\emptyset = P(p)^{-} \neq P(\overline{p}),$$

where $P(\overline{p})$ is not void because of Corollary 2.2.14.

2.4 Central projections

We have already encountered two important properties of central projections – that they are in a 1-1 correspondence with equivalence classes of representations of $\mathfrak{A}$ and that they are regular. The following corollary of Theorem 2.2.2 is, however, the main reason for our interest in this class of projections.

We say that a subset M of $Q(\mathfrak{A})$ is *saturated*, when M has the property

$$\varphi \in M, u \in \mathcal{U}(\mathfrak{A}^{\sim}) \Longrightarrow \varphi(u^* \cdot u) \in M.$$

Note that if $M \subseteq P(\mathfrak{A})$, then M is saturated if and only if M is saturated with respect to equivalence of pure states, cf. Proposition 1.1.1.

Theorem 2.4.1 *Let $\check{H}(p) = H(1-p)$. The mappings L, F and $\check{H}$ are bijections between the class of central closed projections of $\mathfrak{A}^{**}$ and the classes of*

(i) weak closed subspaces of $\mathfrak{A}^*$ that are invariant under $\mathfrak{A}$,*

(ii) weak closed, saturated faces of Q containing 0,*

(iii) ideals of $\mathfrak{A}$,

respectively.

Proof: When p is central, left invariance implies invariance. When $L(p)$ is invariant and $\varphi \in F(p)$, $\varphi(u^* \cdot u) \in L(p)_+ \cap Q = F(p)$ for every $u \in \mathcal{U}(\mathfrak{A}^{\sim})$. And if $F(p)$ is saturated,

$$\begin{aligned} F(u^*pu) &= \{\varphi \in Q | \varphi(1 - u^*pu) = 0\} \\ &= \{\varphi \in Q | \varphi(u^*(1-p)u) = 0\} \\ &= \{\varphi(u \cdot u^*) | \varphi \in F(p)\} \subseteq F(p) \end{aligned}$$

whence $F(u^*pu) = F(p)$ by symmetry and $p = u^*pu$ for every $u \in \mathcal{U}(\mathfrak{A}^\sim)$. We get that p commutes with every unitary of $\mathfrak{A}^\sim$, hence, as the span of those are strongly dense in $\mathfrak{A}^{**}$, with all of $\mathfrak{A}^{**}$.

When $L(p)$ is *weak** closed and right invariant, ${}^\perp L(p)$ is an ideal and $H(1-p) = {}^\perp L(p) = \mathfrak{A}^{**}p \cap \mathfrak{A}$. When $\mathfrak{I}$ is an ideal, it is hereditary with itself as the corresponding left ideal, and $\mathfrak{I}^\perp$ is an invariant *weak** closed subset, hence of the form $L(p)$ for p central. ■

Note that by (i) above, $\overline{p}$ is central when p is. We saw above, that a closed central projection $p \in \mathfrak{A}^{**}$ yields a *weak** closed saturated subset $F(p)$. Also $P(p)$ is saturated, for if φ is pure, we get from $\varphi(u^* \cdot u) = \lambda\psi_0(\cdot) + (1-\lambda)\psi_1(\cdot)$ that $\varphi(\cdot) = \lambda\psi_0(u \cdot u^*) + (1-\lambda)\psi_1(u \cdot u^*)$, showing that $\varphi(u^* \cdot u)$ is pure as well. When $C \subseteq P$ is saturated, we may define

$$\tilde{C} = \{\pi_\varphi | \varphi \in C\} \subseteq \hat{\mathfrak{A}}.$$

When we restrict ourselves to saturated sets, the mapping $C \mapsto \tilde{C}$ is 1-1, cf. Proposition 1.1.1. We put

$$p_C = \overline{\bigoplus_{\pi \in \tilde{C}} p_\pi},$$

where p_π is the central cover of π in $\mathfrak{A}^{**}$. As noted in Section 1.1, the p_π are orthogonal, and p_C is a projection. The following lemma establishes the usefulness of the correspondences $p \mapsto P(p)$ and $C \mapsto p_C$.

Proposition 2.4.2 *The mappings $p \mapsto P(p)$ and $C \mapsto p_C$ are corresponding bijections between the set of closed central projections of $\mathfrak{A}^{**}$ and the set of closed saturated subsets of $P(\mathfrak{A})$.*

Proof: Given a closed central projection $p \in \mathfrak{A}^{**}$, let $C = P(p)$. As the covers p_π of irreducible representations are minimal central projections adding up to z, zp can be written

$$zp = \bigoplus_{\pi \in D} p_\pi$$

and as, clearly, $D = \tilde{C}$, we get that, by Proposition 2.2.16,

$$p = \overline{zp} = \overline{\bigoplus_{\pi \in \tilde{C}} p_\pi} = p_C.$$

Now let C be a *weak** closed saturated subset of P. With

$$q = \bigoplus_{\pi \in \tilde{C}} p_\pi$$

we have that $p_C = \overline{q}$ and hence need to show $P(\overline{q}) = C$. Since obviously $C = P(q)$, and since $q \leq z$ and C is closed in P, this follows from Lemma 2.3.7. ∎

As $\mathfrak{A}^{**}$ is a von Neumann algebra, there is a notion of *central covers* of projections. When M is a subspace of $\mathfrak{H}$, we denote by $[M]$ the projection onto the norm closure of M.

Lemma 2.4.3 *Let $\mathcal{M}$ be a von Neumann algebra, $p \in \mathcal{M}$ a projection. The projection*

$$c(p) = [\mathcal{M}p\mathfrak{H}],$$

lies in $\mathcal{M}$ and is the smallest central projection dominating p. Furthermore, when $\mathcal{U}_0$ is a weakly dense subgroup of $\mathcal{U}(\mathcal{M})$,

$$c(p) = \bigvee_{u \in \mathcal{U}_0} u^* p u.$$

Proof: Existence and the first formula can be found in [17, 5.5]. The second formula is [22, 2.6.3]. ∎

Lemma 2.4.4 *If $p \in \mathfrak{A}^{**}$ is open, so is $c(p)$.*

Proof: By Kaplansky's density theorem [22, 2.3.3], $\mathcal{U}(\mathfrak{A}^{\sim})$ is strongly dense in $\mathcal{U}(\mathfrak{A}^{**})$. From the last observation of the lemma above

$$c(p) = \bigvee_{u \in \mathcal{U}(\mathfrak{A}^{\sim})} u^* p u.$$

Each $u^* p u$ is clearly open by Theorem 2.2.2(iv), and by the open dual of Proposition 2.2.8, so is $c(p)$. ∎

We will also need the notion of the *saturated hull* in the following. When C is a subset of $P(\mathfrak{A})$, this is defined as

$$\operatorname{sat} C = \{\varphi(u^* \cdot u) | \varphi \in C, u \in \mathcal{U}(\mathfrak{A}^{\sim})\}.$$

The saturated hull corresponds to the central cover of a projection in the following fashion:

Lemma 2.4.5 *When $p \in \mathfrak{A}^{**}$ is a projection,* $\operatorname{sat}(P(p)) = P(c(p))$.

Proof: The inclusion from left to right follows by the fact that if $\varphi \in P(p)$, then $\varphi(u^*c(p)u) = \varphi(c(p)) \geq \varphi(p) = 1$. For the other inclusion, take $\varphi \in P(c(p))$ and let q be the minimal projection in $\mathfrak{A}^{**}$ supporting φ. Since $q \leq c(p)$ we have $c(q) \leq c(p)$, and $pc(q) \neq 0$, because $pc(q) = 0$ would lead to $c(q) = c(p)c(q) = 0$ by [17, 5.5.4]. As $pc(q) \leq c(q) \leq z$, there exists a minimal projection r with $r \leq pc(q) \leq p$, and since $c(q)$ is minimal central, $c(r) = c(q)$. The pure state corresponding to r lies in $F(p)$ and is equivalent to φ, cf. [22, 3.8.2, 3.13.4]. ■

3 On tensor products

The purpose of this section is very limited; to write up proofs for two theorems from the theory of tensor products of C^*-algebras, which we will find use for later on. We base the proofs on our standard reference [20], except when it comes to tensor products of von Neumann algebras, in which case we will need to reference [17]. The notation will be based mostly on [20]. Thus, we denote by $\mathfrak{H}\hat{\otimes}\mathfrak{K}$ the Hilbert space tensor product of the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, and write $a\hat{\otimes}b$ for the unique operator in $\mathbb{B}(\mathfrak{H}\hat{\otimes}\mathfrak{K})$ such that

$$(a\hat{\otimes}b)(\xi \otimes \eta) = a\xi \otimes b\eta,$$

where $a \in \mathbb{B}(\mathfrak{H})$, $b \in \mathbb{B}(\mathfrak{K})$. We also use $\hat{\otimes}$ to denote tensor products of representations (in the natural sense, cf. [20, Theorem 6.3.3]), and $\overline{\otimes}$ to denote the spatial tensor product of von Neumann algebras, i.e. the weak closure of the algebra generated by tensored operators on $\mathfrak{H}\hat{\otimes}\mathfrak{K}$. As opposed to [20], however, we denote the algebraic tensor product of C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ by $\mathfrak{A} \odot \mathfrak{B}$. Given a C^*-norm $\|\cdot\|_\beta$ on $\mathfrak{A} \odot \mathfrak{B}$, we denote the completion of $\mathfrak{A} \odot \mathfrak{B}$ under $\|\cdot\|_\beta$ by $\mathfrak{A} \otimes_\beta \mathfrak{B}$. Extensions of tensor products of states and representations will be denoted by $\varphi \otimes_\beta \psi$ and $\pi \otimes_\beta \rho$, when they exist. Following [20], we write $\beta = *$ for the minimal C^*-norm, but for aesthetic reasons we will prefer to write $\beta = \vee$, and not $\beta = \max$, for the maximal one. When $\|\cdot\|_* = \|\cdot\|_\vee$, e.g. when $\mathfrak{A}$ or $\mathfrak{B}$ is nuclear, we may omit the index on $\otimes$ entirely.

3.1 A useful mapping

Most of the results in this section can be found in [15]. In all of this section, $\mathfrak{A}_1, \mathfrak{A}_2$ are C^*-algebras, not necessarily unital. We start by proving a number of simple lemmas.

Lemma 3.1.1 *For every $c \in \mathfrak{A}_1 \otimes_* \mathfrak{A}_2$, we have*

$$\|c\|_* = \sup_{\pi_i \in \hat{\mathfrak{A}}_i} \|\pi_1 \otimes_* \pi_2(c)\|.$$

Proof: The extension of $\pi_1 \hat{\otimes} \pi_2$ to $\mathfrak{A}_1 \otimes_* \mathfrak{A}_2$ exists by [20, Theorem 6.4.4]. By applying first [20, Theorem 6.4.19], then [20, Remark 6.4.1], we get

$$\begin{aligned} \|c\|_* &= \left\| \pi_a^1 \otimes_* \pi_a^2(c) \right\| \\ &= \left\| \Big(\bigoplus_{\pi_1 \in \hat{\mathfrak{A}}_1} \pi_1 \Big) \otimes_* \Big(\bigoplus_{\pi_2 \in \hat{\mathfrak{A}}_2} \pi_2 \Big)(c) \right\| \\ &= \sup_{\pi_i \in \hat{\mathfrak{A}}_i} \|\pi_1 \otimes_* \pi_2(c)\|, \end{aligned}$$

where π_a^1, π_a^2 refer to the reduced atomic representations of $\mathfrak{A}_1, \mathfrak{A}_2$. ■

Lemma 3.1.2 *Let $(\pi_1, \mathfrak{H}_1)$ and $(\pi_2, \mathfrak{H}_2)$ be non-degenerate representations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively. Then*

$$(\pi_1 \otimes_* \pi_2)(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)'' = \pi_1(\mathfrak{A}_1)'' \overline{\otimes} \pi_2(\mathfrak{A}_2)''.$$

If π_1 and π_2 are irreducible, so is $\pi_1 \otimes_* \pi_2$.

Proof: Since $(\pi_1 \hat{\otimes} \pi_2)(\mathfrak{A}_1 \odot \mathfrak{A}_2)$ is dense in $(\pi_1 \otimes_* \pi_2)(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)$ in norm, we get

$$\begin{aligned} (\pi_1 \otimes_* \pi_2)(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)'' &= (\pi_1 \hat{\otimes} \pi_2)(\mathfrak{A}_1 \odot \mathfrak{A}_2)'' \\ &= \pi_1(\mathfrak{A}_1) \overline{\otimes} \pi_2(\mathfrak{A}_2) \\ &\subseteq \pi_1(\mathfrak{A}_1)'' \overline{\otimes} \pi_2(\mathfrak{A}_2)''. \end{aligned}$$

On the other hand, if $x_1 \in \pi_1(\mathfrak{A}_1)''$, by Kaplansky's density theorem, we can take a net (a_λ) in $\mathfrak{A}_1$, such that $\pi_1(a_\lambda) \to x_1$ weakly in $\mathfrak{H}_1$ and $\|\pi_1(a_\lambda)\| \leq \|x_1\|$. Taking an approximate unit (u_μ) for $\mathfrak{A}_2$, $\pi_2(u_\mu) \to 1$ weakly in $\mathfrak{H}_2$, and since, by

$$((a \hat{\otimes} b)(\xi_1 \otimes \eta_1) | \xi_2 \otimes \eta_2) = (a\xi_1|\xi_2)(b\eta_1|\eta_2),$$

and a straightforward approximation in $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$, the mapping $(a, b) \mapsto a \hat{\otimes} b$ is weakly continuous on bounded subsets of $\mathbb{B}(\mathfrak{H}_1) \times \mathbb{B}(\mathfrak{H}_2)$, we get that

$$x_1 \hat{\otimes} 1 = \lim_{\lambda,\mu} \pi_1(a_\lambda) \hat{\otimes} \pi_2(u_\mu),$$

showing that $\pi_1(\mathfrak{A}_1)'' \odot \mathbb{C}1_{\mathfrak{H}_2} \subseteq (\pi_1 \otimes_* \pi_2)(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)''$. A symmetric argument finishes the proof of the first statement, by

$$\pi_1(\mathfrak{A}_1)'' \odot \pi_2(\mathfrak{A}_2)'' = (\pi_1(\mathfrak{A}_1)'' \odot \mathbb{C}1_{\mathfrak{H}_2})(\mathbb{C}1_{\mathfrak{H}_1} \odot \pi_2(\mathfrak{A}_2)'') \subseteq (\pi_1 \otimes_* \pi_2)(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)''.$$

When π_1 and π_2 are both irreducible, we get from the above combined with the commutation theorem [17, 11.2.16] (or a direct argument, cf. [17, 11.2.2]),

$$\begin{aligned} (\pi_1 \otimes_* \pi_2)(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)'' &= \pi_1(\mathfrak{A}_1)'' \overline{\otimes} \pi_2(\mathfrak{A}_2)'' \\ &= (\pi_1(\mathfrak{A}_1)' \overline{\otimes} \pi_2(\mathfrak{A}_2)')' \\ &= (\mathbb{C}1_{\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2})' \\ &= \mathbb{B}(\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2), \end{aligned}$$

and $\pi_1 \otimes_* \pi_2$ is irreducible. ■

Lemma 3.1.3 *If $(\pi_i, \mathfrak{H}_i)$ and $(\rho_i, \mathfrak{K}_i)$ are representations of $\mathfrak{A}_i$ with $\ker \pi_i \subseteq \ker \rho_i$, $i \in \{1,2\}$, we get*

$$\|\pi_1 \otimes_* \pi_2(c)\| \geq \|\rho_1 \otimes_* \rho_2(c)\|$$

for all $c \in \mathfrak{A}_1 \otimes_* \mathfrak{A}_2$.

Proof: The representations extend to $\mathfrak{A}_1 \otimes_* \mathfrak{A}_2$ by [20, Theorem 6.4.4]. Let $\mathfrak{B}_i = \pi_i(\mathfrak{A}_i) \subseteq \mathbb{B}(\mathfrak{H}_i)$ and identify $\mathfrak{B}_1 \otimes_* \mathfrak{B}_2$ with the norm closure of $\mathfrak{B}_1 \odot \mathfrak{B}_2$ in $\mathbb{B}(\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2)$, cf. [20, Theorem 6.4.19]. By assumption, there exist representations $\sigma_i : \mathfrak{B}_i \to \mathbb{B}(\mathfrak{K}_i)$ such that $\rho_i = \sigma_i \circ \pi_i$. When $c = \sum_{i=1}^n a_i^1 \otimes a_i^2 \in \mathfrak{A}_1 \odot \mathfrak{A}_2$, we have

$$\begin{aligned}
(\sigma_1 \otimes_* \sigma_2)(\pi_1 \otimes_* \pi_2)(c) &= (\sigma_1 \otimes_* \sigma_2)(\sum_{i=1}^n \pi_1(a_i^1) \hat{\otimes} \pi_2(a_i^2)) \\
&= \sum_{i=1}^n \sigma_1(\pi_1(a_i^1)) \hat{\otimes} \sigma_2(\pi_2(a_i^2)) \\
&= \rho_1 \otimes_* \rho_2(c).
\end{aligned}$$

As $\|\sigma_1 \otimes_* \sigma_2\| \leq 1$, the claim follows by the density of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ in $\mathfrak{A}_1 \otimes_* \mathfrak{A}_2$. ∎

Proposition 3.1.4 *Let $\mathfrak{A}_1, \mathfrak{A}_2$ be C^*-algebras. There is a mapping*

$$\Pi : \mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2) \to \mathrm{Prim}(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)$$

which is a homeomorphism onto a dense subset.

Proof: We want to define

$$\Pi(\ker \pi_1, \ker \pi_2) = \ker(\pi_1 \otimes_* \pi_2),$$

where $(\pi_1, \mathfrak{H}_1), (\pi_2, \mathfrak{H}_2)$ are irreducible representations of $\mathfrak{A}_1, \mathfrak{A}_2$, respectively. By Lemma 3.1.2, $\pi_1 \otimes_* \pi_2$ is irreducible. To see that the mapping is well-defined, assume that ρ_1, ρ_2 are irreducible representations with $\ker \pi_i = \ker \rho_i$, $i \in \{1,2\}$. By Lemma 3.1.3, $\|\pi_1 \otimes_* \pi_2(c)\|_* = \|\rho_1 \otimes_* \rho_2(c)\|_*$ for all $c \in \mathfrak{A}_1 \otimes_* \mathfrak{A}_2$, and their kernels are the same.

To see that Π is injective, suppose that $\ker(\pi_1 \otimes_* \pi_2) = \ker(\rho_1 \otimes_* \rho_2)$, and assume that $a_1 \in \ker \pi_1$. By definition of $\hat{\mathfrak{A}}_2$, $\rho_2 \neq 0$, so there is an $x_2 \in \mathfrak{A}_2$ such that $\rho_2(x_2) \neq 0$, and since

$$0 = \|\pi_1(a_1)\| \|\pi_2(x_2)\| = \|\pi_1 \otimes_* \pi_2(a_1 \otimes x_2)\|,$$

we have

$$0 = \|\rho_1 \otimes_* \rho_2(a_1 \otimes x_2)\| = \|\rho_1(a_1)\| \|\rho_2(x_2)\|,$$

showing that $a_1 \in \ker \rho_1$. Another three such arguments show that $\ker \pi_i = \ker \rho_i$, $i \in \{1, 2\}$. Also, the range of the mapping is dense, since by Lemma 3.1.1,

$$\begin{aligned}
&\mathrm{Ker}(\Pi(\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2))) = \\
&\qquad \bigcap_{\pi_i \in \hat{\mathfrak{A}}_i} \ker(\pi_1 \otimes_* \pi_2) = \\
&\qquad \{c \in \mathfrak{A}_1 \otimes_* \mathfrak{A}_2 | \forall \pi_1 \in \hat{\mathfrak{A}}_1, \pi_2 \in \hat{\mathfrak{A}}_2 : \|\pi_1 \otimes_* \pi_2(c)\| = 0\} = \{0\},
\end{aligned}$$

so that $\overline{\Pi(\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2))} = \mathrm{Hull}(\{0\}) = \mathrm{Prim}(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)$.

To see that

$$\begin{aligned}
\Pi((\mathrm{Prim}(\mathfrak{A}_1) \backslash \mathrm{Hull}\, \mathfrak{I}_1) \times \mathrm{Prim}\, \mathfrak{A}_2) &= \{\ker(\pi_1 \otimes_* \pi_2) | \pi_i \in \hat{A}_i, \mathfrak{I}_1 \not\subseteq \ker \pi_1\} \\
&= \Pi(\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2)) \backslash \mathrm{Hull}(\mathfrak{I}_1 \odot \mathfrak{A}_2),
\end{aligned}$$

we need to prove the last equation. The inclusion from right to left follows from the fact that if $\mathfrak{I}_1 \odot \mathfrak{A}_2 \not\subseteq \ker \pi_1 \otimes_* \pi_2$ and we take $a_1^i \in \mathfrak{I}_1$, $x_2^i \in \mathfrak{A}_2$ such that $(\pi_1 \otimes_* \pi_2)(\sum_{i=1}^n a_1^i \otimes x_2^i) \neq 0$, then for some i_0, $\pi_1(a_1^{i_0}) \neq 0$. Consequently, $\mathfrak{I}_1 \not\subseteq \ker \pi_1$. The other inclusion can be seen by noting that if $\pi_i \in \hat{\mathfrak{A}}_i$ and $\mathfrak{I}_1 \not\subseteq \ker \pi_1$, we can pick $a_1 \in \mathfrak{I}_1 \backslash \ker \pi_1$ and $x_2 \in \mathfrak{A}_2 \backslash \ker \pi_2$. Then, since

$$\|\pi_1 \otimes_* \pi_2(a_1 \otimes x_2)\| = \|\pi_1(a_1)\| \|\pi_2(x_2)\| > 0,$$

we have that $\ker(\pi_1 \otimes_* \pi_2) \notin \mathrm{Hull}(\mathfrak{I}_1 \odot \mathfrak{A}_2)$. Using the symmetric result in the other variable, we conclude that

$$\begin{aligned}
&\Pi((\mathrm{Prim}(\mathfrak{A}_1) \backslash \mathrm{Hull}\, \mathfrak{I}_1) \times (\mathrm{Prim}(\mathfrak{A}_2) \backslash \mathrm{Hull}\, \mathfrak{I}_2)) = \\
&\quad \Pi((\mathrm{Prim}(\mathfrak{A}_1) \backslash \mathrm{Hull}\, \mathfrak{I}_1) \times \mathrm{Prim}(\mathfrak{A}_2)) \cap \Pi(\mathrm{Prim}(\mathfrak{A}_1) \times (\mathrm{Prim}(\mathfrak{A}_2) \backslash \mathrm{Hull}\, \mathfrak{I}_2)) = \\
&\quad \Pi(\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2)) \backslash (\mathrm{Hull}(\mathfrak{I}_1 \odot \mathfrak{A}_2) \cup \mathrm{Hull}(\mathfrak{A}_1 \odot \mathfrak{I}_2)).
\end{aligned}$$

Since products of open sets form a basis for the topology on $\mathrm{Prim}\, \mathfrak{A}_1 \times \mathrm{Prim}\, \mathfrak{A}_2$, Π is relatively open. That Π is continuous will follow from the commutative diagram

$$\begin{array}{ccc}
P(\mathfrak{A}_1) \times P(\mathfrak{A}_2) & \xrightarrow{\tilde{\Pi}} & P(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2) \\
{\scriptstyle \iota_1 \times \iota_2} \downarrow & & \downarrow {\scriptstyle \iota} \\
\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2) & \xrightarrow{\Pi} & \mathrm{Prim}(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)
\end{array}$$

where $\tilde{\Pi}(\varphi_1, \varphi_2) = \varphi_1 \otimes_* \varphi_2$ in the sense of [20, Theorem 6.4.6] (existence from [20, Corollary 6.4.3]) and the ι are the natural mappings $\varphi \mapsto \ker \pi_\varphi$ described in [22, 4.3.3]. Since, by [20, Theorem 6.4.7], $\pi_{\varphi_1 \otimes_* \varphi_2}$ and $\pi_{\varphi_1} \otimes_* \pi_{\varphi_2}$ are unitarily equivalent, $\varphi_1 \otimes_* \varphi_2$ is pure when φ_1 and φ_2 are, so $\tilde{\Pi}$ is well-defined. Also, since

$$\begin{aligned}
\iota \circ \tilde{\Pi}(\varphi_1, \varphi_2) &= \ker \pi_{\varphi_1 \otimes_* \varphi_2} \\
&= \ker(\pi_{\varphi_1} \otimes_* \pi_{\varphi_2}) \\
&= \Pi \circ (\iota_1 \times \iota_2)(\varphi_1, \varphi_2),
\end{aligned}$$

the diagram is indeed commutative. Finally, $\tilde{\Pi}$ is continuous in the *weak** topology, for if $\varphi_\lambda^i \to \varphi^i$, *weak** in $\mathfrak{A}_i$, $i \in \{1,2\}$, then clearly for every $x \in \mathfrak{A}_1 \odot \mathfrak{A}_2$, $\varphi_\lambda^1 \otimes_* \varphi_\lambda^2(x) \to \varphi^1 \otimes_* \varphi^2(x)$, and since all the functionals involved have norm one, this extends readily to any $x \in \mathfrak{A}_1 \otimes_* \mathfrak{A}_2$. Using the fact that by [22, 4.3.3], ι is continuous and ι_1, ι_2 are open, we get continuity of Π from that of $\tilde{\Pi}$. ■

Lemma 3.1.5 *Let $\mathfrak{B} = C_0(X) \otimes \mathfrak{A}$ for some locally compact Hausdorff space X and some C^*-algebra $\mathfrak{A}$. We have*

(i) $P(\mathfrak{B}) = \{\hat{x} \otimes \varphi | x \in X, \varphi \in P(\mathfrak{A})\}$.

(ii) $\hat{\mathfrak{B}} = \{\hat{x} \otimes \pi | x \in X, \pi \in \hat{\mathfrak{A}}\}$.

Proof: We have seen in Lemma 3.1.2, that the right hand sides are contained in the left hand sides in both cases. In (i), the other inclusion follows by the first paragraph of the proof of [20, Theorem 6.4.15], and as then $\pi_\psi = \pi_{\hat{x}\otimes\varphi} \simeq \hat{x} \otimes \pi_\varphi$, (ii) is a direct consequence. ■

Corollary 3.1.6 *When $\mathfrak{A}_1$ or $\mathfrak{A}_2$ is abelian, Π in Proposition 3.1.4 is onto.*

Proof: Consider the commutative diagram from the proof of Proposition 3.1.4. ■

Remark 3.1.7 Takesaki noted a converse to (i) in the above lemma; if all pure states of $\mathfrak{A} \otimes_* \mathfrak{B}$ have the form $\varphi \otimes_* \psi$ for $\varphi \in P(\mathfrak{A}), \psi \in P(\mathfrak{B})$, then $\mathfrak{A}$ or $\mathfrak{B}$ is abelian. The advantage of using the mapping Π instead of the corresponding mapping of pure states, is that Π is onto in a great number of cases, for instance when both $\mathfrak{A}$ and $\mathfrak{B}$ are separable and one of them is nuclear. See [8] for references giving the proof of this fact.

3.2 On centers of tensor products

By a result of Archbold [7], the equality

$$\mathcal{Z}(\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2) = \mathcal{Z}(\mathfrak{A}_1) \otimes \mathcal{Z}(\mathfrak{A}_2)$$

holds true for any pair of C^*-algebras and any C^*-norm $\|\cdot\|_\beta$ on $\mathfrak{A}_1 \odot \mathfrak{A}_2$, where the right hand side is the closure of $\mathcal{Z}(\mathfrak{A}_1) \odot \mathcal{Z}(\mathfrak{A}_2)$ under $\|\cdot\|_\beta$. As $\mathcal{Z}(\mathfrak{A}_1)$ is nuclear by [20, Theorem 6.4.15], this closure is isomorphic to the unique C^*-completion of $\mathcal{Z}(\mathfrak{A}_1) \odot \mathcal{Z}(\mathfrak{A}_2)$, justifying the choice of notation.

As this result will only be of any use for us when the C^*-algebras in question have a unit, and since non-unital C^*-algebras do complicate matters, we will only carry out the proof of this proposition for $\mathfrak{A}_1, \mathfrak{A}_2$ unital. We will follow [7] rather closely, only adjusting the arguments to use our standard sources and filling in a couple of extra details.

Lemma 3.2.1 *Let $\mathfrak{A}$ be a unital C^*-algebra, $\mathcal{M}$ a von Neumann algebra. For any C^*-norm $\|\cdot\|_\beta$ on $\mathcal{M} \odot \mathfrak{A}$,*

$$\mathcal{Z}(\mathcal{M} \otimes_\beta \mathfrak{A}) = \mathcal{Z}(\mathcal{M}) \otimes \mathcal{Z}(\mathfrak{A}).$$

Proof: First assume that $\mathcal{M}$ is abelian and write $\mathcal{M} = C(X)$ for some compact Hausdorff space X. By [20, Theorem 6.4.17], we have

$$\mathcal{Z}(\mathcal{M} \otimes \mathfrak{A}) \simeq \mathcal{Z}(C(X, \mathfrak{A})) = C(X, \mathcal{Z}(\mathfrak{A})) \simeq \mathcal{M} \otimes \mathcal{Z}(\mathfrak{A})$$

under comparable isomorphisms, proving the lemma in this special case.

In the general case, we denote the center of $\mathcal{M} \otimes_\beta \mathfrak{A}$ by $\mathcal{Z}$. Clearly $\mathcal{Z}(M) \odot \mathcal{Z}(\mathfrak{A}) \subseteq \mathcal{Z}$, and one inclusion follows from the fact that $\mathcal{Z}$ is closed. In the other direction, we claim that for each $c \in \mathcal{M} \otimes_\beta \mathfrak{A}$,

$$\overline{\mathrm{co}}_{\mathcal{M}\otimes 1}(c) \cap (\mathcal{Z}(\mathcal{M}) \otimes \mathfrak{A}) \neq \emptyset, \tag{4}$$

where, by analogy with the Dixmier approximation theory described in [17, 8.3], we define

$$\mathrm{co}_{\mathcal{M}\otimes 1}(c) = \mathrm{co}(\{(u^* \otimes 1)c(u \otimes 1) | u \in \mathcal{U}(\mathcal{M})\}).$$

To see (4), first take $c = \sum_{i=1}^p m_i \otimes a_i \in \mathcal{M} \odot \mathfrak{A}$ and let $\varepsilon > 0$ be given. By the classical Dixmier approximation theorem [17, 8.3.3], there exist $q \in \mathbb{N}$ and $u_j \in \mathcal{U}(\mathcal{M})$, $\lambda_j \geq 0$ with $\sum_{j=1}^q \lambda_j = 1$ and $c_i \in \mathcal{Z}(\mathcal{M})$ such that

$$\left\| \sum_{j=1}^q \lambda_j u_j^* m_i u_j - c_i \right\| \leq \varepsilon, \qquad i \in \{1, \ldots, n\}.$$

Consequently, since β is a subcross-norm ([20, Corollary 6.3.6]),

$$\begin{aligned} \left\| \sum_{j=1}^q \lambda_j (u_j^* \otimes 1) c (u_j \otimes 1) - \sum_{i=1}^p c_i \otimes a_i \right\|_\beta &= \left\| \sum_{i=1}^p (\sum_{j=1}^q \lambda_j u_j^* m_i u_j - c_i) \otimes a_i \right\|_\beta \\ &\leq \sum_{i=1}^p \left\| \sum_{j=1}^q \lambda_j u_j^* m_i u_j - c_i \right\| \|a_i\| \\ &\leq \varepsilon \sum_{i=1}^p \|a_i\|, \end{aligned}$$

wherefrom we deduce that

$$d(\mathrm{co}_{\mathcal{M}\otimes 1}(c), \mathcal{Z}(\mathcal{M}) \odot \mathfrak{A}) = 0 \tag{5}$$

in the sense of distance between sets. If c is an arbitrary element of $\mathcal{M} \otimes_\beta \mathfrak{A}$ and $c_0 = \sum_{i=1}^p m_i \otimes a_i \in \mathcal{M} \odot \mathfrak{A}$ is chosen such that $\|c - c_0\|_\beta < \delta$, then

$$\begin{aligned}
\left\| \sum_{j=1}^q \lambda_j (u_j^* \otimes 1)(c - c_0)(u_j \otimes 1) \right\|_\beta &\leq \sum_{j=1}^q \lambda_j \left\| (u_j^* \otimes 1)(c - c_0)(u_j \otimes 1) \right\|_\beta \\
&\leq \sum_{j=1}^q \lambda_j \delta \\
&= \delta,
\end{aligned}$$

and we easily see that (5) holds true for all $c \in \mathcal{M} \otimes_\beta \mathfrak{A}$. Employing this as in the proof of Dixmier's theorem [17, 8.3.4], we get (4). Let us carry out the details:

We write $\mathcal{D}$ for the set of mappings of the form

$$c \mapsto \sum_{i=1}^p \lambda_i (u_i^* \otimes 1) c (u_i \otimes 1)$$

so that $\mathrm{co}_{\mathcal{M}\otimes 1}(c) = \{\alpha(c) | \alpha \in \mathcal{D}\}$. Note that $\mathcal{D}$ is closed under composition and that for every $\alpha \in \mathcal{D}$, $\|\alpha(c)\| \leq \|c\|$ as seen in the calculation above. Applying (5) recursively, we find $\beta_i \in \mathcal{D}$, $w_i \in \mathcal{Z}(\mathcal{M})$ and $a_i \in \mathfrak{A}$ such that

$$\|\beta_n \circ \cdots \circ \beta_1(c) - w_n \otimes a_n\| \leq 2^{-n}.$$

With $\alpha \in \mathcal{D}$, $w \in \mathcal{Z}(\mathcal{M})$ and $a \in \mathfrak{A}$, clearly $\alpha(w \otimes a) = w \otimes a$, and

$$\begin{aligned}
\|w_{n+1} \otimes a_{n+1} - w_n \otimes a_n\| &\leq \|w_{n+1} \otimes a_{n+1} - \beta_{n+1} \circ \cdots \circ \beta_1(c)\| \\
&+ \|\beta_{n+1} \circ \cdots \circ \beta_1(c) - w_n \otimes a_n\| \\
&\leq 2^{-n-1} + \|\beta_{n+1}(\beta_n \circ \cdots \circ \beta_1(c) - w_n \otimes a_n)\| \\
&\leq 2^{-n-1} + \|\beta_n \circ \cdots \circ \beta_1(c) - w_n \otimes a_n\| \\
&\leq 2^{-n-1} + 2^{-n} \leq 2^{-n+1},
\end{aligned}$$

so that $(w_n \otimes a_n)$ is a Cauchy sequence, converging, say, to $d \in \mathcal{Z}(\mathcal{M}) \otimes \mathfrak{A}$. We get that $d \in \overline{\overline{\mathrm{co}}}_{\mathcal{M}\otimes 1}(c) \cap (\mathcal{Z}(\mathcal{M}) \otimes \mathfrak{A})$.

When $w \in \mathcal{Z}$, $\overline{\overline{\mathrm{co}}}_{\mathcal{M}\otimes 1}(w) = \{w\}$, and $w \in \mathcal{Z}(\mathcal{M}) \otimes \mathfrak{A}$ by (4). Using the special case of the theorem proved above, $w \in \mathcal{Z}(\mathcal{M}) \otimes \mathcal{Z}(\mathfrak{A})$. ■

Lemma 3.2.2 *Let $\mathfrak{A}_1, \mathfrak{A}_2$ be abelian C^*-algebras, and let $\mathfrak{B}_1, \mathfrak{B}_2$ be subalgebras hereof. Then in $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, we have*

$$(\mathfrak{A}_1 \otimes \mathfrak{B}_2) \cap (\mathfrak{B}_1 \otimes \mathfrak{A}_2) = \mathfrak{B}_1 \otimes \mathfrak{B}_2.$$

Proof: Let $\mathfrak{C}$ denote the sub-C^*-algebra of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ on the left side of the equation above. Clearly $\mathfrak{B}_1 \otimes \mathfrak{B}_2 \subseteq \mathfrak{C}$. Since a pure state φ on $\mathfrak{C}$ can be extended to a pure state $\tilde{\varphi}$ on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, and since $\tilde{\varphi} = \varphi_1 \otimes \varphi_2$ for some $\varphi_i \in P(\mathfrak{A}_i)$ by Lemma 3.1.5, we can write any $\varphi \in P(\mathfrak{C})$ in this form.

Assume now that $\varphi, \psi \in P(\mathfrak{C})$ agree on $\mathfrak{B}_1 \otimes \mathfrak{B}_2$. Write $\varphi = \varphi_1 \otimes \varphi_2$, $\psi = \psi_1 \otimes \psi_2$, and note that neither of φ_i, ψ_i vanish on $\mathfrak{B}_i$, $i \in \{1, 2\}$, as this would imply that φ or ψ vanished on $\mathfrak{C}$. Thus, $\|\varphi_i\| = \|\psi_i\| = 1$, $i \in \{1, 2\}$, and we get for $a \in \mathfrak{B}_1$

$$\begin{aligned}
\varphi_1(a) &= \varphi_1(a) \lim_\lambda \varphi_2(u_\lambda) \\
&= \lim_\lambda \varphi_1 \otimes \varphi_2(a \otimes u_\lambda) \\
&= \lim_\lambda \psi_1 \otimes \psi_2(a \otimes u_\lambda) \\
&= \psi_1(a) \lim_\lambda \psi_2(u_\lambda) \\
&= \psi_1(a),
\end{aligned}$$

where u_λ is an approximate unit for $\mathfrak{B}_2$. Likewise, φ_2 and ψ_2 agree on $\mathfrak{B}_2$, and by combining

$$\begin{aligned}
\varphi_1 \otimes \varphi_2(c) &= \varphi_1 \otimes \psi_2(c), && c \in \mathfrak{A}_1 \otimes \mathfrak{B}_2 \\
\varphi_1 \otimes \psi_2(c) &= \psi_1 \otimes \psi_2(c), && c \in \mathfrak{B}_1 \otimes \mathfrak{A}_2,
\end{aligned}$$

we get that φ and ψ must agree on $\mathfrak{C}$. By the Stone-Weierstrass theorem, $\mathfrak{B}_1 \otimes \mathfrak{B}_2 = \mathfrak{C}$. ■

Lemma 3.2.3 *Let $\mathfrak{A}_1, \mathfrak{A}_2$ be C^*-algebras, and let $\|\cdot\|_\beta$ be a C^*-norm on $\mathfrak{A}_1 \odot \mathfrak{A}_2$. There is a C^*-norm $\|\cdot\|_{\beta'}$ on $\mathfrak{A}_1^{**} \odot \mathfrak{A}_2^{**}$ which extends $\|\cdot\|_\beta$.*

Proof: Identifying $\mathfrak{A}_1, \mathfrak{A}_2$ with their images in $\mathbb{B}(\mathfrak{H}_u^1), \mathbb{B}(\mathfrak{H}_u^2)$, we may think of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ and $\mathfrak{A}_1^{**} \odot \mathfrak{A}_2^{**}$ as sitting inside $\mathbb{B}(\mathfrak{H}_u^1 \hat{\otimes} \mathfrak{H}_u^2)$. Clearly, then, $\mathfrak{A}_1 \odot \mathfrak{A}_2 \subseteq \mathfrak{A}_1^{**} \odot \mathfrak{A}_2^{**}$. Let $(\rho_u, \mathfrak{K}_u)$ be the universal representation of $\mathfrak{B} = \mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2$; we denote by ρ_1, ρ_2 the restrictions of ρ_u to $\mathfrak{A}_1, \mathfrak{A}_2$, respectively, cf. [20, Theorem 6.3.5]. The mapping $\rho_0 : \mathfrak{A}_1^{**} \odot \mathfrak{A}_2^{**} \to \mathbb{B}(\mathfrak{K}_u)$ defined by

$$\rho_0(a_1 \otimes a_2) = \rho_1^{**}(a_1)\rho_2^{**}(a_2),$$

is multiplicative, since $\rho_1(\mathfrak{A}_1) \subseteq \rho_2(\mathfrak{A}_2)'$ implies $\rho_1^{**}(\mathfrak{A}_1^{**}) = \rho_1(\mathfrak{A}_1)'' \subseteq \rho_2(\mathfrak{A}_2)' = \rho_2^{**}(\mathfrak{A}_2^{**})'$, cf. [22, 3.7.7]. It is then clearly a $*$-homomorphism.

We are going to show that ρ_0 is 1–1. As we have identified $\mathfrak{A}_i$ with their images in $\mathbb{B}(\mathfrak{H}_u^i)$, the identities $\iota_i : \mathfrak{A}_i \to \mathbb{B}(\mathfrak{H}_u^i)$ are representations, and the representation

$$\sigma = \iota_1 \otimes_\beta \iota_2 : \mathfrak{B} \to \mathbb{B}(\mathfrak{H}_u^1 \hat{\otimes} \mathfrak{H}_u^2)$$

exists by [20, Theorem 6.4.18]. We proceed to show that $\sigma^{**} \circ \rho_0 = \mathrm{id}$. For this, assume that $a \in \mathfrak{A}_1^{**}, b \in \mathfrak{A}_2^{**}$. Letting, by Kaplansky's theorem, $(u_\lambda)_\Lambda, (v_\mu)_M$ be bounded nets in $\mathfrak{A}_1, \mathfrak{A}_2$ converging strongly to a, b, respectively, we have in the strong topology

$$\begin{aligned}
\rho_0(a \otimes b) &= \rho_1^{**}(a)\rho_2^{**}(b) \\
&= (\lim_{\Lambda} \rho_1(u_\lambda))(\lim_{M} \rho_2(v_\mu)) \\
&= \lim_{\Lambda, M} \rho_1(u_\lambda)\rho_2(v_\mu) \\
&= \lim_{\Lambda, M} \rho_u(u_\lambda \otimes v_\mu) \\
&= \lim_{\Lambda, M} u_\lambda \otimes v_\mu
\end{aligned}$$

where $\mathfrak{B}$ is identified with its image in $\mathfrak{B}^{**}$. We have used along the way that normal mappings are strongly continuous on bounded sets (cf. [17, 7.1.15]) as well as the fact that multiplication is strongly continuous on bounded sets. We now have that $\rho_0(a \otimes b) \in \mathfrak{B}^{**}$, and by definition of σ, using this time that $u_\lambda \otimes v_\mu$ is bounded ([20, Corollary 6.3.6]) so that

$$\begin{aligned}
\sigma^{**} \circ \rho_0(a \otimes b) &= \lim_{\Lambda, M} \sigma(u_\lambda \otimes v_\mu) \\
&= \lim_{\Lambda, M} u_\lambda \otimes v_\mu \\
&= a \otimes b,
\end{aligned}$$

where the last equality follows by the boundedness of $(u_\lambda), (v_\mu)$ as in the proof of Lemma 3.1.2. Since ρ_0 is a faithful $*$-homomorphism, $\|c\|_{\beta'} = \|\rho_0(c)\|$ defines a C^*-norm on $\mathfrak{A}_1^{**} \odot \mathfrak{A}_2^{**}$, and as ρ_0 extends ρ_u on $\mathfrak{A}_1 \odot \mathfrak{A}_2$, we have $\|c\|_\beta = \|c\|_{\beta'}$ when $c \in \mathfrak{A}_1 \odot \mathfrak{A}_2$. ∎

Theorem 3.2.4 *[Archbold] Let $\mathfrak{A}_1, \mathfrak{A}_2$ be unital C^*-algebras. For any C^*-norm $\|\cdot\|_\beta$ on $\mathfrak{A}_1 \odot \mathfrak{A}_2$,*

$$\mathcal{Z}(\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2) = \mathcal{Z}(\mathfrak{A}_1) \otimes \mathcal{Z}(\mathfrak{A}_2).$$

Proof: Let $\|\cdot\|_{\beta'}$ be the C^*-norm on $\mathfrak{A}_1^{**} \odot \mathfrak{A}_2^{**}$ constructed in the preceding lemma; recall that it extends $\|\cdot\|_\beta$. Identifying β'-completions with β'-closures within $\mathfrak{A}_1^{**} \otimes_{\beta'} \mathfrak{A}_2^{**}$, we have

$$\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2 \subseteq \mathfrak{A}_1^{**} \otimes_{\beta'} \mathfrak{A}_2 \subseteq \mathfrak{A}_1^{**} \otimes_{\beta'} \mathfrak{A}_2^{**}.$$

Letting $\mathcal{Z} = \mathcal{Z}(\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2)$, we will show that $\mathcal{Z} \subseteq \mathcal{Z}(\mathfrak{A}_1^{**} \otimes_{\beta'} \mathfrak{A}_2)$. To see this, it clearly suffices to prove that if $w \in \mathcal{Z}, a \in \mathfrak{A}_1^{**}, b \in \mathfrak{A}_2$, then

$$w(a \otimes b) = (a \otimes b)w.$$

By the construction of $\|\cdot\|_{\beta'}$ in the previous lemma, $\rho_0 : \mathfrak{A}_1^{**} \odot \mathfrak{A}_2^{**} \to \mathbb{B}(\mathfrak{K}_u)$ is isometric with respect to β', and hence extends to a faithful representation ρ :

$\mathfrak{A}_1^{**} \otimes_{\beta'} \mathfrak{A}_2^{**} \to \mathbb{B}(\mathfrak{K}_u)$. We can pick a net (a_λ) converging σ-weakly to a, and by definition of ρ_0,

$$\begin{aligned} \rho(w(a\otimes b)-(a\otimes b)w) &= \rho(w)\rho_1^{**}(a)\rho_2(b)-\rho_1^{**}(a)\rho_2(b)\rho(w) \\ &= \lim_\lambda (\rho(w)\rho_1(a_\lambda)\rho_2(b)-\rho_1(a_\lambda)\rho_2(b)\rho(w)) \\ &= \lim_\lambda \rho(w(a_\lambda\otimes b)-(a_\lambda\otimes b)w) \\ &= 0. \end{aligned}$$

Since ρ is faithful, $w \in \mathcal{Z}(\mathfrak{A}_1^{**} \otimes_{\beta'} \mathfrak{A}_2)$. By Lemma 3.2.1, $\mathcal{Z} \subseteq \mathcal{Z}(\mathfrak{A}_1^{**}) \otimes \mathcal{Z}(\mathfrak{A}_2)$, and similarly, $\mathcal{Z} \subseteq \mathcal{Z}(\mathfrak{A}_1) \otimes \mathcal{Z}(\mathfrak{A}_2^{**})$. From Lemma 3.2.2, $\mathcal{Z} \subseteq \mathcal{Z}(\mathfrak{A}_1) \otimes \mathcal{Z}(\mathfrak{A}_2)$, and the other implication is clear. ■

4 Excising nets for states

4.1 A brief account

We will need to employ the theory of excising nets of states as developed by Akemann, Anderson, and Pedersen in [4]. This section is as short as possible, only defining the notion and proving the facts we will find use for.

Note that when $\mathfrak{A}$ is not unital, $\mathbb{C} \not\subseteq \mathfrak{A}$ and elements such as $(a-\lambda)b$ only make sense as $ab-\lambda b$. We will allow ourselves this convenience of notation.

Definition 4.1.1 Let $\mathfrak{A}$ be a C^*-algebra. A net $(a_\lambda)_\Lambda$ in $\mathfrak{A}_+$ is said to *excise* $\varphi \in S(\mathfrak{A})$ if $\|a_\lambda\| = 1$ for all $\lambda \in \Lambda$ and

$$\|a_\lambda(a-\varphi(a))a_\lambda\| \to 0, \qquad \forall a \in \mathfrak{A}.$$

As an example, consider $\mathfrak{A} = C_0(X)$ for some locally compact Hausdorff space X, and let $\varphi = \hat{x}$, the character at some $x \in X$. Let Λ be the lattice of open neighborhoods of x ordered by reverse inclusion, and let f_G be a function satisfying $0 \le f_G \le 1$ and $f_G(x) = 1$, vanishing outside of G. For every $g \in C_0(X)$,

$$\|f_G(g-\varphi(g))f_G\| = \left\|(g-g(x))f_G^2\right\| \to 0,$$

for if $\varepsilon > 0$ is given, we can find $G \in \Lambda$ such that $|g(x)-g(y)| < \varepsilon$ for all $y \in G$.

The troika of [4] proves that $\varphi \in S(\mathfrak{A})$ can be excised if and only if $\varphi \in P(\mathfrak{A})^-$. We will content ourselves with much less.

Proposition 4.1.2 *Every $\varphi \in P(\mathfrak{A})$ has an excising net $(a_\lambda)_\Lambda$ which is decreasing and satisfies $\varphi(a_\lambda) = 1$, $\lambda \in \Lambda$. If $\mathfrak{A}$ is separable, the net can be chosen as a sequence.*

Proof: Applying Kadison's transitivity theorem, there is an element $d \in \mathfrak{A}_+$, $\|d\| = 1$ such that $\pi_\varphi(d)\xi_\varphi = \xi_\varphi$, and $\varphi(d) = (\pi_\varphi(d)\xi_\varphi|\xi_\varphi) = 1$. We get that every polynomial in $\pi_\varphi(d)$ maps $\mathbb{C}\{\xi_\varphi\}$ into itself, hence that every element of $C^*(1, \pi_\varphi(d))$ has this property. Writing $\pi_\varphi(d^{1/2})\xi_\varphi = \mu\xi_\varphi$, we have that $\mu \geq 0$ and $1 = \varphi(d) = \|\pi_\varphi(d^{1/2})\xi_\varphi\|^2 = |\mu|^2$, so that $\mu = 1$ and

$$\varphi(d^{1/2}xd^{1/2}) = (\pi_\varphi(x)\pi_\varphi(d^{1/2})\xi_\varphi|\pi_\varphi(d^{1/2})\xi_\varphi) = (\pi_\varphi(x)\xi_\varphi|\xi_\varphi) = \varphi(x)$$

for all $x \in \mathfrak{A}$.

Recall that $L_\varphi = \{x \in \mathfrak{A}|\varphi(x^*x) = 0\}$ is a left ideal of $\mathfrak{A}$, so that $L_\varphi \cap L_\varphi^*$ is a hereditary subalgebra. Note that by [22, 3.13.6],

$$L_\varphi \cap L_\varphi^* \subseteq L_\varphi + L_\varphi^* = \ker\varphi. \tag{6}$$

Letting (u_λ) be an approximate unit for $L_\varphi \cap L_\varphi^*$, cf. [22, 1.4.2], we put

$$a_\lambda = d^{1/2}(1 - u_\lambda)d^{1/2}$$

and note that since $u_\lambda \in \ker\varphi$ by (6), $\varphi(a_\lambda) = 1$. As, on the other hand, $a_\lambda \leq d$, we get $1 \leq \|a_\lambda\| \leq \|d\| = 1$. For every $a \in \mathfrak{A}$,

$$\varphi(d^{1/2}(a - \varphi(a))d^{1/2}) = \varphi(a) - \varphi(a)\varphi(d) = 0,$$

and by (6), we can write

$$d^{1/2}(a - \varphi(a))d^{1/2} = x + y^*,$$

where $x, y \in L_\varphi$. But then

$$\begin{aligned}
\|a_\lambda(a - \varphi(a))a_\lambda\| &\leq \|(1 - u_\lambda)(x + y^*)(1 - u_\lambda)\| \\
&\leq \|x(1 - u_\lambda)\| + \|(1 - u_\lambda)y^*\| \\
&= \|(1 - u_\lambda)x^*x(1 - u_\lambda)\|^{1/2} + \|(1 - u_\lambda)y^*y(1 - u_\lambda)\|^{1/2} \\
&\leq \|x^*x(1 - u_\lambda)\|^{1/2} + \|y^*y(1 - u_\lambda)\|^{1/2} \\
&\to 0
\end{aligned}$$

as $x^*x, y^*y \in L_\varphi \cap L_\varphi^*$.

When $\mathfrak{A}$ is separable, we may take (u_λ), and hence (a_λ), as a sequence. ■

The following result gives a slight generalization of an observation in the paper mentioned above, cf. the proof of [4, 2.3].

Lemma 4.1.3 *Let $(a_\mu)_M$ be an excising net for the state ψ, with the property that $\psi(a_\mu) = 1$, $\mu \in M$. Then for a net of states $(\varphi_\lambda)_\Lambda$ we have that $\varphi_\lambda \to \psi$ in the weak* topology if and only if*

$$\varphi_\lambda(a_\mu) \to 1, \qquad \forall \mu \in M.$$

Proof: Since all a_μ lie in $\mathfrak{A}$, the "only if" part is obvious. In the other direction, first note that if (a_μ) excises ψ on $\mathfrak{A}$ then, automatically, it will excise ψ's extension to $\mathfrak{A}^\sim$, because

$$a_\mu(a+\lambda 1-\tilde{\psi}(a+\lambda 1))a_\mu = a_\mu(a-\psi(a))a_\mu.$$

Furthermore, since the extensions are states, they are all 1 on the identity and thus converge *weak** on all of $\mathfrak{A}^\sim$, so without any loss of generality, we may assume that $\mathfrak{A}$ is unital.

Given $a \in \mathfrak{A}$ and $\varepsilon > 0$. We may assume that $\varepsilon^2 < 288\|a\|^2$ and pick μ_0 such that

$$\|a_{\mu_0}(a-\psi(a))a_{\mu_0}\| \leq \frac{\varepsilon}{2}.$$

According to the hypotheses, we can find λ_0 such that

$$\lambda \geq \lambda_0 \Longrightarrow \varphi_\lambda(a_{\mu_0}) \geq 1-\frac{\varepsilon^2}{288\|a\|^2}.$$

For any such λ, let $(\pi_\lambda, \mathfrak{H}_\lambda, \xi_\lambda)$ be the GNS triple of φ_λ. We have

$$\begin{aligned}
\|\pi_\lambda(1-a_{\mu_0})\xi_\lambda\|^2 &= \|\xi_\lambda - \pi_\lambda(a_{\mu_0})\xi_\lambda\|^2 \\
&= 1-2\varphi_\lambda(a_{\mu_0}) + \|\pi_\lambda(a_{\mu_0})\xi_\lambda\|^2 \\
&\leq 2-2\left(1-\frac{\varepsilon^2}{288\|a\|^2}\right) \\
&= \frac{\varepsilon^2}{144\|a\|^2},
\end{aligned}$$

from which we get, estimating liberally,

$$\begin{aligned}
|\varphi_\lambda(a)-\psi(a)| &= |\varphi_\lambda(a-\psi(a)1)| \\
&= |(\pi_\lambda(a-\psi(a)1)\xi_\lambda|\xi_\lambda)| \\
&= |(\pi_\lambda(a-\psi(a)1)\pi_\lambda(a_{\mu_0}+(1-a_{\mu_0}))\xi_\lambda|\pi_\lambda(a_{\mu_0}+(1-a_{\mu_0}))\xi_\lambda)| \\
&\leq |(\pi_\lambda(a_{\mu_0}(a-\psi(a))a_{\mu_0})\xi_\lambda|\xi_\lambda)| \\
&+ 2\|a-\psi(a)1\|\|a_{\mu_0}\|\|\pi_\lambda(1-a_{\mu_0})\xi_\lambda\| \\
&+ \|a-\psi(a)1\|\|\pi_\lambda(1-a_{\mu_0})\xi_\lambda\|^2 \\
&\leq \|a_{\mu_0}(a-\psi(a))a_{\mu_0}\| + (4\|a\|+2\|a\|)\frac{\varepsilon}{12\|a\|} \\
&\leq \varepsilon.
\end{aligned}$$

■

5 Introduction to Liapounoff theorems

This project was started as an investigation of Liapounoff theorems in the dual of a C^*-algebra, and although the focus has shifted somewhat since then, we include this section to explain the lines of thought behind it and to introduce some terminology.

5.1 A reformulation

Recall the following theorem from measure theory:

Theorem 5.1.1 *Liapounoff]liap:classic Let μ be an $\mathbb{R}^n$-valued non-atomic vector measure on the measure space $(X, \mathbb{E})$. Then $\mu(\mathbb{E})$ is compact and convex.*

Akemann and Anderson note in [3] that the theorem can be stated as follows, using terms from the theory of operator algebras.

Theorem 5.1.2 *Let $\mathcal{M}$ be a commutative non-atomic von Neumann algebra and $\Psi : \mathcal{M} \to \mathbb{R}^n$ a σ-weakly continuous linear map. Then*

$$\Psi((\mathcal{M}_+)_1) = \Psi(\mathrm{ext}((\mathcal{M}_+)_1)).$$

We will prefer to think of the σ-weak topology on $L^\infty(X, \mathbb{E}, \mu)$ as the *weak** topology induced by $L^1(X, \mathbb{E}, \mu)$.

Remark 5.1.3 As both of these statements are true, they are of course logically equivalent. It is, however, enlightening to see how one statement follows from the other. First note that from operator theory ([23, 2.5.6(j)]) or direct computations, we get that $\mathrm{ext}((\mathcal{M}_+)_1) = \mathcal{P}(\mathcal{M})$, the set of projections in $\mathcal{M}$.

Assuming Theorem 5.1.2, let μ be a non-atomic vector measure. Letting $\nu = |\mu_1| + \cdots + |\mu_n|$, we note that ν is also non-atomic, and $\mathcal{M} = L^\infty(X, \mathbb{E}, \nu)$ is a non-atomic commutative von Neumann algebra. We put $\psi_i(f) = \int_X f d\mu_i$, $\Psi = (\psi_1, \ldots, \psi_n)$ and see that since $\mu_i \ll \nu$, we can write $d\mu_i = g_i d\nu$ for some $g_i \in L^1(X, \mathbb{E}, \nu)$ by the Radon-Nikodym theorem, showing that every ψ_i, and hence Ψ, is *weak** continuous[4]. As $(\mathcal{M}_+)_1$ is convex and compact, so is $\Psi((\mathcal{M}_+)_1)$, and we can finish the argument by writing out

$$\Psi(\mathrm{ext}(\mathcal{M}_+)_1) = \Psi(\{1_E | E \in \mathbb{E}\}) = \mu(\mathbb{E}), \tag{7}$$

[4] We have $(L^\infty)_* = L^1$.

where we have used that the projections of $\mathcal{M}$ are exactly the characteristic functions.

In the other direction, we write $\mathcal{M} = L^\infty(X, \mathbb{E}, \nu)$ for some non-atomic measure ν. When $\Psi = (\psi_1, \ldots, \psi_n)$ is *weak** continuous, so is every ψ_i, and there exists $g_i \in L^1(X, \mathbb{E}, \nu)$ such that $\psi_i(f) = \int_X f g_i d\nu$. Letting $\mu = (g_1 \cdot \nu, \ldots, g_n \cdot \nu)$, where "$\cdot$" denotes measure with density, we get a non-atomic vector measure, and by the classical Liapounoff theorem, $\mu(\mathbb{E})$ is convex and compact. By Lemma A.2.2 combined with the Krein-Milman theorem and (7), we get

$$\begin{aligned}
\Psi((\mathcal{M}_+)_1) &= \overline{\mathrm{co}}(\mathrm{ext}(\Psi((\mathcal{M}_+)_1))) \\
&\subseteq \overline{\mathrm{co}}(\Psi(\mathrm{ext}((\mathcal{M}_+)_1))) \\
&= \overline{\mathrm{co}}(\mu(\mathbb{E})) \\
&= \mu(\mathbb{E}) \\
&= \Psi(\mathrm{ext}((\mathcal{M}_+)_1)).
\end{aligned}$$

We now widen our scope of interest to include all affine maps $\Psi : Q \to M$, where Q is a convex set and M a normed space. We say that we have a *Liapounoff theorem*, when certain conditions allow us to conclude that

$$\Psi(\mathrm{ext}\, Q) = \Psi(Q).$$

Such theorems for convex sets associated with operator algebras are considered as Liapounoff theorems of type 1 in [3]. Liapounoff theorems of other types, giving weaker results showing that $\Psi(\mathrm{ext}\, Q)$ is "big" in $\Psi(Q)$ in some sense, are also dealt with here.

5.2 The abstract Liapounoff theorem

The following theorem plays an important rôle in [3].

When Q is a convex set, sitting in a real linear space M, we say that $v \in Q$ is *internal relative to* x, when there exists an $\alpha > 0$ such that

$$(1 + \alpha)v - \alpha x \in Q. \tag{8}$$

Loosely speaking, v is internal relative to x when the line from x to v can be continued within Q.

Definition 5.2.1 $G(Q, v) = \{x \in Q | v \text{ is internal relative to } x\}$

Lemma 5.2.2 *$G(Q, v)$ is the minimal face of Q containing v.*

Proof: We write $G = G(Q, v)$. Obviously $v \in G$ because (8) holds with any $\alpha > 0$. When $x \in G$, select $\alpha > 0$ such that (8) holds. We see that when $0 < \beta < \alpha$, $(1 + \beta)v - \beta x \in Q$ from

$$(1 + \beta)v - \beta x = \frac{\beta}{\alpha}[(1 + \alpha)v - \alpha x] + (1 - \frac{\beta}{\alpha})v \in Q.$$

Thus, for $y_0, y_1 \in G(Q, v)$ we can find $\alpha > 0$ such that (8) holds for $x = y_i, i \in \{0, 1\}$. For any $\lambda \in (0, 1)$, (8) holds for $\lambda y_0 + (1 - \lambda)y_1$ as well, since

$$\begin{gathered}(1 + \alpha)v - \alpha[\lambda y_0 + (1 - \lambda y_1)] = \\ \lambda[(1 + \alpha)v - \alpha y_0] + (1 - \lambda)[(1 + \alpha)v - \alpha y_1] \in Q.\end{gathered}$$

Consequently, G is convex. To see that G is a face, let $y_0, y_1 \in Q$ and assume that $w = \frac{1}{2}y_0 + \frac{1}{2}y_1 \in G$. Take $\alpha > 0$ for w in (8), let $\beta = \alpha/2 + \alpha$ and note that

$$\begin{aligned}(1 + \beta)v - \beta y_1 &= \frac{2 + 2\alpha}{2 + \alpha}v - \frac{\alpha}{2 + \alpha}y_0 - \frac{\alpha}{2 + \alpha}y_1 + \frac{\alpha}{2 + \alpha}y_0 \\ &= \frac{2}{2 + \alpha}[(1 + \alpha)v - \alpha w] + (1 - \frac{2}{2 + \alpha})y_0 \in Q\end{aligned}$$

from convexity, as $0 < 2/2 + \alpha < 1$. We get that $y_1 \in G$; that $y_0 \in G$ follows similarly.

Now let F be a face of Q containing v. When $x \in G$, we pick $\alpha > 0$ in (8) and write

$$v = \frac{1}{1 + \alpha}[(1 + \alpha)v - \alpha x] + (1 - \frac{1}{1 + \alpha})x$$

to see that $x \in F$. ■

As noted in Appendix A, $v \in Q$ is extreme if and only if $\{v\}$ is a face, and according to the above lemma, $\{v\}$ is extreme if and only if $G(Q, v) = \{v\}$. The following definition depends on the observation that if Q has more than one point, it has (infinitely many) composite points.

Definition 5.2.3 Let Q be a convex set with more than one point. The *facial dimension* of Q is defined by

$$\dim_{\mathcal{F}}(Q) = \min\{\dim G(Q, v) | v \text{ composite point of } Q\},$$

where we write $\dim G(Q, v)$ for the linear dimension of $G(Q, v) - v$.

Example 5.2.4 *Using symbols to represent convex subsets of* $\mathbb{R}^2$, $\dim_{\mathcal{F}}(\bullet) = 2$ *and* $\dim_{\mathcal{F}}(\blacksquare) = \dim_{\mathcal{F}}(\blacktriangle) = 1$.

It turns out that the facial dimension of Q can be used in some cases to prove the existence of Liapounoff theorems for mappings defined on Q. The following, very general result is found by combining pure[5] convexity theory with the Krein-Milman theorem.

Theorem 5.2.5 *If Q is a compact subset of the locally convex space M such that $\dim_{\mathcal{F}}(Q) > n$, and $\Psi : Q \to \mathbb{R}^n$ is a continuous, affine map, then $\Psi(\operatorname{ext} Q) = \Psi(Q)$.*

Proof: This is [3, 1.7]. ■

Let us see how the classical Liapounoff theorem follows from this.

Proof of Theorem 5.1.2: Write $\mathcal{M} = L^\infty(X, \mathbb{E}, \mu)$ for some non-atomic measure μ. According to Theorem 5.2.5, we need only prove that $\dim_{\mathcal{F}}((L^\infty(\mu)_+)_1) = \infty$. A composite point of $(L^\infty(\mu)_+)_1$ can be given by $f \in \mathcal{L}^\infty(\mu)$ with $0 \le f \le 1$ such that

$$\mu(\{x \in X | 0 < f(x) < 1\}) > 0.$$

With

$$E_n = \{x \in X | 2^{-n} \le f(x) \le 1 - 2^{-n}\},$$

we can find $n \in \mathbb{N}$ such that $\mu(E_n) > 0$ and note that for $g \in L^\infty(\mu)$ with $0 \le g \le 1$ and $\operatorname{supp}(g) \subseteq E_n$,

$$(1 + 2^{-n})f - 2^{-n}g \in (L^\infty(\mu)_+)_1.$$

Consequently, $G(Q, f) - f$ contains a copy of $(L^\infty(E_n, \mathbb{E}_{E_n}, \mu_{E_n})_+)_1$, and since μ_{E_n} is non-atomic, this is an infinite-dimensional set. ■

[5] i.e. non-topological

Part II

Connectivity

Non-commutative topology, in the sense associated with C^*-algebras, is based on the fact that every abelian C^*-algebra is of the form $C_0(X)$ for X a locally compact Hausdorff space, so that we may think of a C^*-algebra as a generalized function algebra. Work in this branch almost always goes by the following recipe: suppose (†) is some topological property that we want to study. One proceeds by redefining "X has (†)" as "$C_0(X)$ has (‡)" and then examining different generalizations of (‡) to a C^*-algebra $\mathfrak{A}$. Most often, there will be more than one reasonable definition, because many of the objects associated with C^*-algebras are very different, even though they coincide for $\mathfrak{A}$ abelian.

This part of the thesis is such an examination, regarding the topological term "connected", cf. Appendix B. We start out by breaking the ground for a definition of connected C^*-algebras, proving that most of the natural generalizations are equivalent. Then we carry the notion of components through in this setting, and look at the number of components of algebras constructed from other algebras. Finally, we look at a couple of other topological notions related to connectivity.

6 Connected C^*-algebras

6.1 A basic notion

In this section we will focus on one notion of connectedness for a C^*-algebra and demonstrate how it relates to the Liapounoff type problems that are discussed in [3].

As usual, we turn to an abelian C^*-algebra $C_0(X)$, where X is a locally compact Hausdorff space, to make connections to (commutative) topology. If X is connected, then for every $f \in C_0(X)_{sa}$, $f(X)$ is a connected (or, equivalently, convex) set in $\mathbb{R}$. Since we have restricted ourselves to functions vanishing at infinity, this is not in general a sufficient condition for X to be connected:

Example 6.1.1 *A commutative C^*-algebra $C_0(X)$ with disconnected underlying space X such that $f(X)$ is connected for every $f \in C_0(X)$.*

We endow $X = \mathbb{R} \times \mathbb{R}$ with discrete topology in the first coordinate and the usual one in the second. In this case X is far from connected, but since $f \in C_0(X)$ vanishes at infinity on X then, from continuity, $f(\{x_1\} \times \mathbb{R}) \cup \{0\}$ is connected for every $x_1 \in \mathbb{R}$. Since $f(\cdot, 0)$ has to vanish outside a countable set, $0 \in f(\mathbb{R} \times \{0\})$ and we can write

$$f(\mathbb{R} \times \mathbb{R}) = \bigcup_{x_1 \in \mathbb{R}} f(\{x_1\} \times \mathbb{R}) = \bigcup_{x_1 \in \mathbb{R}} (f(\{x_1\} \times \mathbb{R}) \cup \{0\})$$

thus demonstrating $f(X)$ to be a union of connected sets with one mutual point, hence itself connected by Proposition B.1.2(i). ■

If $C_0(X)$ contains a strictly positive element g, however, and we assume that $X = U \cup V$ with disjoint U, V both non-void and open, then obviously $f = g1_U - g1_V$ is an element of $C_0(X)$ such that $0 \notin f(X)$ and $f(X)$ contains both positive and negative elements. So when the abelian C^*-algebra is σ-unital, the underlying space is connected if and only if

$$\forall a \in \mathfrak{A}_{sa} : \hat{a}(P(\mathfrak{A})) \text{ is convex.} \tag{1}$$

Now let $\mathfrak{A}$ be any C^*-algebra. We will prove in the next section, that if $\mathfrak{A}$ is σ-unital, condition (1) relates to connectivity of $\mathfrak{A}$ in more than one natural sense, but our preoccupation in the rest of this section will be to relate condition (1) to the theory of Liapounoff type theorems as defined in Section 5.1. By means of arguments involving only basics of the theory of C^*-algebras we will see that $\hat{a}(P(\mathfrak{A}))$ is convex if and only if the Liapounoff type condition $\hat{a}(P(\mathfrak{A})) = \hat{a}(S(\mathfrak{A}))$ holds. We must argue for the unital and non-unital cases separately, but the ways of reasoning are essentially the same.

Note that when $\mathfrak{A}$ is unital, (1) is equivalent to requiring that $\hat{a}(P(\mathfrak{A}))$ is convex for all $a \in \mathfrak{A}_+$, since $(a + \|a\|1)\hat{}(P(\mathfrak{A})) = \hat{a}(P(\mathfrak{A})) + \|a\|$. That is not generally the case, as the example $\mathfrak{A} = C_0(\mathbb{R} \backslash \{0\})$ shows.

Lemma 6.1.2 *Let $\mathfrak{A}$ be unital and take $a \in \mathfrak{A}_{sa}$. We have*

(i) $\mathrm{sp}(a) \subseteq \hat{a}(P(\mathfrak{A}))$.

(ii) $\hat{a}(S(\mathfrak{A})) = \mathrm{co}(\mathrm{sp}(a))$.

(iii) If $\mathrm{sp}(a)$ *is convex, then* $\hat{a}(P(\mathfrak{A})) = \hat{a}(S(\mathfrak{A}))$.

(iv) $\hat{a}(P(\mathfrak{A})) = \hat{a}(S(\mathfrak{A}))$ *if and only if* $\hat{a}(P(\mathfrak{A}))$ *is convex .*

Proof: For (i), let γ be a point of $\mathrm{sp}(a)$. We can find a homomorphism φ on $C^*(1, a)$, such that $\varphi(a) = \gamma$. Since φ is pure on $C^*(1, a)$, it has a extension to a pure state of $\mathfrak{A}$. Since $\hat{a}(S)$ is convex as S is, $\mathrm{co}(\mathrm{sp}(a)) \subseteq \hat{a}(S)$ follows from (i). The other inclusion can be dealt with by splitting up in cases: If both a_+ and a_- are non-zero, then $\mathrm{co}(\mathrm{sp}(a)) = [-\|a_-\|, \|a_+\|]$ and the inclusion is clear. Assume $a \in \mathfrak{A}_+$. If a is not invertible, $\mathrm{co}(\mathrm{sp}(a)) = [0, \|a\|]$, if it is, $\mathrm{co}(\mathrm{sp}(a)) = [\|a^{-1}\|^{-1}, \|a\|]$. For this last case, we use that when $\varphi \in S$,

$$1 = \varphi(1) = \varphi(a^{\frac{1}{2}}a^{-\frac{1}{2}}) \leq \varphi(a)^{\frac{1}{2}}\varphi(a^{-1})^{\frac{1}{2}}$$

and $\varphi(a) \geq \varphi(a^{-1})^{-1} \geq \|a^{-1}\|^{-1}$. When $\mathrm{sp}(a)$ is convex we get (iii) from

$$\hat{a}(S) = \mathrm{co}(\mathrm{sp}(a)) = \mathrm{sp}(a) \subseteq \hat{a}(P) \subseteq \hat{a}(S).$$

In particular, $\hat{a}(P)$ is convex. For (iv), we may similarly write

$$\hat{a}(S) = \mathrm{co}(\mathrm{sp}(a)) \subseteq \mathrm{co}(\hat{a}(P)) = \hat{a}(P) \subseteq \hat{a}(S),$$

assuming that $\hat{a}(P)$ is convex. ■

Now we turn our attention to non-unital C^*-algebras $\mathfrak{A}$.

Lemma 6.1.3 *We have for every $a \in \mathfrak{A}_+$*

$$0 \in \hat{a}(S(\mathfrak{A})) \Longleftrightarrow 0 \in \hat{a}(P(\mathfrak{A})).$$

If $\mathfrak{A}$ is non-unital, then in any case $0 \in \hat{a}(P(\mathfrak{A}))^=$.

Proof: We may assume that $\|a\| = 1$. Suppose $\varphi(a) = 0$ for some $\varphi \in S$ and recall that $L_\varphi = \{x \in \mathfrak{A} | \varphi(x^*x) = 0\}$. Since $0 \leq a^2 \leq a$, $a \in L_\varphi$, and since $\|\varphi\| = 1$, $L_\varphi \neq \mathfrak{A}$. From [22, 3.13.5], we can take $\psi \in P$ such that $L_\varphi \subseteq L_\psi$. From [22, 3.13.6], $a \in L_\psi + L_\psi^* = \ker \psi$.

For the second assertion we assume $\hat{a}(P) \subseteq [\varepsilon, 1]$ for $\varepsilon > 0$. This shows that $\pi_a(a)$ is bounded away from zero, hence invertible. But since a faithful representation preserves spectrum, this would imply $0 \notin \mathrm{sp}(a)$. ■

Lemma 6.1.4 *Let $\mathfrak{A}$ be non-unital and take $a \in \mathfrak{A}_{sa}$. We have*

(i) $\mathrm{sp}(a)\backslash\{0\} \subseteq \hat{a}(P(\mathfrak{A}))$.

(ii) $\mathrm{co}(\mathrm{sp}(a)\backslash\{0\}) \subseteq \hat{a}(S(\mathfrak{A})) \subseteq \mathrm{co}(\mathrm{sp}(a))$.

(iii) If $\mathrm{sp}(a)$ is convex and $0 \in \hat{a}(P(\mathfrak{A}))$, then $\hat{a}(P(\mathfrak{A})) = \hat{a}(S(\mathfrak{A}))$.

(iv) $\hat{a}(P(\mathfrak{A})) = \hat{a}(S(\mathfrak{A}))$ if and only if $\hat{a}(P(\mathfrak{A}))$ is convex.

Proof: The first two statements are proven as in Lemma 6.1.2, taking into consideration that the homomorphism at 0 is identically zero, and therefore not a state. For (iii), note that under the given hypotheses

$$\hat{a}(P) \subseteq \hat{a}(S) \subseteq \mathrm{co}(\mathrm{sp}(a)) = \mathrm{sp}(a) \subseteq \hat{a}(P).$$

Finally, to see (iv) in the non-unital case, assume that $\hat{a}(P)$ is convex. If $0 \in \hat{a}(P)$, we can write

$$\hat{a}(P) \subseteq \hat{a}(S) \subseteq \mathrm{co}(\mathrm{sp}(a)) \subseteq \mathrm{co}(\hat{a}(P)) = \hat{a}(P).$$

If $0 \notin \hat{a}(P)$ then by convexity we may assume that $\hat{a}(P)$ lies entirely within $\mathbb{R}_+$. Then $a \geq 0$, and from Lemma 6.1.3 we have that

$$\hat{a}(P) \subseteq \hat{a}(S) \subseteq \mathrm{co}(\mathrm{sp}(a)) \backslash \{0\} \subseteq]0, \|a\|] = \hat{a}(P).$$

■

6.2 The main theorem

A topological space is connected when it admits only the trivial separation into two disjoint, clopen sets. One way of carrying this through to the non-commutative case is to invoke the theory of open and closed projections of $\mathfrak{A}^{**}$, and it will be seen in the following section that a C^*-algebra $\mathfrak{A}$, having the property that every separation of the identity of $\mathfrak{A}^{**}$ into a sum of two clopen central projection is trivial, has some properties that relate to connectedness is a reasonable sense.

The first lemma in this section is absolutely crucial for our results. It gives us a way to pass to saturated subsets of $P(\mathfrak{A})$ in arguments involving connectedness, thereby allowing us to use the correspondence between closed saturated subsets of $P(\mathfrak{A})$ and closed central projections found in Proposition 2.4.2.

Lemma 6.2.1 *Let $\varphi \in P(\mathfrak{A})$. The set*

$$\mathrm{sat}\{\varphi\} = \{\varphi(u^* \cdot u) | u \in \mathcal{U}(\mathfrak{A}^\sim)\}$$

is arcwise connected in $P(\mathfrak{A})$.

Proof: It suffices to find a path from φ to an arbitrary element ψ of $\mathrm{sat}\{\varphi\}$. Letting p be the common cover of π_φ and π_ψ, cf. Lemma 1.1.1, we can find unit vectors ξ, η in $p\mathfrak{H}_\mathfrak{u}$ such that

$$\varphi(a) = (\pi(a)\xi|\xi), \qquad \psi(a) = (\pi(a)\eta|\eta).$$

Choose a continuous path of unit vectors ζ_t in $p\mathfrak{H}_\mathfrak{u}$ such that $\zeta_0 = \xi, \zeta_1 = \eta$. From Kadison's transitivity theorem [22, 2.7.5] we can find unitaries u_t in $\mathfrak{A}^\sim$ such that $\pi(u_t)\xi = \zeta_t$ and

$$\varphi(u_t^* a u_t) = (\pi(u_t^* a u_t)\xi|\xi) = (\pi(a)\zeta_t|\zeta_t).$$

This path is clearly *weak** continuous in $P(\mathfrak{A})$. ■

Theorem 6.2.2 *Let $\mathfrak{A}$ be a C^*-algebra. The following conditions are equivalent*

(i) If $\mathfrak{A} = \mathfrak{I}_0 \oplus \mathfrak{I}_1$ with $\mathfrak{I}_0, \mathfrak{I}_1$ ideals of $\mathfrak{A}$, then $\{\mathfrak{I}_0, \mathfrak{I}_1\} = \{\{0\}, \mathfrak{A}\}$.

(ii) If $1 = p_0 + p_1$ in $\mathfrak{A}^{**}$ with p_0, p_1 central open projections, then $\{p_0, p_1\} = \{0, 1\}$.

(iii) $P(\mathfrak{A})$ is connected.

(iv) $\mathrm{Prim}(\mathfrak{A})$ is connected.

and imply

(v) $\forall a \in \mathfrak{A} : \hat{a}(P(\mathfrak{A}))$ is connected.

(vi) $\forall a \in \mathfrak{A}_{sa} : \hat{a}(P(\mathfrak{A})) = \hat{a}(S(\mathfrak{A}))$.

If $\mathfrak{A}$ is σ-unital, all the conditions are equivalent.

Definition 6.2.3 A C^*-algebra $\mathfrak{A}$ is called *connected* if it satisfies (i)–(iv) above.

Proof of Theorem 6.2.2: That (i)$\Longleftrightarrow$(ii) follows by Theorem 2.4.1. For the implication (ii)$\Longrightarrow$(iii), assume that $P(\mathfrak{A}) = P$ is separated by $\{U_0, U_1\}$, i.e. P is the disjoint union of these two clopen sets. For every $\varphi \in U_0$, $\mathrm{sat}\{\varphi\}$ is connected by Lemma 6.2.1, and must hence be contained in U_0. We get that U_0 must be saturated, whence so is U_1.

Applying Proposition 2.4.2 we get central closed projections p_{U_0} and p_{U_1} that are orthogonal since $P(p_{U_0}p_{U_1}) = P(p_{U_0}) \cap P(p_{U_1}) = U_0 \cap U_1 = \emptyset$. For by Proposition 2.2.8, $p_{U_0}p_{U_1}$ is closed, and Proposition 2.4.2 applies. Furthermore,

$$P(p_{U_0} + p_{U_1}) \supseteq P(p_{U_0}) \cup P(p_{U_0}) = U_0 \cup U_1 = P,$$

and by Proposition 2.2.9, $p_{U_0} + p_{U_1}$ is closed. As above, we can conclude $p_{U_0} + p_{U_1} = 1$. From the hypothesis, we may assume that $p_{U_0} = 0, p_{U_1} = 1$ and by Proposition 2.4.2 again, $U_0 = \emptyset, U_1 = P$.

That (iii)$\Longrightarrow$(iv) follows from the fact that $\varphi \mapsto \ker \pi_\varphi$ is a continuous function from $P(\mathfrak{A})$ onto $\mathrm{Prim}(\mathfrak{A})$, cf. [22, 4.3.3], and Proposition B.1.2(iii). For (iv)$\Longrightarrow$(ii) let $\{p_0, p_1\}$ be as in (ii). Since both projections must be open and closed, we have from Theorem 2.2.2 and [22, 3.12.9] that $p_0, p_1 \in M(\mathfrak{A})$. By the Dauns-Hofmann theorem [22, 4.4.8] the center of $M(\mathfrak{A})$ is isomorphic to $C^b(\mathrm{Prim}(\mathfrak{A}))$, and the images of p_0, p_1 are projections here, i.e. characteristic functions for clopen sets. Since $\mathrm{Prim}(\mathfrak{A})$ is connected, $\{p_0, p_1\} = \{0, 1\}$. That (iii)$\Longrightarrow$(v) follows by continuity of $\hat{a} : P(\mathfrak{A}) \to \mathbb{C}$, and (v)$\Longrightarrow$(vi) follows by Lemma 6.1.2 when $\mathfrak{A}$ has a unit, Lemma 6.1.4 when it does not, since a connected subset of $\mathbb{R}$ is automatically convex.

Finally, assume that $\mathfrak{A}$ has a strictly positive element c and that (ii) does not hold. Then there exist non-trivial open central projections p_0, p_1, and as above, we note that $p_0, p_1 \in M(\mathfrak{A})$. By definition of the multiplier algebra, $a = cp_0 - cp_1 \in \mathfrak{A}$, and we claim that $\hat{a}(P)$ is not convex. When $\varphi \in P$, we denote by p_φ the cover of π_φ and note since π_φ is irreducible, p_φ is minimal central and $p_0 p_\varphi$ is either 0 or p_φ. In the first case,

$$\varphi(a) = \varphi(cp_0) - \varphi(cp_1) = -\varphi(c) < 0,$$

in the second

$$\varphi(a) = \varphi(cp_0) - \varphi(cp_1) = \varphi(c) > 0.$$

We conclude that $0 \notin \hat{a}(P)$. But since p_0 and p_1 are non-zero, $\mathrm{sp}(a) \subseteq \hat{a}(P)$ contains both positive and negative elements. ■

Remark 6.2.4 The class of connected C^*-algebras has been considered by Kaplansky in [18] under the name "non-decomposable" C^*-algebras. In this paper, (i)$\Longleftrightarrow$(iv) is proved directly by the correspondence between ideals of $\mathfrak{A}$ and open sets of $\mathrm{Prim}(\mathfrak{A})$. The usage of open and closed projections in the proof above seems not to be essential, but we choose this approach in anticipation of following sections, where "components" will be discussed.

Note that when $\mathfrak{A}$ is abelian, say $\mathfrak{A} = C_0(X)$, $\mathfrak{A}$ is connected if and only if $X = P(\mathfrak{A})$ is. Also note that if $\mathfrak{A}$ contains a non-trivial central projection, it is not connected, for such a projection is automatically clopen.

We now state and prove a few easy corollaries, furnishing us with some examples of connected C^*-algebras. Recall that a C^*-algebra is *prime* if its zero ideal is prime, i.e. if $\mathfrak{I}\mathfrak{J} = \{\mathfrak{o}\}$ only if $\mathfrak{I}$ or $\mathfrak{J}$ is the zero ideal.

Corollary 6.2.5 *If $\mathfrak{A}$ is prime, $\mathfrak{A}$ is connected. In particular, if $\mathfrak{A}$ is primitive, it is connected.*

Proof: If $\mathfrak{I}_\mathfrak{o} \oplus \mathfrak{I}_\mathfrak{1} = \mathfrak{A}$ then $\mathfrak{I}_\mathfrak{o}\mathfrak{I}_\mathfrak{1} = \{\mathfrak{o}\}$ and as we may then assume that $\mathfrak{I}_\mathfrak{o} = \{\mathfrak{o}\}$, $\mathfrak{A}$ is connected. That primitive C^*-algebras are prime follows by [22, 3.13.10]. ■

In the case of primitive C^*-algebras, we might also have noted that the subset $\{\{0\}\}$ of $\mathrm{Prim}(\mathfrak{A})$ is connected and dense and have seen that $\mathrm{Prim}(\mathfrak{A})$ is connected by Proposition B.1.2(ii). But non-separable primitive algebras are not known to be prime.

Corollary 6.2.6 *Assume that $\mathfrak{A}$ is unital. Then*

(i) If $\mathfrak{A}$ is commutative, $\mathfrak{A}$ is connected if and only if $\mathfrak{A}$ contains no non-trivial projections.

(ii) $\mathfrak{A}$ is connected if and only if its center is.

(iii) $\mathfrak{A}$ is connected if and only if it contains no non-trivial central projections.

(iv) If $\mathfrak{A}$ is a von Neumann algebra, it is connected if and only if it is a factor.

Proof: For (i), let $\mathfrak{A} = C(X)$ for X some compact Hausdorff space. Clearly X is connected if and only if $C(X)$ has no non-trivial projections. For (ii), note that when $\mathfrak{A}$ is unital, $\mathfrak{A} = M(\mathfrak{A})$ and $\mathcal{Z}(\mathfrak{A}) = \mathcal{Z}(M(\mathfrak{A})) \simeq C^b(\mathrm{Prim}(\mathfrak{A}))$. Clearly, then, $\mathcal{Z}(\mathfrak{A})$ is connected if and only $\mathrm{Prim}(\mathfrak{A})$ is. Statement (iii) follows from (i) and (ii).

For (iv), note that the center of a von Neumann algebra is itself a von Neumann algebra, hence generated by its projections. Therefore, if it has no non-trivial projections, it is trivial. ■

It is now clear that $\mathbb{K}(\mathfrak{H})$ is connected when $\mathfrak{H}$ is a separable Hilbert space, but that $\mathfrak{A} = \mathbb{K}(\mathfrak{H}) \oplus \mathbb{K}(\mathfrak{H})$ is not. However, $\mathfrak{A}$ clearly has trivial center, thus demonstrating that (ii) and (iii) do not hold for non-unital C^*-algebras.

Corollary 6.2.7 *Let $\mathfrak{I}$ be an essential ideal in the C^*-algebra $\mathfrak{A}$. If $\mathfrak{I}$ is connected, so is $\mathfrak{A}$.*

Proof: Assume that $\mathfrak{A} = \mathfrak{J}_0 \oplus \mathfrak{J}_1$. Clearly $(\mathfrak{I} \cap \mathfrak{J}_0) \cap (\mathfrak{I} \cap \mathfrak{J}_1) = \{0\}$. By existence of square root for positive elements, $\mathfrak{I}^2 = \mathfrak{I}$, where $\mathfrak{I}^2$ denotes the (closed) ideal generated by elements of the form ab, where $a, b \in \mathfrak{I}$. We get

$$\mathfrak{I} = \mathfrak{I}^2 \subseteq \mathfrak{I}(\mathfrak{J}_0 + \mathfrak{J}_1) \subseteq \mathfrak{I}\mathfrak{J}_0 + \mathfrak{I}\mathfrak{J}_1 \subseteq (\mathfrak{I} \cap \mathfrak{J}_0) + (\mathfrak{I} \cap \mathfrak{J}_1),$$

so $\mathfrak{I} = (\mathfrak{I} \cap \mathfrak{J}_0) \oplus (\mathfrak{I} \cap \mathfrak{J}_1)$. By hypothesis, we may assume that $\mathfrak{I} \cap \mathfrak{J}_0 = \{0\}, \mathfrak{I} \cap \mathfrak{J}_1 = \mathfrak{I}$, and since $\mathfrak{I}$ is essential, $\mathfrak{J}_0 = \{0\}$ follows. ■

We might also have used, that $\mathrm{Prim}(\mathfrak{I})$ sits as a dense subset of $\mathrm{Prim}(\mathfrak{A})$, cf. Proposition B.1.2(ii).

Remark 6.2.8 Assume that $\mathfrak{A}$ is unital. The above theorem determines when a Liapounoff theorem holds true for *all* mappings $\hat{a} : S(\mathfrak{A}) \to \mathbb{R}$. The set $S(\mathfrak{A})$ is *weak** compact, and by definition, $\hat{a}$ is continuous in this topology. If $\mathfrak{A}$ is not connected, by Theorem 6.2.2, we get $\hat{a}(S(\mathfrak{A})) \neq \hat{a}(\mathrm{ext}(S(\mathfrak{A})))$ for some $a \in \mathfrak{A}_{sa}$, and by the abstract Liapounoff theorem 5.2.5 we conclude that $\dim_{\mathcal{F}}(S(\mathfrak{A})) = 1$.

We can actually exhibit a one-dimensional face: Take non-trivial open central projections p, q as in Theorem 6.2.2 and note that pz and qz are central. Also, pz and qz are non-zero by Corollary 2.2.14. Let p', q' be minimal projections under pz, qz respectively. The set

$$F(p'+q') \cap S(\mathfrak{A}) = \{\varphi \in S(\mathfrak{A}) | \varphi(p'+q') = 1\}$$

is a face of $S(\mathfrak{A})$, isomorphic to the set of states on $(p'+q')\mathfrak{A}^{**}(p'+q')$. And since $[\mathfrak{A}^{**}p'\mathfrak{H}_{\mathfrak{u}}] = \mathfrak{c}(\mathfrak{p}') \leq \mathfrak{p}$,$[\mathfrak{A}^{**}q'\mathfrak{H}_{\mathfrak{u}}] = \mathfrak{c}(\mathfrak{q}') \leq \mathfrak{q}$, we have

$$(p'+q')\mathfrak{A}^{**}(p'+q') = p'\mathfrak{A}^{**}p' \oplus q'\mathfrak{A}^{**}q' \simeq \mathbb{C} \oplus \mathbb{C}.$$

Clearly the state space of $\mathbb{C} \oplus \mathbb{C}$ is isomorphic to the 1-dimensional space $[0, 1]$. Connectedness of $\mathfrak{A}$ does not imply that $\dim_{\mathcal{F}}(S(\mathfrak{A})) > 1$, as any abelian connected C^*-algebra $\mathfrak{A} \neq \mathbb{C}$ will show. In fact, we can show that $\dim_{\mathcal{F}}(S(\mathfrak{A})) = 1$ as above, as soon as $\mathfrak{A}$ has two inequivalent irreducible representations.

6.3 Connectedness and unitizations

Let us assume that $\mathfrak{A}$ is non-unital. Even in the σ-unital commutative case we can not conclude that $\mathfrak{A}$ is connected when $\mathfrak{A}^{\sim}$ is, corresponding to the fact that the one-point compactification of a non-connected topological space can be connected, viz. $(-\infty, -1] \cup [1, \infty)$. We can, however, easily establish a non-commutative result in the opposite direction.

Proposition 6.3.1 *If $\mathfrak{A}$ is connected, so is $\mathfrak{A}^{\sim}$.*

Proof: Clearly, $\mathfrak{A}$ is essential in $\mathfrak{A}^{\sim}$. Apply Corollary 6.2.7. ■

In fact, when a *unitization* of $\mathfrak{A}$ is defined as a C^*-algebra containing $\mathfrak{A}$ as an essential ideal, any unitization of a connected C^*-algebra must be connected.

It is well-known that the multiplier algebra of a non-unital C^*-algebra can be thought of as a non-commutative equivalent of the Stone-Čech compactification. Let X be a locally compact Hausdorff space. If we use the description of βX as the unique compact Hausdorff space such that $C^b(X) = C(\beta X)$ it easy to see that X is connected if and only if βX is. As a direct consequence of Theorem 6.2.2, this extends to the non-commutative case.

Proposition 6.3.2 *$\mathfrak{A}$ is connected if and only if $M(\mathfrak{A})$ is.*

Proof: We employ Theorem 6.2.2 to $\mathfrak{A}$ and $M(\mathfrak{A})$ to see that if any of $\mathfrak{A}$, $M(\mathfrak{A})$, and $\mathcal{Z}(M(\mathfrak{A})) \simeq C^b(\mathrm{Prim}(\mathfrak{A}))$ is connected, so are the other two. ■

7 Components

As we saw in Section 6.2, we need to assume that the C^*-algebras we look at are σ-unital in order to assure that all the different notions of connectivity agree. In this section we will develop a notion of connectivity for projections in $\mathfrak{A}^{**}$, thereby taking the step from $\mathfrak{A}$ to any of its hereditary subalgebras, and in order to employ Theorem 6.2.2 in full force to the results beneath, we need that not only $\mathfrak{A}$, but all of its hereditary subalgebras, are σ-unital. There are σ-unital C^*-algebras that do not have this property, e.g. $\mathfrak{A}^\sim$ when $\mathfrak{A}$ is any non-σ-unital algebra. However, all subalgebras of a separable C^*-algebra are separable, hence σ-unital.

The reader may thus prefer to think of the C^*-algebras appearing in this section as separable, but he should note that with our standing definition of connectedness, there is no formal problems in carrying any C^*-algebra through the following arguments.

7.1 Components of a C^*-algebra

Definition 7.1.1 Let $p \in \mathfrak{A}^{**}$ be a projection. We say that p is *connected* if whenever q_0, q_1 are central open projections of $\mathfrak{A}^{**}$ such that

$$p = pq_0 + pq_1, \tag{2}$$

then $\{pq_0, pq_1\} = \{0, p\}$.

Note that by Theorem 6.2.2, $\mathfrak{A}$ is connected if and only $1 \in \mathfrak{A}^{**}$ is. A pair $\{q_0, q_1\}$ of central open projections is called a *separation* of p when (2) holds. It is said to be *trivial* (with respect to p) if $\{pq_0, pq_1\} = \{0, p\}$. In these words, p is connected when all separating pairs are trivial. Note that a minimal projection is automatically connected, as is a minimal central projection. Also, $p = 0$ is connected.

The definition is made such that the following property carries over from the commutative case:

Lemma 7.1.2 *If p is connected and q is central and open, then*

$$\left.\begin{array}{l} pq \neq 0 \\ p(1-q) \neq 0 \end{array}\right\} \Longrightarrow p\partial q \neq 0.$$

Proof: If $p\partial q = 0$, we can write

$$p\overline{q} = pq,$$

and p is separated by $\{q, 1-\overline{q}\}$. We conclude that either $pq = 0$ or $p(1-q) = 0$. ■

Proposition 7.1.3 *If $p \in \mathfrak{A}^{**}$ is an open projection, then p is connected if and only if $H(p)$ is.*

Proof: By Theorem 2.2.2(vi), the strong closure $H(p)^-$ relative to $\mathfrak{A}^{**}$ is $p\mathfrak{A}^{**}p$. Furthermore, $H(p)^-$ and $H(p)^{**}$ are isomorphic under a normal isometry that preserves $H(p)$ (cf. [22, 3.7.9]), so, using Theorem 2.2.2(iv), we may identify the open projections of $H(p)^{**}$ with those of $\mathfrak{A}^{**}$ that lie under p.

Assume that $H(p)$ is connected and let $\{q_0, q_1\}$ be a separation of p. By Proposition 2.2.9, since the q_i are central, pq_0 and pq_1 are open, and since these are clearly central as elements of $p\mathfrak{A}^{**}p$, we get (via the isomorphism mentioned above) that the separation is trivial. When p is connected, a separation of the identity in $H(p)^{**}$ gives a separation

$$p = p_0 + p_1,$$

where p_0, p_1 are open central projections of $p\mathfrak{A}^{**}p$. For each $i \in \{0, 1\}$, $c(p_i)$ is open by Lemma 2.4.4, so by proving that $pc(p_i) = p_i$ we get that $\{c(p_0), c(p_1)\}$ is a separation of p and $\{p_0, p_1\}$ is trivial. To see this, we need only prove that $pc(p_i) \leq p_i$. Since $c(p_i) = [\mathfrak{A}^{**}p_i\mathfrak{H}_{\mathrm{u}}]$ by Lemma 2.4.3, we get this from the fact that for every $a \in \mathfrak{A}^{**}, \xi \in \mathfrak{H}_{\mathrm{u}}$,

$$pap_i\xi = (pap)p_i\xi = p_i(pap)\xi \in p_i\mathfrak{H}_{\mathrm{u}}.$$

■

Example 7.1.4 *If $\mathfrak{A}$ is primitive, every open projection is connected.*

Proof: By [22, 4.1.5], every hereditary subalgebra of $\mathfrak{A}$ is primitive. Combine Proposition 7.1.3 and Corollary 6.2.5. ■

Lemma 7.1.5 *Let $p \in \mathfrak{A}^{**}$ be a projection.*

(i) p is connected if and only if $c(p)$ is.

(ii) If p is connected, so is $\overline{p}$.

Proof: For the first assertion, note that the mapping $ap \mapsto ac(p)$ is a 1–1 isomorphism of $\mathfrak{A}'p$ onto $\mathfrak{A}'c(p)$, cf. [22, 2.6.7]. Thus, if $c(p)$ is connected and $\{q_0, q_1\}$ is a separation of p then, by applying the isomorphism to the equation $1p = (q_0 + q_1)p$ we get $1c(p) = (q_0 + q_1)c(p)$ whence $\{q_0, q_1\}$ is a separation of $c(p)$ and we may assume $c(p)q_0 = 0, c(p)q_1 = c(p)$. Applying the isomorphism again, $pq_0 = 0, pq_1 = p$. The other implication follows similarly.

Now assume that $\{q_0, q_1\}$ separates $\overline{p}$. We may assume that $pq_0 = 0, pq_1 = p$, and since q_0 is open, $1 - q_0$ is closed, where $p \leq 1 - q_0$ by centrality. Then also $\overline{p} \leq 1 - q_0$, and $q_0\overline{p} = 0$. ∎

Proposition 7.1.6 *Let $(p_i)_{i\in I}$ be a family of connected projections of $\mathfrak{A}^{**}$ and assume that there exists a minimal central projection $q \in \mathfrak{A}^{**}$ such that for all $i \in I$,*

$$p_i q \neq 0.$$

Then $\bigvee_I p_i$ is connected.

Proof: We may assume that $I \neq \emptyset$. Let $p = \bigvee_I p_i$ and assume that $\{r_0, r_1\}$ separates p. Since $p \geq p_i$ for any i, $pq \neq 0$, so $(r_0 + r_1)q \neq 0$. Since q is minimal central, $r_k q \in \{0, q\}$ for $k \in \{0, 1\}$, and having $r_0 q = r_1 q = q$ would lead to the contradiction

$$pq = pr_0 q + pr_1 q = 2pq,$$

so we may assume that $r_0 q = 0, r_1 q = q$. For any $i \in I$, $p_i r_1 \geq p_i q \neq 0$, and since $\{r_0, r_1\}$ is trivial with respect to p_i, $p_i r_0 = 0$ and $p_i r_1 = p_i$. As i was arbitrary, $pr_0 = 0, pr_1 = p$. ∎

Corollary 7.1.7 *Let $(p_i)_{i\in I}$ be a family of connected projections of $\mathfrak{A}^{**}$ and assume that there exists $\varphi \in P$ such that for all $i \in I$,*

$$\varphi(p_i) > 0.$$

Then $\bigvee_I p_i$ is connected.

Proof: Apply the proposition above to the central cover of π_φ. ∎

Definition 7.1.8 A *component* of $\mathfrak{A}$ is a maximal connected projection of $\mathfrak{A}^{**}$.

Proposition 7.1.9 *Any connected projection is dominated by a component.*

Proof: The case $p = 0$ is trivial. Let $p \in \mathfrak{A}^{**}\backslash\{0\}$ be connected and consider

$$\mathcal{A}_p = \{q \in \mathfrak{A}^{**} | q \text{ connected}, p \leq q\}$$

as a partially ordered set under the usual order on $\mathfrak{A}^{**}_{sa}$. Then $\mathcal{A}_p$ is inductively ordered, for if $\{q_i\}_I$ is a totally ordered subset, then $q = \bigvee_I q_i$ is a majorant in $\mathfrak{A}^{**}$.

It dominates p, and to see that it is connected, let $\{r_0, r_1\}$ be a separation. Since this is a separation of each q_i, we can find $(b_i)_I$ such that

$$q_i r_{b_i} = 0, \qquad q_i r_{1-b_i} = q_i$$

where $b_i \in \{0,1\}$ is uniquely defined since $q_i \geq p > 0$. Given i, j we may assume that $q_i \geq q_j$. If $(b_i, b_j) = (1,0)$ we would get the contradiction

$$q_j = q_j r_1 \leq q_i r_1 = 0,$$

if $(b_i, b_j) = (0,1)$, likewise

$$q_j = q_j r_0 \leq q_j r_0 = 0,$$

so b_i is constant, say identically 0, from which we infer

$$q r_0 = 0, q r_1 = q.$$

We have seen that $\mathcal{A}_p$ is inductively ordered, and may, since $p \in \mathcal{A}_p$, apply Zorn's lemma to find a maximal $q \in \mathcal{A}_p$. A connected $r \geq q$ would lie in $\mathcal{A}_p$, and q is a component. ■

When the connected projection in question is minimal, we can actually write down a component containing it by means of Corollary 7.1.7. Let p be minimal in $\mathfrak{A}^{**}$ and take $\varphi \in P(\mathfrak{A})$ supporting it. Define

$$c_\varphi = \bigvee \{q \in \mathfrak{A}^{**} | q \text{ connected}, \varphi(q) > 0\}$$

and note that $c_\varphi \geq p$. By Corollary 7.1.7, c_φ is connected, and if $q \geq c_\varphi$ is connected then since

$$\varphi(q) \geq \varphi(c_\varphi) \geq \varphi(p) = 1,$$

we have $q \leq c_\varphi$ and c_φ is maximal.

We are now ready to collect our results about components.

Proposition 7.1.10 *A component of $\mathfrak{A}$ is closed and central. If two components are different, they are orthogonal. The sum of all components of $\mathfrak{A}$ dominates z as well as all minimal central projections.*

Proof: Let p be a component of $\mathfrak{A}$. From Proposition 7.1.5(i) we have that $c(p) = p$, so p is central, and by 7.1.5(ii), $\overline{p} = p$ and p is closed. If p, q are components and $pq \neq 0$ then, since $pq = p \wedge q$ is closed, $pqz \neq 0$ from Corollary 2.2.14 and we can find $\varphi \in P$ such that $\varphi(pq) > 0$. Consequently $\varphi(p), \varphi(q) > 0$ and by Corollary 7.1.7, $p \vee q$ is connected. By maximality, $p = p \vee q = q$.

We have proved (twice!) that all minimal projections of $\mathfrak{A}^{**}$ are dominated by a component. Consequently, z is dominated by the supremum of all components, which in turn can be realized as the sum of all different components. Also, since minimal central projections are connected, this sum dominates each of those. ■

Corollary 7.1.11 *$\mathfrak{A}$ is connected if and only if $\mathfrak{A}$ only has one component.*

Proof: We have seen that $\mathfrak{A}$ is connected if and only $1 \in \mathfrak{A}^{**}$ is. If $\mathfrak{A}$ has only one component c, then, since $c \geq z$ and c is closed, $c = 1$ is connected. ■

Note that although we can have connected projections which do not meet z, e.g. the cover of a type II factor representation[1] we now know that all components must meet z, cf. Proposition 2.2.14. It is this fact that gives us hope to associate the component structure of $\mathfrak{A}$ to the ideal structure of $\mathfrak{A}$, or to the component structure of $P(\mathfrak{A})$. The slightly unpleasant situation where the components do not add up to the unit of $\mathfrak{A}^{**}$ can not be avoided, however:

Example 7.1.12 *Let X be the Cantor "middle fourths" set[2]. Then $\mathfrak{A} = C(X)$ is an abelian, unital C^*-algebra $\mathfrak{A}$ such that the sum of all components is a proper projection in $\mathfrak{A}^{**}$.*

We postpone the proof of this to Section 7.3, and end instead this section by an example regarding C^*-algebras that are "almost" commutative. Let $\mathfrak{A} = C_0(X) \otimes \mathbb{K}(\mathfrak{H}) = \mathfrak{C}_\circ(\mathfrak{X}, \mathbb{K}(\mathfrak{H}))$, where $\dim \mathfrak{H} \leq \aleph_\circ$. It is well-known[3] that every irreducible representation of $\mathbb{K}(\mathfrak{H})$ is unitarily equivalent to the identity representation. Applying Proposition 3.1.5(ii) and regarding an element of $\mathfrak{A}$ as a $\mathbb{K}(\mathfrak{H})$-valued function, every irreducible representation of $\mathfrak{A}$ is unitarily equivalent to one of the form $\pi_x : \mathfrak{A} \to \mathbb{B}(\mathfrak{H})$,

$$\pi_x(f) = f(x).$$

Now π_a is unitarily equivalent to $\bigoplus_X \pi_x$, and

$$\mathfrak{A}^{**} z \simeq \pi_a(\mathfrak{A})'' \simeq \bigoplus_{x \in X} \mathbb{B}(\mathfrak{H}) \simeq \mathcal{B}(X, \mathbb{B}(\mathfrak{H})),$$

the set of bounded $\mathbb{B}(\mathfrak{H})$-valued functions on X. We will apply this identification tacitly in the following.

Lemma 7.1.13 *When $\dim \mathfrak{H} \leq \aleph_\circ$, $G \subseteq X$ is open and $p \in \mathbb{B}(\mathfrak{H})$ is a projection, there is a unique open projection $q \in \mathfrak{A}^{**}$ such that*

$$qz = p1_G.$$

If p is central, then so is q.

[1] And such projections exist whenever $\mathfrak{A}$ is separable and not type I, cf. [22, 6.8.7]

[2] Sic! The point is that this set has positive Lebesgue measure.

[3] Or it follows from [22, 6.8.7](vi) combined with [22, 3.5.4] and simplicity of $\mathbb{K}(\mathfrak{H})$

Proof: For existence, choose a net $f_\lambda \in C_{00}(X)$ such that $f_\lambda \nearrow 1_G$ and $a_n \in \mathbb{K}(\mathfrak{H})_+$ such that $a_n \nearrow p$ (the a_n may be chosen as finite rank projections). As $\|f_\lambda\|$ and $\|a_n\|$ are dominated by 1, so is $\|f_\lambda \otimes a_n\|$, and the doubly indexed net increases to $q \in \mathfrak{A}^{**}$. As

$$\pi_x(f_\lambda \otimes a_n) = a_n f_\lambda(x) \nearrow p 1_G(x),$$

we must have $qz = p1_G$. Clearly $q^2 - q \in U(\mathfrak{A})$ by Proposition 2.2.11, and as $(q^2 - q)z = 0$, we get from Theorem 2.2.12, that q is a projection; by Theorem 2.2.2, it is open. Uniqueness also follows by Theorem 2.2.2, for if q, r are open with $qz = p1_G = rz$, then $q - r \in U(\mathfrak{A})$ with $(q - r)z = 0$. If p is central, $c(q)z = p1_G$, and as $c(q)$ is open by Lemma 2.4.4, we get that q is central by $qz = c(q)z$. ■

Note that by parallel reasoning, or passing to complements, we may replace "open" by "closed" in the above lemma. We define, for $a \in \mathfrak{A}^{**}$,

$$\mathrm{s}(a) = \{x \in X | \pi_x^{**}(a) \neq 0\};$$

"s" is for "support", disregarding that usually a support is a closed set. Indeed, note that when p is open, since we can choose $a_\lambda \in \mathfrak{A}_+$ with $a_\lambda \nearrow p$, so is $\mathrm{s}(p) = \bigcup_\lambda \mathrm{s}(a_\lambda)$. Again, the dual version holds. We are now ready to prove:

Proposition 7.1.14 *Let $\mathfrak{A} = C_0(X) \otimes \mathbb{K}(\mathfrak{H})$, $\dim \mathfrak{H} \leq \aleph_0$ and let $p \in \mathfrak{A}^{**}$ be an open or closed projection. Then p is connected if and only if $\mathrm{s}(p)$ is.*

Proof: First assume that p is open. When $\mathrm{s}(p)$ is connected, suppose that $p = pq_0 + pq_1$ for central open projections q_0, q_1. For all $x \in X$ we have

$$\pi_x^{**}(p) = \pi_x^{**}(p)\pi_x^{**}(q_0) + \pi_x^{**}(p)\pi_x^{**}(q_1),$$

whence, by irreducibility, $\{\pi_x^{**}(q_0), \pi_x^{**}(q_1)\} = \{0, 1\}$, and $\{\mathrm{s}(q_0), \mathrm{s}(q_1)\}$ forms a separation of $\mathrm{s}(p)$. As these sets are open, we may assume $\mathrm{s}(q_0) \cap \mathrm{s}(p) = \emptyset$, $\mathrm{s}(p) \subseteq \mathrm{s}(q_1)$. We get that $pq_0 z = 0$, and since by Proposition 2.2.9, pq_0 is open, $pq_0 = 0$ as desired. Suppose now that p is connected and that $\mathrm{s}(p)$ is separated by the open sets U_0, U_1. We can define open central projections q_0, q_1 by $q_i z = 1_{U_i}$, $i \in \{0, 1\}$ according to the above lemma. By Proposition 2.2.9, $pq_0 \wedge pq_1 = pq_0 q_1$ is open, and as $pq_0 q_1 z = 0$ by definition, pq_0 and pq_1 are orthogonal. Obviously, then

$$(pq_0 \vee pq_1)z = (pq_0 + pq_1)z = 1_{U_0} + 1_{U_1} = pz,$$

and as $pq_0 \vee pq_1 = pq_0 + pq_1$ is open by Propositions 2.2.8 and 2.2.9, p is in fact separated by $\{q_0, q_1\}$. We may thus assume that $pq_0 = 0$, whence $\mathrm{s}(p) \cap U_0 = \emptyset$ as desired.

The case where p is closed is dealt with similarly, using that when $\{q_0, q_1\}$ is an open separation of p, $\{1 - q_0, 1 - q_1\}$ is a closed separation of p. ■

7.2 Components of the set of pure states

Next, we try to link the component structure of $\mathfrak{A}$ to that of the set of pure states $P(\mathfrak{A})$. We will proceed as in the proof of Theorem 6.2.2, using the results and the terminology from Section 2.4.

Note that when $p \in \mathfrak{A}^{**}$ is central and $\varphi \in P(\mathfrak{A})$, then $\varphi(p) \in \{0,1\}$. For the cover p_φ of π_φ is a minimal central projection, whence $pp_\varphi \in \{0, p_\varphi\}$ and $\varphi(p) = \varphi(pp_\varphi) \in \{0,1\}$. We conclude that when p is central,

$$P(p) \cup P(1-p) = P(\mathfrak{A}), \qquad P(p) \cap P(1-p) = \emptyset. \tag{3}$$

Proposition 7.2.1 *Let C be a closed saturated subset of $P(\mathfrak{A})$. Then C is connected if and only if p_C is.*

Proof: First assume that $C \subseteq P$ is connected and that $\{q_0, q_1\}$ separates p_C. As noted above, we have

$$\begin{aligned} V_i &= \{\varphi \in C | \varphi(p_C q_i) = 0\} \\ &= C \cap F(p_C q_i) \\ &= C \cap P(q_i) \\ &= C \backslash P(1-q_i), \qquad i \in \{0,1\} \end{aligned}$$

by centrality, and we clearly get a disjoint partition $\{V_0, V_1\}$ of C this way. Furthermore, $\{V_0, V_1\}$ are relatively open sets since the $P(1-q_i)$ are closed, so the sets actually separate C and we may assume that $V_0 = \emptyset, V_1 = C$. We learn that $P(1-q_0) \supseteq C = P(p_C)$ whence, since both projections are closed and central, we may infer that $p_C \leq 1 - q_0$ by Proposition 2.4.2. Consequently, $p_C q_0 = 0$.

Conversely, if $q = p_C$ is connected and C is separated by $\{U_0, U_1\}$ we may argue as in the proof of (ii)$\Longrightarrow$(iii) in Theorem 6.2.2 to conclude from Lemma 6.2.1 that U_0 and U_1 are both saturated. By Proposition 2.4.2 and $P(p_{U_0} \wedge p_{U_1}) = U_0 \cap U_1 = \emptyset$, the closed central projections p_{U_0} and p_{U_1} are orthogonal, so $p_{U_0} + p_{U_1}$ is a projection. We have

$$P(p_{U_0} + p_{U_1}) \supseteq P(p_{U_0}) \cup P(p_{U_1}) = U_0 \cup U_1 = P(q),$$

and since the projections on both sides are closed and central, $p_{U_0} + p_{U_1} \geq q$ as above. Now

$$q = 2q - q = 2q - (p_{U_0} + p_{U_1})q = (1 - p_{U_0})q + (1 - p_{U_1})q$$

and we may assume that $(1 - p_{U_0})q = 0, (1 - p_{U_1})q = q$. When $p_{U_1} q = 0$ we must have

$$\emptyset = P(p_{U_1} q) = P(p_{U_1}) \cap P(q) = U_1.$$

■

Corollary 7.2.2 *The mappings $p \mapsto P(p)$, $C \mapsto p_C$ are corresponding bijections between the set of components of $\mathfrak{A}$ and the set of components of $P(\mathfrak{A})$. They take the set of open components of $\mathfrak{A}$ to the set of open components of $P(\mathfrak{A})$.*

Proof: If C is a component of P it is saturated by Lemma 6.2.1, and by the proposition above, p_C is connected. Let q be the component dominating p_C, cf. Proposition 7.1.9. Since q is connected, so is $P(q)$, and since the mapping $p \mapsto P(p)$ is order-preserving, $C \subseteq P(q)$, whence $P(p_C) = C = P(q)$ and $p_C = q$. On the other hand, if q is a component of $\mathfrak{A}^{**}$, it is central and closed by Proposition 7.1.10, and by Proposition 2.4.2, we can write $q = p_C$ for some $C \subseteq P$ which is connected by Proposition 7.2.1. With D the component containing C, $p_C \leq p_D$ and since p_D is connected, $q = p_C = p_D$.

Let C be an open component of P. Since $P \backslash C$ is closed and saturated, we get from Proposition 2.4.2 a closed and central projection $p_{P\backslash C}$ such that $P(p_{P\backslash C}) = P\backslash C$. Since p_C is also closed, we conclude that $p_C + p_{P\backslash C} = 1$, and p_C is open. When c is an open component of $\mathfrak{A}$, then from (3), $P(c) = P\backslash F(1-c)$, showing that $P(c)$ is open. ■

7.3 Totally disconnected C^*-algebras

In the commutative case, connectivity relates to dimension of a topological space X in a natural way – X has low dimension if its components are small in an appropriate sense. The purpose of this section is to discuss how our notion of non-commutative connectivity and components relates to the non-commutative dimension theory to be found, for instance, in [11]. Unfortunately, we will not have much nice to say, and the subject will be abandoned at the end of this section without us even reaching an adequate definition. The section is included to demonstrate how the fact that our notion of connectivity is essentially central limits its usefulness.

We start out with a closer look at the commutative case, but first a number of definitions: We define the *covering dimension* of a topological space X as follows: a finite cover $(G_1, \ldots, G_m)$, not all empty, is said to have *redundancy of order* n, if n is the largest integer such that $n+1$ of the G_i's have non-empty intersection. A cover of $\emptyset$ is said to have redundancy of order -1. The covering dimension $\dim X$ is the least integer n such that every open cover of X has an open refinement of order not exceeding n. Finally, a C^*-algebra $\mathfrak{A}$, is called *approximately finite-dimensional*, or *AF*, when it can be written in the form

$$\mathfrak{A} = \overline{\overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}},$$

where all the subalgebras $\mathfrak{A}_n$ are finite-dimensional. The reader can find more on covering dimension in [21, 3], on AF algebras in [20, 6.1].

Proposition 7.3.1 *Let X be a non-empty, second countable, compact Hausdorff space. The following conditions are equivalent:*

(i) X is totally disconnected.

(ii) There is a base for the topology on X consisting of clopen sets.

(iii) $\dim X = 0$.

(iv) $C(X)$ is an AF algebra.

Proof: That (i)$\Longrightarrow$(ii) follows from Lemma B.1.10. For (ii)$\Longrightarrow$(iii), let $(G_1, \ldots, G_n)$ be an open covering of X. For every $x \in G_1$, take a clopen set H_x with $x \in H_x \subseteq G_1$. As

$$X \backslash (\bigcup_{i=2}^{n} G_i) \subseteq G_1,$$

and the left hand side is closed, hence compact, it can be covered by $H_{x_1}, \ldots, H_{x_m}$. Letting

$$H_1 = \bigcup_{j=1}^{m} H_{x_j},$$

we achieve a clopen set contained in G_1 such that $(H_1, G_2, \ldots, G_n)$ is a cover of X. Proceeding by induction, we find a refinement $(H_1, \ldots, H_n)$ consisting of clopen sets, and with $K_i = H_i \backslash (H_1 \cup \cdots \cup H_{i-1})$ we get an open cover without redundancy, i.e. of order 0.

By [23, 1.7.9], X is metrizable, say with d. Assuming (iii), we can cover X by a finite number of disjoint, clopen sets of diameter less than ε for every $\varepsilon > 0$. Let $(G_1^n, \ldots, G_{m_n}^n)$ be such a covering of diameter less that n^{-1}, and arrange the clopen sets as

$$(H_1, \ldots) = (G_1^1, \ldots, G_{m_1}^1, \ldots, G_1^i, \ldots, G_{m_i}^i, \ldots).$$

We observe that $1_{H_i} \in C(X)$, and with $\mathfrak{A}_n = \mathrm{span}(1_{H_1}, \ldots, 1_{H_n})$ we get from the Stone-Weierstrass theorem that $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ is dense, because it separates points of X. Indeed, when $x \neq y$, we have $d(x, y) \geq n^{-1}$ for some n, and for some $1 \leq i \leq m_n$, $1_{G_i^n}(x) = 0 \neq 1 = 1_{G_i^n}(y)$. We get that $C(X)$ is AF. For (iv)$\Longrightarrow$(i), assume that $C(X) = (\bigcup_{n=1}^{\infty} \mathfrak{A}_n)^{=}$ where the $\mathfrak{A}_n$ are finite-dimensional. Take different $x, y \in X$. The corresponding characters $\hat{x}, \hat{y}$ are continuous and must differ on every dense subset of $C(X)$, hence on some $\mathfrak{A}_n$. As a finite C^*-algebra is spanned by its projections, there exists a projection $p \in \mathfrak{A}_n \subset C(X)$ such that

$$\hat{x}(p) \neq \hat{y}(p).$$

We conclude that for some clopen set G, $1_G(x) \neq 1_G(y)$, and no set containing x and y can be connected. As x, y were arbitrary, X is totally disconnected. ■

Proposition 7.3.2 *When X is a compact Hausdorff space, X is totally disconnected if and only if all components of $\mathfrak{A} = C(X)$ are minimal projections.*

Proof: When $X = P(\mathfrak{A})$ is totally disconnected, all components have the form $p_{\{\psi\}}$ according to Corollary 7.2.2, and these are minimal projections by [22, 3.13.3]. When $P(\mathfrak{A})$ is not totally disconnected, some component C here has two elements φ and ψ, and the supports p, q of these states are different and lie under p_C. ■

Proof of Example 7.1.12: As X is a Cantor set, X is totally disconnected, and by the above, all components are minimal and hence dominated by z. We get that z is the sum of all components in $\mathfrak{A}^{**}$. As X has positive Lebesgue measure, there is a non-atomic probability measure on X, and $z \neq 1$. ■

Remark 7.3.3 When $\mathfrak{A}$ is abelian, all minimal projections are minimal closed projections and vice versa, cf. Proposition 2.2.7, and inspired by the proposition above, we might try to define that $\mathfrak{A}$ is totally disconnected if

(i) All components are minimal projections.

(ii) All components are minimal central projections.

(iii) All components are minimal closed central projections.

and try to relate this to some class of low dimensional C^*-algebras, e.g. the AF algebras or those of real rank zero.

The first condition seems to be far too strong, excluding for instance $\mathbb{K}(\mathfrak{H})$. Actually, as (i) also implies that every minimal projection is central (because every minimal projection is a component), this condition implies that every irreducible representation is central and $\mathfrak{A}$ is abelian. On the other hand, condition (iii) is too weak to give any results pertaining to the dimension of $\mathfrak{A}$. Indeed, according to (iii), every simple C^*-algebra would be totally disconnected. The second condition seems better, but here many AF algebras would turn out not to be totally disconnected. The class would include $\mathbb{K}(\mathfrak{H})$, but already $\mathbb{K}(\mathfrak{H})^\sim$ would be excluded, as the component 1 dominates the cover of $\mathbb{K}(\mathfrak{H})$. In the other direction, all separable C^*-algebras enjoying property (ii) must be of type I, as when p is a cover of a type II factor representation, p can not be closed, and $p < c$, where c is the component dominating p.

Bratteli's discovery [9] that every separable abelian C^*-algebra appears as the center of an AF algebra gives a further indication of why we seem to have no luck relating the notions in question. Indeed, $\mathfrak{A}$ can be AF with a center $\mathcal{Z}(\mathfrak{A}) = C_0(X)$ with X of arbitrarily large dimension.

7.4 Counting components

Let X be a compact Hausdorff space and consider for a while how the component structure of X is contained in $C(X)$. Since components are, in general, not open, there might not be elements in $C(X)$ corresponding to a given component in any reasonable sense. We can, however, count the number of components as the maximal number of non-zero projections adding up to 1, i.e. the maximum of

$$\{n \in \mathbb{N} | \sum_{i=1}^{n} f_i = 1, \forall i, j : f_i f_j = \delta_{ij} f_j, f_i \in C(X)\backslash\{0\}\}, \tag{4}$$

at least when we content ourselves with only one notion of infinity. Here δ_{ij} denotes Kronecker delta. In fact, as we can see by looking at the Cantor set, this method does not allow us to determine the cardinality of the set of components, even if we were to allow infinite sums in (4). Indeed, a claim to fame for this set is that it has transfinitely many components (points), but since it is second countable, such a sum could only have countably many non-zero summands.

The discussion above outlines some of the problems we must expect when we try to pass from notions of components involving point set topology to notions involving continuous functions. To make life simpler for ourselves, we will focus on the case with only one notion of infinity in most of this thesis. Extending the terminology from Appendix B, we denote by $\mathcal{C}(\mathfrak{A})$ the set of components of a C^*-algebra $\mathfrak{A}$. We write $\#_c\mathfrak{A}$ to denote the number of components, ranging in $\mathbb{N} \cup \{\infty\}$, of any $\mathfrak{A}$, and $\mathfrak{c}_\mathfrak{c}\mathfrak{A}$ for the cardinal of $\mathcal{C}(\mathfrak{A})$.

Lemma 7.4.1 $\mathfrak{c}_\mathfrak{c}\mathfrak{A} = \mathfrak{c}_\mathfrak{c}(\mathfrak{P}(\mathfrak{A})) \geq \mathfrak{c}_\mathfrak{c}(\mathrm{Prim}(\mathfrak{A}))$.

Proof: The first equality follows from Corollary 7.2.2. And since the mapping $\varphi \mapsto \ker \pi_\varphi$ is continuous and onto according to [22, 4.3.3], the inequality follows from Lemma B.1.4. ■

Proposition 7.4.2 *The numbers*

(i) $\#_c\mathfrak{A}$

(ii) $\#_c P(\mathfrak{A})$

(iii) $\#_c \operatorname{Prim}(\mathfrak{A})$

(iv) $\sup\{n \in \mathbb{N} | \sum_{i=1}^n f_i = 1, \forall i,j : f_i f_j = \delta_{ij} f_j, f_i \in \mathcal{Z}(M(\mathfrak{A}))\backslash\{0\}\}$

are all the same. If $\mathfrak{A}$ is σ-unital, they agree with

(v) $\sup\{\#_c \hat{a}(P(\mathfrak{A})) | a \in \mathfrak{A}\}$.

Proof: From Lemma 7.4.1 we already know that $\#_c\mathfrak{A} = \#_c P \geq \#_c \operatorname{Prim}(\mathfrak{A})$. Assume that $\#_c\mathfrak{A} = n < \infty$ and let c_i be an enumeration of the components. The sum $\sum_{i=1}^n c_i$ is 1 since it is closed and dominates z, and since $c_j = 1 - \sum_{i\neq j} c_i$, all components are open. By [22, 3.12.9], all c_i lie in $\mathcal{Z}(M(\mathfrak{A}))$ and the value m_1 of the supremum in (iv) is at least n. By the Dauns-Hofmann theorem [22, 4.4.8], $\mathcal{Z}(M(\mathfrak{A})) \simeq C^b(\operatorname{Prim}(\mathfrak{A}))$, and we conclude

$$n \leq m_1 \leq \#_c \operatorname{Prim}(\mathfrak{A}) \leq \#_c\mathfrak{A} = n.$$

Since $\hat{a} : P \to \mathbb{C}$ is continuous, the value m_2 of the supremum in (v) is dominated by n, cf. Lemma B.1.4, and if we have a strictly positive element c we achieve this maximum at

$$a = \sum_{j=1}^n e^{\frac{i\pi j}{n}} c c_j$$

where $a \in \mathfrak{A}$ since $c_j \in M(\mathfrak{A})$. For by centrality, when $\varphi \in P$ it is 1 on exactly one component c_{j_0} and 0 on the rest, and since all c_j are closed, $P(c_j) \neq \emptyset$. Therefore, $\hat{a}(P)$ meets each of the n components of

$$\{te^{\frac{i\pi j}{n}} | j \in \{1,\ldots,n\}, t \in (0, \|c\|]\}.$$

When $\#_c\mathfrak{A} = \#_c P = \infty$, we can apply an inductive argument to find that 1 can be written as a sum of n orthogonal, non-zero, central clopen projection for any $n \in \mathbb{N}$. For since $\mathfrak{A}$ is not connected we can write $1 = p_1 + p_2$ where p_1, p_2 can not be connected because if they were, they would be the components of $\mathfrak{A}$, contradicting the hypothesis. By arguing as above, we get that $n \leq m_1 \leq \#_c \operatorname{Prim}(\mathfrak{A})$ and $n \leq m_2$, showing that each of the remaining numbers are at least n for every $n \in \mathbb{N}$. ■

Corollary 7.4.3 *When $\mathfrak{A}$ is unital, $\#_c A = \#_c(\mathcal{Z}(\mathfrak{A}))$.*

Proof: Apply (iv) above. ■

Remark 7.4.4 Every topological space X gives rise to a lattice, namely the set of projections in $C^b(X)$. We denote this by $\mathrm{Lat}(C^b(X))$ and note that if X is connected, $\mathrm{Lat}(C^b(X)) = \{0, 1\}$. Letting $\Psi : P(\mathfrak{A}) \to \mathrm{Prim}(\mathfrak{A})$ be the mapping mentioned in the proof of Lemma 7.4.1, we can prove that the transpose of Ψ induces an order isomorphism between $\mathrm{Lat}(C^b(P(\mathfrak{A})))$ and $\mathrm{Lat}(C^b(\mathrm{Prim}(\mathfrak{A})))$, hence preserves supremum and infimum. When

$$\mathrm{Lat}(C^b(X)) \simeq \mathrm{Lat}(C^b(Y)) \tag{5}$$

in the sense of order isomorphisms, $\#_c X = \#_c Y$, and knowing (5) appears to be much stronger than knowing $\mathfrak{c}_\mathfrak{c}\mathfrak{X} = \mathfrak{c}_\mathfrak{c}\mathfrak{Y}$. For instance, with X the Cantor set and Y some discrete set of cardinality $\aleph$, $\mathrm{Lat}(C^b(Y))$ has minimal non-zero elements, $\mathrm{Lat}(C^b(X))$ does not, so (5) does not hold in this case, although $\mathfrak{c}_\mathfrak{c}\mathfrak{X} = \mathfrak{c}_\mathfrak{c}\mathfrak{Y}$. However, the author knows of no theorem allowing us to conclude from (5) that $\mathfrak{c}_\mathfrak{c}\mathfrak{X} = \mathfrak{c}_\mathfrak{c}\mathfrak{Y}$, and we do not know whether $\mathfrak{c}_\mathfrak{c}(\mathrm{Prim}(\mathfrak{A})) = \mathfrak{c}_\mathfrak{c}(\mathfrak{P}(\mathfrak{A}))$ or not.

8 Components of constructions with C^*-algebras

This section contains results relating the number of components of a C^*-algebra constructed from other C^*-algebras to the number of components of these "building blocks". As described in section 7.4, we have two different kinds of results; sometimes we are able to identify the number of components up to cardinality of sets, at other times we must content ourselves with only one notion of infinity.

8.1 Sums and limits

When $(\mathfrak{A}_i)_I$ is a family of C^*-algebras, we denote by $\mathfrak{A} = \sum_I \mathfrak{A}_i$ the set of all tuples $(a_i)_I$ vanishing at infinity in the sense that for every $\varepsilon > 0$, the set

$$\{i \in I | \|a_i\| \geq \varepsilon\}$$

is finite. Endowing $\mathfrak{A}$ with maximum-norm, it is easy to see that it is a C^*-algebra in its own right.

When $(X_i)_I$ is a family of topological spaces, the *topological direct sum* $X = \bigcup_I X_i$ is a disjoint union of the X_i's with $G \subseteq X$ open if and only if all $G \cap X_i$ are open. The following result unfortunately requires a rather long proof.

Lemma 8.1.1 *The topological direct sum $\bigcup_I P(\mathfrak{A}_i)$ is homeomorphic to $P(\sum_I \mathfrak{A}_i)$.*

Proof: We first note that when $\mathfrak{B} = \mathfrak{B}_0 \oplus \mathfrak{B}_1$, and $\varphi \in \mathfrak{B}_0^*$ is given, we may extend φ to $\tilde{\varphi} \in \mathfrak{B}^*$ by

$$\tilde{\varphi}(b_0, b_1) = \varphi(b_0).$$

On the other hand, when $\psi \in \mathfrak{B}^*$, $\psi_{\mathfrak{B}_0} \in \mathfrak{B}_0^*$ is given by

$$\psi_{\mathfrak{B}_0}(b_0) = \psi(b_0, 0).$$

We have that $\tilde{\varphi} \in P(\mathfrak{B})$ when $\varphi \in P(\mathfrak{B}_0)$, for when $\tilde{\varphi} = \lambda\psi + (1-\lambda)\chi$ on $\mathfrak{B}$, $\lambda \in (0,1)$, then as φ is extreme in $Q(\mathfrak{B}_0)$, $\varphi = \psi_{\mathfrak{B}_0} = \chi_{\mathfrak{B}_0}$. Since $\psi_{\mathfrak{B}_1} = \chi_{\mathfrak{B}_1} = 0$ by positivity, $\tilde{\varphi} = \psi = \chi$.

Now let $\mathfrak{A} = \sum_I \mathfrak{A}_i$ and define $\Psi : \bigcup_I P(\mathfrak{A}_i) \to P(\mathfrak{A})$ by $\Psi(\varphi) = \tilde{\varphi}$. Ψ is clearly 1–1, and if $\varphi \in P(\mathfrak{A})$ is given and we write $\mathfrak{A} = \mathfrak{B}_0 \oplus \mathfrak{B}_1$, then we can write $\varphi_{\mathfrak{B}_0} = \lambda_0\psi_0, \varphi_{\mathfrak{B}_1} = \lambda_1\psi_1$ for $\psi_i \in P(\mathfrak{B}_i)$ and $\lambda_i \in [0,1]$ according to [22, 4.1.6]. By linearity

$$\varphi = (\varphi_{\mathfrak{B}_0})^{\sim} + (\varphi_{\mathfrak{B}_1})^{\sim} = \lambda_0\tilde{\psi}_0 + \lambda_1\tilde{\psi}_1$$

and by evaluating on an approximate unit, we get that $\lambda_0 + \lambda_1 = 1$, wherefrom $\lambda_0 \in \{0,1\}$ since φ is extreme. We conclude that every $\varphi \in P(\mathfrak{A})$ lives on one of the ideals $\mathfrak{B}_0$ and $\mathfrak{B}_1$. When I is finite, surjectivity of Ψ follows from this fact by induction. When I is infinite, we may take $a = (a_i)_I \in \mathfrak{A}_+$ with $\varphi(a) \geq 1$ and let $I_0 = \{i \in I | \|a_i\| \geq 1/2\}$. Putting $\mathfrak{B}_0 = \bigoplus_{I_0} \mathfrak{A}_i$, $\mathfrak{B}_1 = \sum_{I \setminus I_0} \mathfrak{A}_i$, φ can not vanish on $\mathfrak{B}_0$, as then $\varphi(a) = \varphi(a') \leq \|a'\| \leq 1/2$, where

$$a'_i = \begin{cases} 0, & i \in I_0 \\ a_i, & i \in I \setminus I_0 \end{cases},$$

and we have reduced to the finite case.

For continuity of Ψ, note that if $\varphi_\lambda \to \varphi$ *weak** in $P(\mathfrak{A}_i)$, then obviously for every $(a_i)_I \in \mathfrak{A}$ we have

$$\Psi(\varphi_\lambda)((a_i)) = \varphi_\lambda(a_{i_0}) \to \varphi(a_{i_0}) = \Psi(\varphi)((a_i)).$$

On the other hand, if $\psi_\lambda \to \psi$ in $P(\mathfrak{A})$, we know that $\psi_{\mathfrak{A}_{i_0}} \neq 0$ for some $i_0 \in I$. Take $a \in (\mathfrak{A}_{i_0} \oplus \{0\})_{sa}$ with $\psi(a) > 0$ and assume, as we may, that $\varphi_\lambda(a) > 0$ for every λ. From the above, $\Psi^{-1}(\psi_\lambda) \in P(\mathfrak{A}_{i_0})$, and obviously $\Psi^{-1}(\varphi_\lambda) \to \Psi^{-1}(\psi)$. ■

Proposition 8.1.2 *Let $\mathfrak{A}_i$, $i \in I$, be C^*-algebras. We have*

$$\mathfrak{c}_\mathfrak{c}(\sum_\mathfrak{I} \mathfrak{A}_\mathfrak{i}) = \sum_\mathfrak{I} \mathfrak{c}_\mathfrak{c}\mathfrak{A}_\mathfrak{i}.$$

Proof: From the above, $P(\sum_I \mathfrak{A}_i) \simeq \bigcup_I P(\mathfrak{A}_i)$. As every $P(\mathfrak{A}_i)$ sits in $\bigcup_I P(\mathfrak{A}_i)$ as a clopen set, each component of $\bigcup_I P(\mathfrak{A}_i)$ must be contained in one of these; furthermore, $X \subseteq P(\mathfrak{A}_i)$ is connected in $\bigcup_I P(\mathfrak{A}_i)$ if and only if it is connected in $P(\mathfrak{A}_i)$. We have established a 1–1 correspondence between $\mathcal{C}(\bigcup_I P(\mathfrak{A}_i))$ and $\bigcup_I \mathcal{C}(P(\mathfrak{A}_i))$, from which the claim follows. ■

Proposition 8.1.3 *Let $\mathfrak{A}$ be a unital C^*-algebra such that*

$$\mathfrak{A} = \overline{\overline{\bigcup_{\lambda \in \Lambda} \mathfrak{A}_\lambda}},$$

where $(\mathfrak{A}_\lambda)_{\lambda \in \Lambda}$ is an upward directed set of subalgebras. Then

$$\#_c \mathfrak{A} \leq \liminf_{\lambda \in \Lambda} \#_c \mathfrak{A}_\lambda.$$

Proof: Assume that

$$1 = \sum_{i=1}^{n} p_i, \tag{6}$$

where all p_i are non-zero central projections of $\mathfrak{A}$. For fixed i, we take a net $(a_\lambda)_\Lambda$ converging to p_i, $a_\lambda \in \mathfrak{A}_\lambda$. Since Re $a_\lambda \to$ Re $p_i = p_i$, we may assume that every a_λ is self-adjoint, and since $a_\lambda^2 \to p_i^2 = p_i$, there exists λ_i such that when $\lambda \geq \lambda_i$,

$$\left\| a_\lambda^2 - a_\lambda \right\| \leq \left\| a_\lambda^2 - p_i \right\| + \| p_i - a_\lambda \| < \frac{1}{4}.$$

By spectral theory, we conclude that $\mathrm{sp}(a_\lambda) \subseteq (-1/4, 1/2) \cup (1/2, 5/4)$ for $\lambda \geq \lambda_i$, and we can take a projection $q_\lambda = 1_{[1/2,5/4]}(a_\lambda)$ in $\mathfrak{A}_\lambda$ such that $\|a_\lambda - q_\lambda\| < 1/2$, and, consequently, $\|p_i - q_\lambda\| < 1$. By [20, Lemma 6.2.1], there exist $u_\lambda \in \mathcal{U}(\mathfrak{A})$ such that $q_\lambda = u_\lambda^* p_i u_\lambda$, and since p_i is central, $p_i = q_\lambda$. Letting λ_0 be a common successor of $\lambda_1, \ldots, \lambda_n$, we thus have that $p_i \in \mathfrak{A}_\lambda$ for every $i \in \{1, \ldots, n\}$ and $\lambda \geq \lambda_0$. As (6) holds true in $\mathfrak{A}_\lambda$ for all such λ, we get from Proposition 7.4.2(iv) that

$$n \leq \#_c \mathfrak{A}_\lambda, \qquad \lambda \geq \lambda_0,$$

whence $n \leq \liminf_\Lambda \#_c \mathfrak{A}_\lambda$. Applying the proposition again, the claim follows. ■

Remark 8.1.4 Equality does not hold in the above result. Indeed, when $\mathfrak{H}$ is a separable Hilbert space, $\mathbb{K}(\mathfrak{H})^\sim = \mathbb{K}(\mathfrak{H}) + \mathbb{C}\mathbf{1} \subseteq \mathbb{B}(\mathfrak{H})$ is connected by Proposition 6.3.1. By embedding $M_n(\mathbb{C})$ via a fixed orthogonal basis for $\mathfrak{H}$, we have

$$\overline{\overline{\bigcup_{i=1}^{\infty} (M_n(\mathbb{C}) + \mathbb{C}1)}} = \mathbb{K}(\mathfrak{H}) + \mathbb{C}\mathbf{1},$$

and $\#_c(M_n(\mathbb{C}) + \mathbb{C}1) = 2$ for each $n \in \mathbb{N}$.

Also, from the commutative case, we see that the statement does not hold true with cardinals. Indeed, when X is the totally disconnected Cantor “middle thirds” set, we get from Proposition 7.3.1 that $\mathfrak{A} = C(X)$ is AF, so that $\mathfrak{A} = (\bigcup_{n=1}^{\infty} \mathfrak{A}_n)^=$ with $\mathfrak{A}_n$ finite-dimensional and $\liminf_n \mathfrak{c}_c \mathfrak{A}_n \leq \aleph_0$. On the other hand, as noted in Section 7.4, $\mathfrak{c}_c \mathfrak{A} = \mathfrak{c}_c \mathfrak{X} = \aleph$.

Corollary 8.1.5 *Let $\mathfrak{A}$ be a unital C^*-algebra such that*

$$\mathfrak{A} = \overline{\overline{\bigcup_{\lambda\in\Lambda} \mathfrak{A}_\lambda}},$$

where $(\mathfrak{A}_\lambda)_{\lambda\in\Lambda}$ is an upward directed set of connected subalgebras. Then $\mathfrak{A}$ is connected.

The assumption that $\mathfrak{A}$ has a unit is essential for the proof of the proposition above, and we do not have a more general result involving non-unital limits of C^*-algebras. Under the extra assumption that the subalgebras are hereditary, an analogue of Corollary 8.1.5 is easy to come by:

Proposition 8.1.6 *Let $\mathfrak{A}$ be a C^*-algebra such that*

$$\mathfrak{A} = \overline{\overline{\bigcup_{\lambda\in\Lambda} \mathfrak{A}_\lambda}},$$

where $(\mathfrak{A}_\lambda)_{\lambda\in\Lambda}$ is an upward directed set of connected hereditary subalgebras. Then $\mathfrak{A}$ is connected.

Proof: From Corollary 2.2.4, we can find open projections $p_\lambda \in \mathfrak{A}^{**}$ such that $\mathfrak{A}_\lambda = H(p_\lambda)$. When $\lambda \leq \mu$, $\mathfrak{A}_\lambda \subseteq \mathfrak{A}_\mu$, and by Theorem 2.2.2, $p_\lambda \in H(p_\lambda)^{-} \subseteq H(p_\mu)^{-} = p_\mu \mathfrak{A}^{**} p_\mu$ so that $p_\lambda \leq p_\mu$.

Suppose that $\bigvee_\Lambda p_\lambda \neq 1$ and take $\xi \in \mathfrak{H}_\mathfrak{u}$ orthogonal to all p_λ. We get that $a_\lambda \xi = p_\lambda a_\lambda p_\lambda \xi = 0$ for all $a_\lambda \in \mathfrak{A}_\lambda$, hence $a\xi = 0$ for all $a \in \mathfrak{A}$, contradicting the fact that π_u is non-degenerate by definition. Letting q be some minimal central projection of $\mathfrak{A}^{**}$ we may thus assume that $p_\lambda q \neq 0$ for all $\lambda \geq \lambda_0$, and since every p_λ is connected by Proposition 7.1.3, we get from Proposition 7.1.6 that $1 = \bigvee_{\lambda\geq\lambda_0} p_\lambda$, and hence $\mathfrak{A}$, is connected. ■

8.2 Tensor products

Lemma 8.2.1 *Let X and Y be locally compact Hausdorff spaces. Then*

$$\mathfrak{c}_\mathfrak{c}(\mathfrak{C}_\mathfrak{o}(\mathfrak{X}) \otimes \mathfrak{C}_\mathfrak{o}(\mathfrak{Y})) = \mathfrak{c}_\mathfrak{c}\mathfrak{C}_\mathfrak{o}(\mathfrak{X})\mathfrak{c}_\mathfrak{c}\mathfrak{C}_\mathfrak{o}(\mathfrak{Y}).$$

Proof: That $C_0(X) \otimes C_0(Y) \simeq C_0(X \times Y)$ follows from Lemma 3.1.5(i) (or can be seen directly). As $\mathfrak{c}_\mathfrak{c}\mathfrak{C}_\mathfrak{o}(\mathfrak{X}) = \mathfrak{c}_\mathfrak{c}\mathfrak{X}$, the claim follows from Lemma B.1.5. ■

Proposition 8.2.2 *Let $\mathfrak{A}_1, \mathfrak{A}_2$ be unital C^*-algebras. For any C^*-norm β on $\mathfrak{A}_1 \odot \mathfrak{A}_2$, we have*

$$\#_c(\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2) = \#_c\mathfrak{A}_1 \#_c\mathfrak{A}_2.$$

Proof: As $\mathcal{Z}(\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2) = \mathcal{Z}(\mathfrak{A}_1) \otimes \mathcal{Z}(\mathfrak{A}_2)$ by Theorem 3.2.4, we get from Corollary 7.4.3 and the lemma above, that

$$\begin{aligned}
\#_c(\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2) &= \#_c\mathcal{Z}(\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2) \\
&= \#_c(\mathcal{Z}(\mathfrak{A}_1) \otimes \mathcal{Z}(\mathfrak{A}_2)) \\
&= \#_c\mathcal{Z}(\mathfrak{A}_1)\#_c\mathcal{Z}(\mathfrak{A}_2) \\
&= \#_c\mathfrak{A}_1 \#_c\mathfrak{A}_2.
\end{aligned}$$

■

Proposition 8.2.3 *When $\mathfrak{A}_1, \mathfrak{A}_2$ are C^*-algebras, we have*

$$\#_c(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2) = \#_c\mathfrak{A}_1 \#_c\mathfrak{A}_2.$$

Proof: Denoting the universal representation of $\mathfrak{A}_i$ by $(\pi_u^i, \mathfrak{H}_u^i)$, $M(\mathfrak{A}_1) \otimes_* M(\mathfrak{A}_2)$ is the operator-norm closure of $M(\mathfrak{A}_1) \odot M(\mathfrak{A}_2)$ in $\mathbb{B}(\mathfrak{H}_u^1 \hat{\otimes} \mathfrak{H}_u^2)$, and as the $\mathfrak{A}_i$ are ideals of $M(\mathfrak{A}_i)$, $(\pi_u^1(\mathfrak{A}_1) \odot \pi_u^2(\mathfrak{A}_2))^= \simeq \mathfrak{A}_1 \otimes_* \mathfrak{A}_2$ is an ideal of $M(\mathfrak{A}_1) \otimes_* M(\mathfrak{A}_2)$. Furthermore, as $\pi_u^i(\mathfrak{A}_i)$ acts non-degenerately on $\mathfrak{H}_i$, $\pi_u^1(\mathfrak{A}_1) \odot \pi_u^2(\mathfrak{A}_2)$ acts non-degenerately on $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$, and $\mathfrak{A}_1 \otimes_* \mathfrak{A}_2$ sits as an essential ideal of $M(\mathfrak{A}_1) \otimes_* M(\mathfrak{A}_2)$. According to [22, 3.12.8], there is an embedding $\iota : M(\mathfrak{A}_1) \otimes_* M(\mathfrak{A}_2) \hookrightarrow M(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)$ leaving $\mathfrak{A}_1 \otimes_* \mathfrak{A}_2$ fixed. We identify $M(\mathfrak{A}_1) \otimes_* M(\mathfrak{A}_2)$ with $\iota(M(\mathfrak{A}_1) \otimes_* M(\mathfrak{A}_2))$. When $n_i \leq \#_c\mathfrak{A}_i$, $n_i \in \mathbb{N}$, $i \in \{1, 2\}$, we can write

$$1 = \sum_{j_1=1}^{n_1} p_{j_i}^1, \qquad 1 = \sum_{j_2=1}^{n_2} p_{j_2}^2,$$

where the $p_{j_i}^i$ are non-zero central projections in $M(\mathfrak{A}_i)$. As every $p_{j_1}^1 \otimes p_{j_2}^2$ commutes with all elements of $\mathfrak{A}_1 \odot \mathfrak{A}_2$, hence with every element of $\mathfrak{A}_1 \otimes_* \mathfrak{A}_2$ and every element of $(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)^{**}$, we get $p_{j_1}^1 \otimes p_{j_2}^2 \in \mathcal{Z}(M(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2))$, and each such element is a non-zero central projection. In $M(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)$ we have

$$\begin{aligned}
1 &= (\sum_{j_1=1}^{n_1} p_{j_1}^1) \otimes (\sum_{j_2=1}^{n_2} p_{j_2}^2) \\
&= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} p_{j_1}^1 \otimes p_{j_2}^2,
\end{aligned}$$

showing that $n_1 n_2 \leq \#_c(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)$ by Proposition 7.4.2. Applying the proposition again, we get that $\#_c\mathfrak{A}_1 \#_c\mathfrak{A}_2 \leq \#_c(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)$.

In the other direction, we get that $\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2)$ has a dense homeomorphic image in $\mathrm{Prim}(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2)$ by Proposition 3.1.4, and by Lemma B.1.6 combined with Proposition 7.4.2, we get from this

$$\begin{aligned}
\#_c(\mathfrak{A}_1 \otimes_* \mathfrak{A}_2) &\leq \#_c(\mathrm{Prim}(\mathfrak{A}_1) \times \mathrm{Prim}(\mathfrak{A}_2)) \\
&= \#_c \mathrm{Prim}(\mathfrak{A}_1) \#_c \mathrm{Prim}(\mathfrak{A}_2) \\
&= \#_c \mathfrak{A}_1 \#_c \mathfrak{A}_2.
\end{aligned}$$

■

Remark 8.2.4 Note that we have *not* established $\#_c(\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2) = \#_c\mathfrak{A}_1\#_c\mathfrak{A}_2$ for non-unital $\mathfrak{A}_i$, unless $\beta = *$. When $\beta \neq *$, $\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2$ has ideals "coming from the completion", and the tensor product of connected C^*-algebras might conceivably be non-connected. We know that even though $\mathfrak{A}_1, \mathfrak{A}_2$ are simple, $\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2$ is only simple when $\beta = *$, but as centers are preserved, connectivity behaves more nicely than simplicity in this respect, at least when the C^*-algebras in question have a unit. In the non-unital case, we need to involve multiplier algebras or other external objects in our discussion, and this complicates matters so that I have not been able to answer the question: Is the tensor product of connected C^*-algebras always connected?

Recent correspondence with professor R.J. Archbold has given the following partial answer: When $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are simple, β is some C^*-norm, and $\rho : \mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2 \to \mathfrak{A}_1 \otimes_* \mathfrak{A}_2$ is the canonical mapping, $\ker \rho$ contains every ideal of $\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2$, which must then be connected. On the other hand, examples can be given where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are simple, but $\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2$ is not prime. So we may get orthogonal ideals, but they can not add up to the entire algebra.

Remark 8.2.5 We have not taken any particular interest in associating number of components up to cardinality in this section, but include here a couple of words on the problem: It is quite easy to see that (ii) of Lemma 3.1.5 implements a homeomorphism from $P(C_0(X) \otimes \mathfrak{A})$ to $X \times P(\mathfrak{A})$, from which we infer

$$\mathfrak{c}_\mathfrak{c}(\mathfrak{C}_\mathfrak{o}(\mathfrak{X}) \otimes \mathfrak{A}) = \mathfrak{c}_\mathfrak{c}\mathfrak{X}\mathfrak{c}_\mathfrak{c}\mathfrak{A}$$

according to Lemma B.1.5. As noted in Remark 3.1.7, the set

$$\{\varphi \otimes_* \psi | \varphi \in P(\mathfrak{A}), \psi \in P(\mathfrak{B})\} \simeq P(\mathfrak{A}) \times P(\mathfrak{B})$$

is only all of $P(\mathfrak{A} \otimes_* \mathfrak{B})$ when $\mathfrak{A}$ or $\mathfrak{B}$ is abelian, but by Lemma 3.1.1 combined with the double polar theorem A.1.2 and Proposition A.2.1, we get that it is dense in $P(\mathfrak{A} \otimes_* \mathfrak{B})$, because

$$\begin{aligned}
\{\varphi \otimes_* \psi | \varphi \in P(\mathfrak{A}), \psi \in P(\mathfrak{B})\}^\circ &= \\
\{a \in \mathfrak{A} \otimes_* \mathfrak{B} | \forall \varphi \in P(\mathfrak{A}), \psi \in P(\mathfrak{B}) : \varphi \otimes_* \psi(a) \geq -1\} &= \\
\{a \in \mathfrak{A} \otimes_* \mathfrak{B} | \forall \varphi \in P(\mathfrak{A}), \psi \in P(\mathfrak{B}) : \pi_{\varphi \otimes_* \psi}(a) \geq -1\} &= \\
\{a \in \mathfrak{A} \otimes_* \mathfrak{B} | a \geq -1\} &= Q^\circ,
\end{aligned}$$

since $\sum_{P(\mathfrak{A})\times P(\mathfrak{B})} \pi_{\varphi\otimes_*\psi}$ is unitarily equivalent to $\sum_{P(\mathfrak{A})\times P(\mathfrak{B})} \pi_\varphi \otimes_* \pi_\psi$, and this is faithful by Lemma 3.1.1.

This furnishes an alternative – and simpler – proof of Proposition 8.2.3. The reason why we present the more complicated approach involving primitive spectra is the following: According to Example B.1.7 we can not conclude that $\mathfrak{c}_c\mathfrak{P}(\mathfrak{A}\otimes_*\mathfrak{B}) = \mathfrak{c}_c\mathfrak{P}(\mathfrak{A})\mathfrak{c}_c\mathfrak{P}(\mathfrak{B})$ from the fact that $P(\mathfrak{A}) \times P(\mathfrak{B})$ has a dense homeomorphic image in $P(\mathfrak{A}\otimes_*\mathfrak{B})$. As we do have $\mathrm{Prim}(\mathfrak{A}\otimes_*\mathfrak{B}) \simeq \mathrm{Prim}(\mathfrak{A}) \times \mathrm{Prim}(\mathfrak{B})$ when both of $\mathfrak{A}$ and $\mathfrak{B}$ are separable and one of them is nuclear as mentioned in Remark 3.1.7, the path over primitive ideal spectra leaves us with

$$\mathfrak{c}_c\,\mathrm{Prim}(\mathfrak{A}\otimes_*\mathfrak{B}) = \mathfrak{c}_c\,\mathrm{Prim}(\mathfrak{A})\mathfrak{c}_c\,\mathrm{Prim}(\mathfrak{B}).$$

However, as long as $\mathfrak{c}_c\,\mathrm{Prim}(\mathfrak{A}) = \mathfrak{c}_c\mathfrak{A}$ is not established, cf. Remark 7.4.4, the value of this result is questionable.

9 Further notions of connectivity

In this section we consider a couple of notions from the theory of topology which are naturally related to connectivity, cf. Appendix B.

9.1 Arcwise connectivity

We state and prove a proposition regarding *arcwise* connectivity of the set of pure states of a certain class of connected, σ-unital C^*-algebras.

Proposition 9.1.1 *When $\mathfrak{A}$ is a primitive, separable C^*-algebra, $P(\mathfrak{A})$ is arcwise connected.*

Proof: Let π_0 be a faithful irreducible representation, p_0 its central cover (cf. Section 1.1), φ_0 a pure state supported here. As π_0 is faithful and $1-\overline{p_0}$ supports an ideal in its kernel, $\overline{p_0} = 1$. By Lemma 2.3.7, $P(p_0)$ is dense in $P(\overline{p_0}) = P$. Let $\varphi \in P$; we construct a *weak** continuous path from φ_0 to φ in four steps.

1°: Denote by p the central cover of the irreducible representation π_φ. If $p = p_0$ then φ and φ_0 are equivalent, and we may apply Lemma 6.2.1 to find a path from φ to φ_0, so we may assume that $pp_0 = 0$ by the comment following Lemma 1.1.1. Since the *weak** topology on Q stems from a metric d when $\mathfrak{A}$ is separable, we can find a sequence $(\psi_n)_{n=2}^\infty$ in $P(p_0)$ such that $\psi_n \to \varphi$. Let $\psi_1 = \varphi_0$. When q is a minimal projection in $\mathfrak{A}^{**}$ such that $q \le p_0$, we claim that $\psi_m(q) \to 0$ when m tends to infinity. To see this, first note that since q is closed (Proposition 2.2.7) we can find a net (a_λ) in $\mathfrak{A}_{sa}$ such that $a_\lambda \searrow q$, and because φ is normal on $\mathfrak{A}^{**}$

$$\varphi(a_\lambda) \searrow \varphi(q) = \varphi(pq) = \varphi(pp_0) = 0.$$

Given $\varepsilon > 0$, take λ_0 such that $|\varphi(a_{\lambda_0})| < \varepsilon/2$ and m_0 such that

$$m \geq m_0 \Longrightarrow |\varphi(a_{\lambda_0}) - \psi_m(a_{\lambda_0})| < \varepsilon/2.$$

Then for $m \geq m_0$,

$$0 \leq \psi_m(q) \leq \psi_m(a_{\lambda_0}) < \varepsilon$$

and we have proved that $\psi_m(q) \to 0$.

2°: We use this fact to construct a new sequence of states in $P(p_0)$, converging to φ, such that the support projections of these pure states are mutually orthogonal. Thinking more explicitly than usual of elements of $\mathfrak{A}$ (and $\mathfrak{A}^{**}$) as operators on $\mathfrak{H}_\mathfrak{u}$, we can find a unit vector $\xi_n \in p_0\mathfrak{H}_\mathfrak{u}$ for all n such that

$$\psi_n(a) = (a\xi_n|\xi_n), \qquad a \in \mathfrak{A}^{**},$$

and we proceed to construct a new sequence $\tilde{\xi}_n$ of mutually orthogonal unit vectors in $p_0\mathfrak{H}_\mathfrak{u}$ such that when $\tilde{\psi}_n$ are the corresponding pure states, $\tilde{\psi}_n \to \varphi$ as well.

Let $\delta = d(\psi_1, \varphi)$ and $\tilde{\xi}_1 = \xi_1$. Then certainly

$$d(\tilde{\psi}_k, \varphi) \leq \delta/k \tag{7}$$

for $k = 1$. Let q_1 be the support projection of ψ_1 and note that ξ_1 lies in its range. By [22, 3.13.6], q_1 is minimal, and the observation above yields that $\psi_n(q_1) \to 0$. We can pick n_0 such that $\psi_n(q_1) \leq 1/2$ for $n \geq n_0$. For such n we let

$$\eta_n = \psi_n(1 - q_1)^{-1/2}(1 - q_1)\xi_n$$

and see that

$$\begin{aligned} \|\eta_n\|^2 &= \psi_n(1 - q_1)^{-1}((1 - q_1)\xi_n|\xi_n) = 1 \\ (\tilde{\xi}_1|\eta_n) &= \psi_n(1 - q_1)^{-1/2}(q_1\xi_1|(1 - q_1)\xi_n) = 0. \end{aligned}$$

Then, for every $a \in \mathfrak{A}$

$$\begin{aligned} |\varphi(a) - (a\eta_n|\eta_n)| &\leq |\varphi(a) - \psi_n(a)| \\ &+ |(a\xi_n|\xi_n) - \psi_n(1 - q_1)^{-1}((1 - q_1)a(1 - q_1)\xi_n|\xi_n)| \\ &\leq |\varphi(a) - \psi_n(a)| + |1 - \psi_n(1 - q_1)^{-1}||\psi_n(a)| \\ &+ |\psi_n(1 - q_1)^{-1}|(2\|a\| + \|aq_1\|)|\psi_n(q_1)| \\ &\leq |\varphi(a) - \psi_n(a)| + |\psi_n(q_1)/1 - \psi_n(1 - q_1)|\|a\| \\ &+ 3|\psi_n(1 - q_1)^{-1}|\|a\||\psi_n(q_1)| \\ &\leq |\varphi(a) - \psi_n(a)| + 2\|a\||\psi_n(q_1)| + 6\|a\||\psi_n(q_1)| \\ &\to 0, \qquad n_0 \leq n \to \infty. \end{aligned}$$

and we can pick n such that with $\tilde{\xi}_2 = \eta_n$, (7) holds for $k = 2$ as well. Now assume $\tilde{\xi}_1, \dots, \tilde{\xi}_l$ have been chosen, mutually orthogonal and satisfying (7) for all $k \in \{1, \dots, l\}$. The smallest projection q_l in $\mathfrak{A}^{**}$ covering the span of these vectors is a finite sum of minimal projections, so again we get $\psi_m(q_l) \to 0$ and can find $\tilde{\xi}_{n+1}$ by applying the above arguments to q_n. In particular, we get

$$(\tilde{\xi}_k | \psi_n(1-q_l)^{-1/2}(1-q_l)\xi_n) = \psi_n(1-q_l)^{-1/2}(q_l\tilde{\xi}_k|(1-q_l)\xi_n) = 0$$

for all $k \in \{1, \dots, l\}$.

3°: According to the above, we may assume that the sequence (ψ_n) is given by a sequence (ξ_n) of mutually orthogonal unit vectors in $p_0\mathfrak{H}_{\mathfrak{u}}$. Also, $\psi_1 = \varphi_0$. For any n, we define a norm continuous path of unit vectors in $p_0\mathfrak{H}_{\mathfrak{u}}$ from ξ_n to ξ_{n+1}, thus creating a *weak** continuous path of pure states from ψ_n to ψ_{n+1}, in the following fashion. Using $\{\xi_n, \xi_{n+1}\}$ as an orthonormal basis for $M_n = \operatorname{span}\{\xi_n, \xi_{n+1}\}$ we let

$$\eta_t^n = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \cos t\xi_n + \sin t\xi_{n+1}$$

and note that this simple construction has an important property: Let $a \in \mathfrak{A}_{sa}$ be such that $\|a\| = 1$, write a^n for the restriction of a to M_n and note that a^n is self-adjoint. In case that $0 < \varepsilon < 1$ and ε satisfies

$$(a^n\eta_0^n|\eta_0^n) \geq 1-\varepsilon^2, \qquad (a^n\eta_1^n|\eta_1^n) \geq 1-\varepsilon^2$$

we claim

$$(a^n\eta_t^n|\eta_t^n) \geq 1-4\varepsilon, \qquad \forall t \in [0,1].$$

To see this, write $a^n = [\alpha_{ij}]$. By assumption α_{11} and α_{22} are real numbers, greater than $1-\varepsilon^2$. Because $\|a^n\| \leq \|a\| = 1$, we have

$$1 \geq \left\| a^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \end{bmatrix} \right\|^2 = \alpha_{11}^2 + |\alpha_{21}|^2,$$

yielding, since $\alpha_{11}^2 \geq 1 - 2\varepsilon^2 + \varepsilon^4$, that $|\alpha_{21}|^2 \leq 2\varepsilon^2 - \varepsilon^4 \leq 2\varepsilon^2$. Finally,

$$\begin{aligned} (a^n\eta_t^n|\eta_t^n) &= \cos^2 t\alpha_{11} + \sin^2 t\alpha_{22} + 2\sin t\cos t \operatorname{Re}\ \alpha_{21} \\ &\geq 1 - \varepsilon^2 - 2\sqrt{2}\varepsilon \\ &\geq 1 - 4\varepsilon. \end{aligned}$$

4°: By defining $\Xi : [0,1) \to p_0\mathfrak{H}_{\mathfrak{u}}$ as

$$\Xi(1 - 2^{1-n} + 2^{-n}t) = \eta_t^n$$

when $n \in \mathbb{N}$ and $t \in [0,1]$, we get a function with unit vector values, well-defined and continuous since $\eta_1^n = \xi_{n+1} = \eta_0^{n+1}$. The corresponding function with pure state values $\Psi : [0,1) \to P$ has the property

$$\Psi(1 - 2^{n-1}) = \psi_n,$$

and we have arranged so that $\Psi(0) = \psi_1 = \varphi_0$, so all we need to prove is that when Ψ is extended with φ at 1, then Ψ is *weak** continuous on all of $[0, 1]$. To see this, let t_n be an arbitrary sequence in $[0, 1)$ tending to 1. By Proposition 4.1.2, φ has an excising sequence a_m such that $\varphi(a_m) = 1$, and by Lemma 4.1.3, we only need to prove that, for any m and ε, there exists t_0 such that

$$\Psi(t)(a_m) = (a_m \Xi(t)|\Xi(t)) \geq 1 - \varepsilon$$

whenever $t \geq t_0$. By Lemma 4.1.3, we can find k_0 such that when $k \geq k_0$, $\psi_k(a_m) \geq 1 - \varepsilon^2/16$. Using the property of the η_t^n's from 3°, $\Psi(t)(a_m) = (a_m \Xi(t)|\Xi(t)) \geq 1 - \varepsilon$ whenever $t \geq 1 - 2^{1-k_0}$, from which the proposition follows. ■

There seems to be at least two essentially different reasons why a given connected C^*-algebra $\mathfrak{A}$ has a set of pure states that is not arcwise connected. The most obvious one is that the underlying central structure of $\mathfrak{A}$, as it is found in $\mathcal{Z}(M(\mathfrak{A}))$, for example, is not arcwise connected. For example, $\mathfrak{A}$ might be $C(X)$ where X is some connected, but not arcwise connected, compact set, cf. Remark B.2.2. Another problem arises when $P(\mathfrak{A})$ is too big for us to expect that a mapping defined on the second countable space $[0, 1]$ can "reach" from one end to another. The following example demonstrates this.

Example 9.1.2 *A simple C^*-algebra $\mathfrak{A}$ such that $P(\mathfrak{A})$ is not arcwise connected.*

Let $\mathfrak{A}$ be a factor von Neumann algebra on a separable Hilbert space $\mathfrak{H}$. When $\mathfrak{A}$ is of type II_1 or III, $\mathfrak{A}$ is simple according to, e.g., [17, 6.8.4&6.8.5]. Also, $\mathfrak{A}$ has exactly $2^{\aleph}$ irreducible representations; this can be seen by noting that the arguments in [17, 10.4.15] apply to any von Neumann algebra on $\mathfrak{H}$ containing infinitely many orthogonal projections. By letting p be the central cover in $\mathfrak{A}^{**}$ of any such representation, we thus have $0 < p < z$, and by Theorem 2.1.3, combined with (3) of Section 7.2, $\{P_0, P_1\}$ is a non-trivial separation of P into disjoint norm closed sets, where $P_0 = P(p), P_1 = P(1 - p)$.

Since P is connected in the *weak** topology, the P_i can not be closed in this topology. However, we only need to know that the sets are *sequentially* closed, as the following argument will show. Take $\varphi_0 \in P_0$, $\varphi_1 \in P_1$ and assume that φ_t is a *weak** continuous path from φ_0 to φ_1. Let

$$t_0 = \inf\{t \in [0, 1]|\varphi_t \in P_1\}$$

and assume that $\varphi_{t_0} \in P_1$. Since $t_0 > 0$ we can find a sequence $t_n \nearrow t_0$, and by continuity, $\varphi_{t_n} \to \varphi_{t_0}$ in the *weak** topology. By [1, 5], $\varphi_{t_n} \to \varphi_{t_0}$ in norm, contradicting the fact that P_0 is norm closed. Similarly, $\varphi_{t_0} \in P_1$ leads to a contradiction, and such a path can not exist. ■

Remark 9.1.3 It should be possible to relax the condition that $\mathfrak{A}$ is primitive in Proposition 9.1.1 to a condition assuring that the center of $M(\mathfrak{A})$ is "small" in a less restrictive sense. When $\mathfrak{A}$ is primitive, $\mathrm{Prim}(\mathfrak{A})$ is arcwise connected as the function $\varphi : [0,1] \to \mathrm{Prim}(\mathfrak{A})$ given by

$$\varphi(x) = \begin{cases} \{0\}, & x \in [0,1) \\ \mathfrak{I}, & x = 1 \end{cases}$$

is continuous for every $\mathfrak{I} \in \mathrm{Prim}(\mathfrak{A})$, but it seems a bit too much to hope that it suffices to know that $\mathrm{Prim}(\mathfrak{A})$ is arcwise connected. In any case, a completely different approach to such a result appears to be required.

9.2 Local connectivity

We can mimic the idea of local connectivity, cf. Definition B.2.4, in the setting of C^*-algebras. First an important lemma:

Lemma 9.2.1 *Let $\mathfrak{A}$ be a C^*-algebra. The following conditions are equivalent.*

(i) All components of $\mathfrak{A}$ are open projections.

(ii) All components of $P(\mathfrak{A})$ are open sets.

(iii) All components of $\mathrm{Prim}(\mathfrak{A})$ are open sets.

(iv) $\mathfrak{A} \simeq \sum_I \mathfrak{A}_i$, where every $\mathfrak{A}_i$ is connected.

Proof: That (i) and (ii) are equivalent follows from Corollary 7.2.2. For (ii)$\Longrightarrow$(iii), we note that since $\varphi \mapsto \ker \pi_\varphi$ is continuous and onto, every component of $\mathrm{Prim}(\mathfrak{A})$ is a union of images of components of $P(\mathfrak{A})$. As $\varphi \mapsto \ker \pi_\varphi$ is an open mapping, every component of $\mathrm{Prim}(\mathfrak{A})$ is open when every component of $P(\mathfrak{A})$ is. For (iii)$\Longrightarrow$(iv), let C_i be the set of components of $\mathrm{Prim}(\mathfrak{A})$ and note that since $1_{C_i} \in C^b(\mathrm{Prim}(\mathfrak{A}))$, these functions correspond to $c_i \in \mathcal{Z}(M(\mathfrak{A}))$ by the Dauns-Hoffman theorem [22, 4.4.8]. The proof of this theorem gives that

$$c_i z = \sum_{\ker \pi \in C_i} p_\pi, \tag{8}$$

where p_π is the central cover of π in $\mathfrak{A}^{**}$. For every $i \in I$, $\mathfrak{A}_i = \mathfrak{A}c_i$ is an ideal of $\mathfrak{A}$, and we may define $\Phi : \mathfrak{A} \to \sum_I \mathfrak{A}_i$ by

$$\Phi(a) = (ac_i)_{i \in I}.$$

To see that these tuples vanish at infinity, we will need to consider the mappings $\check{a} : \mathrm{Prim}(\mathfrak{A}) \to [0, \infty)$ defined by $\check{a}(\ker \pi) = \|\pi(a)\|$ for every $a \in \mathfrak{A}_{sa}$, cf. [22, 4.4.2]. We have, for every self-adjoint $a \in \mathfrak{A}$,

$$\|ac_i\| = \|ac_i z\| = \sup_{\ker \pi \in C_i} \|ap_\pi\| = \sup_{t \in C_i} \check{a}(t)$$

according to (8) above. Given $\varepsilon > 0$, the set $\{t \in \mathrm{Prim}(\mathfrak{A}) | \check{a}(t) \geq \varepsilon\}$ is compact by [22, 4.4.4] and is thus covered by finitely many of the C_i. We conclude that the set $\{i \in I | \|ac_i\| \geq \varepsilon\}$ is finite whenever $a \in \mathfrak{A}_{sa}$ and may extend that conlusion to all of $\mathfrak{A}$ by decomposition into real and imaginary parts. These methods also yield that Φ is an isometry, as

$$\|\Phi(a)\| = \sup_{i \in I} \|ac_i\| = \sup_{i \in I} \sup_{t \in C_i} \check{a}(t) = \sup_{\pi \in \hat{A}} \|\pi(a)\| = \|a\|,$$

and since $\Phi(\mathfrak{A})$ contains all tuples with only a finite number of non-zero terms, and this set is dense, Φ is an isometric isomorphism. Finally, (iv)$\Longrightarrow$(ii) follows from Lemma 8.1.1. ■

For the next result, recall that when $\mathfrak{B}$ is a hereditary subalgebra of $\mathfrak{A}$, we can write $\mathfrak{B} = H(p)$ for some open projection p, and as seen in the proof of Proposition 7.1.3, $\mathfrak{B}^{**} \simeq p\mathfrak{A}^{**}p$ under an isomorphism that preserves openness of projections. As p might not be closed in $\mathfrak{A}^{**}$, closedness is not preserved. When $q \leq p$ we have

$$P_{\mathfrak{B}}(q) = \{\varphi|_{\mathfrak{B}} \, | \varphi \in P(q)\},$$

where the subscribed $\mathfrak{B}$ means that we are talking about the pure states in this algebra. To see that equality, note that by [22, 4.1.6], $\varphi|_{\mathfrak{B}}$ is a multiple of a pure state for every $\varphi \in P(\mathfrak{A})$, and as $\varphi(q) = 1$ in $\mathfrak{B}^{**}$, $\varphi|_{\mathfrak{B}}$ is itself pure. On the other hand, every $\psi \in P(\mathfrak{B})$ has a (unique) extension to a pure state of $\mathfrak{A}$ by [22, 3.1.6].

Theorem 9.2.2 *Let $\mathfrak{A}$ be a C^*-algebra. The following conditions are equivalent:*

(i) If $p \in \mathfrak{A}^{**}$ is an open projection and $\varphi \in P(p)$, there exists a connected open projection $q \in \mathfrak{A}^{**}$ such that $q \leq p$, $\varphi(q) = 1$.

(ii) When $\mathfrak{B}$ is a hereditary subalgebra of $\mathfrak{A}$, all components of $\mathfrak{B}$ are open projections.

(iii) If $p \in \mathfrak{A}^{**}$ is an open central projection and $\varphi \in P(p)$, there exists a connected open central projection $q \in \mathfrak{A}^{**}$ such that $q \leq p$, $\varphi(q) = 1$.

(iv) When $\mathfrak{I}$ is an ideal of $\mathfrak{A}$, all components of $\mathfrak{I}$ are open projections.

(v) $\mathrm{Prim}(\mathfrak{A})$ *is locally connected.*

Definition 9.2.3 A C^*-algebra having these properties is called *locally connected.*

Proof of Theorem 9.2.2: We prove (i)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$(v)$\Longrightarrow$(ii)$\Longrightarrow$(i). For (i)$\Longrightarrow$(iii), let p and φ be given. Choosing q in (i), we may use $c(q)$ in (iii) according to Lemma 2.4.4 and 7.1.5. In (iii)$\Longrightarrow$(iv), first recall that we can write $\mathfrak{I} = \mathfrak{H}(\mathfrak{p})$ for some open and central $p \in \mathfrak{A}^{**}$ by Theorem 2.4.1. Let c be a component of $\mathfrak{I}$ and identify c with a central projection in $\mathfrak{A}^{**}p$. For all $\varphi \in P(c)$ take, by assumption, $q_\varphi \leq p$ connected, open and central such that $\varphi(q_\varphi) = 1$. From Corollary 7.1.7, $q_\varphi \vee c$ is connected, and q_φ is dominated by c. Then so is the central projection

$$q = \bigvee_{\varphi \in P(c)} q_\varphi.$$

By Proposition 2.2.8, q is open, and we only need to prove that $q \geq c$ to see that c is open. Since q is open and c is closed in $\mathfrak{B}$, we need only prove $qz_{\mathfrak{B}} \geq cz_{\mathfrak{B}}$ according to Corollary 2.2.15. By centrality, this will follow by $P_{\mathfrak{B}}(c) \subseteq P_{\mathfrak{B}}(q)$, and by definition of q, if $\varphi \in P_{\mathfrak{B}}(c)$, then $\varphi(q) \geq \varphi(q_\varphi) = 1$.

An open set of $\mathrm{Prim}(\mathfrak{A})$ has the form $\mathrm{Prim}(\mathfrak{A})\setminus \mathrm{Hull}(\mathfrak{I})$ for some ideal $\mathfrak{I}$ according to [22, 4.1.3], and by [22, 4.1.10], that is homeomorphic to $\mathrm{Prim}(\mathfrak{I})$. Applying Lemma 9.2.1, we get (iv)$\Longrightarrow$(v). For (v)$\Longrightarrow$(ii), note that $\mathrm{Prim}(\mathfrak{B})$ is homeomorphic to the open set $\mathrm{Prim}(\mathfrak{A})\setminus \mathrm{Hull}(\mathfrak{B})$ according to [22, 4.1.10] and apply Lemma 9.2.1 again. Finally, for (ii)$\Longrightarrow$(i), let p and φ be given. Since φ restricts to a pure state on $\mathfrak{B} = H(p)$ by the discussion preceding this theorem, there is a component q of $\mathfrak{B}$ such that $\varphi(q) = 1$, because the minimal projection supporting φ is dominated by a component, cf. Proposition 7.1.9. We may think of q as an element of $p\mathfrak{A}^{**}p$, and when q is open in $\mathfrak{B}^{**}$, q is open in $\mathfrak{A}^{**}$. If $\{r_0, r_1\}$ separates q in $\mathfrak{A}^{**}$, $\{r_0 p, r_1 p\}$ separates q in $p\mathfrak{A}^{**}p \simeq \mathfrak{B}^{**}$ and we may assume that $0 = r_0 pq = r_0 q$, showing that q is also connected in $\mathfrak{A}^{**}$. ■

Corollary 9.2.4 *A hereditary subalgebra of a locally connected C^*-algebra is locally connected.*

Proof: Let $\mathfrak{A}$ be locally connected, $\mathfrak{B}$ a hereditary subalgebra of $\mathfrak{A}$, $\mathfrak{C}$ a hereditary subalgebra of $\mathfrak{B}$. As $\mathfrak{C}$ is also hereditary as a subalgebra of $\mathfrak{A}$, the claim follows from Theorem 9.2.2(ii). ■

Corollary 9.2.5 *When $\mathfrak{A}$ is a locally connected C^*-algebra, $\mathfrak{A} \simeq \sum_I \mathfrak{A}_i$ for some connected and locally connected $\mathfrak{A}_i$.*

Proof: By Lemma 9.2.1, $\mathfrak{A}$ can be written as a sum of connected C^*-algebras, and as these sit as ideals of $\mathfrak{A}$ they are locally connected by the above corollary. ■

Example 9.2.6

> *(i) A primitive C^*-algebra is locally connected. In particular, a simple C^*-algebra is locally connected.*
>
> *(ii) $C_0(X)$ is locally connected if and only if X is.*

Proof: By Example 7.1.4, every open projection of a primitive C^*-algebra is connected, so we may use $q = p$ in Theorem 9.2.2(i). The second statement follows directly by (v) of Theorem 9.2.2. ■

Proposition 9.2.7 *If X and $\mathfrak{A}$ are both locally connected, then so is $C_0(X) \otimes \mathfrak{A}$.*

Proof: By Corollary 3.1.6, $\mathrm{Prim}(C_0(X) \otimes \mathfrak{A}) \simeq X \times \mathrm{Prim}(\mathfrak{A})$, and this set is locally connected by Theorem 9.2.2(v) and Lemma B.2.7. ■

Proposition 9.2.8 *If $P(\mathfrak{A})$ is locally connected, so is $\mathfrak{A}$.*

Proof: The mapping $\varphi \mapsto \ker \pi_\varphi$ is continuous, open and onto according to [22, 4.3.3]. Apply Lemma B.2.6. ■

Remark 9.2.9 We suspect that there exist locally connected C^*-algebras with $P(\mathfrak{A})$ not locally connected, but have no examples.

Part III

End theory

10 Ends in topological spaces

In this section we will describe the notion of *ends* in topological spaces, as found in [14], with the purpose of preparing the way for a later generalization to C^*-algebras. The exposition is rather different than in the classical treatise mentioned above; simplified and altered in order to facilitate the implementation of a non-commutative version. Anticipating possible parallels in the case of C^*-algebras, we supplement the exposition from [14] with a comparison between the compactification due to Freudenthal and that of Stone and Čech.

10.1 Sequences determining ends

We will confine ourselves to the following class of topological spaces, which may not be the largest class in which the arguments go through. However, as the goal is to generalize the theory to a non-commutative setting, there are many good reasons for wanting to keep everything as simple as possible.

Definition 10.1.1 A Hausdorff space X is called a *Raum* if X is

(i) connected

(ii) locally connected

(iii) locally compact

(iv) σ-compact.

When also X is second countable, it is metrizable according to [23, 1.7.9], and we say that X is a *metric Raum.*

Let X be a Raum. We say that a decreasing sequence (G_k) of non-empty, connected, open subsets of X with compact boundaries ∂G_k *determines an end* if

$$\bigcap_{n=1}^{\infty} \overline{G_k} = \emptyset. \tag{1}$$

Note that (1) implies that no $\overline{G_k}$ can be compact, for by the finite intersection property, this would imply that some $\overline{G_k}$ were empty.

Definition 10.1.2 Let (G_k) be a sequence determining an end of X. When H is a subset of X we say that $(G_k) \subseteq H$ if there exists a $k \in \mathbb{N}$ such that $G_k \subseteq H$.

Lemma 10.1.3 *Let (G_k) and (H_k) be sequences determining ends of the Raum X. The following conditions are equivalent.*

(i) $\forall l \in \mathbb{N} : G_l \cap H_l \neq \emptyset$.

(ii) $\forall l \in \mathbb{N} : (G_k) \subseteq H_l$.

(iii) $\forall l \in \mathbb{N} : (H_k) \subseteq G_l$.

Definition 10.1.4 If (G_k) and (H_k) satisfy (i)–(iii) above, we say that the sequences are *equivalent* and write $(H_k) \approx (G_k)$.

Proof of Lemma 10.1.3: By symmetry, it suffices to show (i)$\Longleftrightarrow$(ii). Given l, if $(G_k) \subseteq H_l$, then for some $k \in \mathbb{N}$, $G_l \cap H_l \supseteq G_k \cap H_l = G_k \neq \emptyset$. On the other hand, if (i) holds true and l is given, then since ∂H_l is compact,

$$\bigcap_{k=1}^{\infty} (\partial H_l \cap \overline{G_k}) \subseteq \bigcap_{k=1}^{\infty} \overline{G_k} = \emptyset$$

yields that for some $k \in \mathbb{N}$, $\partial H_l \cap \overline{G_k} = \emptyset$. *A fortiori*, $\partial H_l \cap G_k = \emptyset$, and as with $m = \max(k, l)$,

$$H_l \cap G_k \supseteq H_m \cap G_m \neq \emptyset,$$

we conclude from Lemma B.1.9, that $G_k \subseteq H_l$. ■

Using (i) for reflexivity and symmetry, (ii) for transitivity, we see that "$\approx$" is an equivalence relation. We write $[(G_k)]$ for the equivalence class represented by (G_k), and use $\mathfrak{e}$ as a symbol for such a class. Using (ii) above, we may define $\mathfrak{e} \subseteq \mathfrak{H}$ as $(G_k) \subseteq H$, when $\mathfrak{e} = [(\mathfrak{G}_\mathfrak{k})]$.

Definition 10.1.5 Let X be a Raum. An *end* of X is an element of

$$\mathfrak{E} = \{[(\mathfrak{G}_\mathfrak{k})] | (\mathfrak{G}_\mathfrak{k}) \text{ determines an end of } X\}.$$

The number of ends of X, with values in $\mathbb{N} \cup \{\infty\}$, is denoted by $\#_e X$.

Example 10.1.6 It is easy to see that $\mathbb{R}^n$ is a Raum for all $n \in \mathbb{N}$. The sequence $(\{x \in \mathbb{R}^n | \|x\| > k\})_{k=1}^{\infty}$ determines an end for all $n > 1$, but not in $\mathbb{R}$. Here, $((k, \infty))_{k=1}^{\infty}$ and $((-\infty, -k))_{k=1}^{\infty}$ determine inequivalent ends. It will be obvious from the following, that in fact $\#_e\mathbb{R} = 2$, $\#_e\mathbb{R}^n = 1, n > 1$.

It is our goal to prove that when X is not compact, there exists at least one end of X, and to devise a way to find all there might be. The following lemma contains an essential consequence of our assumption that the X are Räume, in particular, that they are locally connected.

Lemma 10.1.7 *Let X be a Raum, and assume that K is compact, G is open such that $K \subseteq G \subseteq X$. Writing*

$$X \backslash K = \bigcup_{i \in I} C_i,$$

where the C_i are the components of $X \backslash K$, we have that

$$I_0 = \{i \in I | C_i \not\subseteq G\}$$

is finite.

Proof: As X is locally compact, there is an open set H containing K, such that $\overline{H}$ is compact, and because it clearly suffices to prove the lemma with $G \cap H$ instead of G, we may assume without loss of generality that G itself has compact closure.

If $K = \emptyset$, I has only one element, so assume further that $K \neq \emptyset$. From the local connectedness of X, cf. Lemma B.2.3, each C_i is open in $X \backslash K$ and X. As they are components, they are closed in $X \backslash K$, but since X is connected, they are not closed here. We conclude that with closures taken in X, $\overline{C_i} \cap K \neq \emptyset$ for all $i \in I$, from which we further deduce that

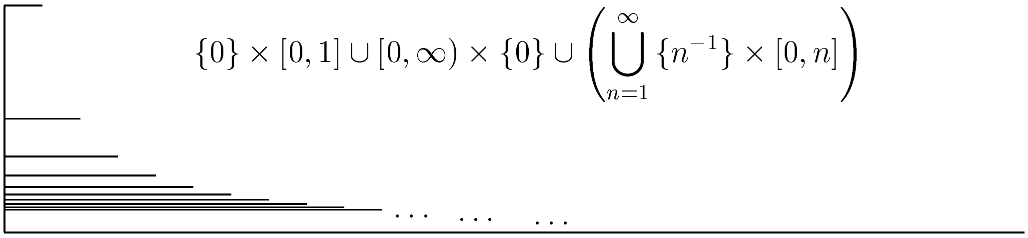
$$C_i \cap G \neq \emptyset, \qquad \forall i \in I.$$

When $i \in I_0$, $C_i \cap (X \backslash G) \neq \emptyset$ follows by definition, and by Lemma B.1.9,

$$C_i \cap \partial G \neq \emptyset, \qquad \forall i \in I_0. \tag{2}$$

As ∂G is compact by assumption, there is a finite set $I_1 \subseteq I$ such that $\partial G \subseteq \bigcup_{I_1} C_i$. The C_i are disjoint, whence from (2), $I_0 \subseteq I_1$. ■

Example 10.1.8 The subset X of $\mathbb{R}^2$ given by



$$\{0\} \times [0,1] \cup [0,\infty) \times \{0\} \cup \left(\bigcup_{n=1}^{\infty} \{n^{-1}\} \times [0,n] \right)$$

is not locally connected, for instance because the complement of the vertical line has a non-closed component. It is hence not a Raum, and it does not have the property of the above lemma; indeed, when K is the vertical line and G is chosen with $\overline{G}$ compact, then an infinite number of components of $X \backslash K$ are not contained in G.

From the σ-compactness combined with the local compactness of a Raum X, we can choose compact sets L_k such that

$$(\dagger) \qquad L_1^\circ \subseteq L_1 \subseteq L_2^\circ \subseteq \cdots \subseteq L_k \subseteq L_{k+1}^\circ \subseteq L_{k+1} \subseteq \cdots, \qquad \bigcup_{k=1}^{\infty} L_k = X.$$

We call such a sequence a *compact nest* for X, and will assume in the following that one is fixed, using the dagger ($\dagger$) to refer to the chosen nest. We will also, for brevity, classify open sets as *bounded* or *unbounded*, according to whether or not their closures are compact. Finally, for $Y \subseteq X$ open, we will denote by $\#_{uc}Y$ the number of unbounded components of Y.

Proposition 10.1.9 *Let X be a non-compact Raum with compact nest ($\dagger$). Then*

(i) There is at least one, and at most a finite number of unbounded components of $X \backslash L_k$.

(ii) Let C be an unbounded component of $X \backslash L_k$. At least one unbounded component of $X \backslash L_{k+1}$ is contained in C.

(iii) Every unbounded component of $X \backslash L_{k+1}$ is contained in one of $X \backslash L_k$.

(iv) With W_k the union of all unbounded components of $X \backslash L_k$, $X \backslash W_k$ is compact.

Proof: Write $X \backslash L_k$ out in components $(C_i)_I$. By Lemma 10.1.7, the subset

$$I_0 = \{i \in I | C_i \not\subseteq L_{k+1}^\circ\}$$

is finite, and as C_i for $i \in I \backslash I_0$ is contained in the compact set L_{k+1}, there is a finite number of unbounded components. On the other hand, as the unbounded set

$X\backslash L_{k+1}$ is contained in $\bigcup_{I_0} C_i$, at least one C_i, $i \in I_0$ must itself be unbounded, proving (i). Letting

$$I_1 = \{i \in I | C_i \text{ is unbounded}\}$$

we have seen that $I_1 \subseteq I_0$, whence in the notation of (iv),

$$X\backslash W_k = X\backslash \bigcup_{i\in I_1} C_i = L_k \cup \bigcup_{i\notin I_1} C_i \subseteq L^\circ_{k+1} \cup \bigcup_{i\in I_0\backslash I_1} C_i.$$

As $X\backslash W_k$ is closed and contained in a finite union of bounded sets, it is compact, proving (iv).

For (ii), let C be an unbounded component of $X\backslash L_k$ and note that by Lemma B.1.8, we can write

$$C \cap (X\backslash L_{k+1}) = C\backslash L_{k+1} = \bigcup_{i\in I} D_i,$$

where the D_i are components of $X\backslash L_{k+1}$. As above, all but a finite number of the D_i are contained in L°_{k+2}, and at least one of these must be unbounded, as $C\backslash L_{k+1}$ is. Finally, every component D of $X\backslash L_{k+1}$ must be contained in one of $X\backslash L_k$, say C, and if D is unbounded, so is C. ■

Remark 10.1.10 We have from (ii) above, that $\#_{uc}(X\backslash L_k) \leq \#_{uc}(X\backslash L_{k+1})$ whenever X is a Raum. Considering again the set X of Example 10.1.8, if we let L_1 =$\{(0,0)\}$ and $L_2 = \{0\} \times [0,1]$, then

$$\#_{uc}(X\backslash L_1) = 2 > 1 = \#_{uc}(X\backslash L_2).$$

10.2 The Freudenthal compactification

Let X be a locally compact Hausdorff space and recall the definition of a compactification of X from [23, 1.7.2]. We will assume all compactifications to be Hausdorff, and define a relation on these by saying that $(X \overset{\iota_1}{\to} Y_1) \geq (X \overset{\iota_2}{\to} Y_2)$ when there exists a continuous mapping $\kappa : Y_1 \to Y_2$ respecting the embeddings, i.e. with $\iota_2 = \kappa \circ \iota_1$. Note that κ is then automatically onto, and that if $\lambda : Y_2 \to Y_1$ exists with $\iota_1 = \lambda \circ \iota_2$, then κ is a homeomorphism with $\kappa^{-1} = \lambda$. The defined relation can thus be seen to be an order on the set of compactifications identified under embedding-preserving homeomorphism. The (homeomorphism class of the) one-point compactification, αX, is clearly minimal in this respect; the Stone-Čech compactification βX is maximal according to [23, 4.3.19]. We will use γX as a generic symbol for a compactification of X, and write $(X \hookrightarrow \gamma X)$ to indicate that X is identified with its image here. In this case, we write $\overline{Y}^\gamma$ to specify closure within γX.

The purpose of this section is to define another compactification of the Raum X, formally adjoining the ends to the space. For this, we first note that Proposition 10.1.9 allows us to arrange the unbounded components of $X\backslash L_k$ in decreasing sequences by means of multi-indices. The unbounded components of $X\backslash L_1$ are denoted by $H_1, \ldots, H_m$, and we then group the unbounded components of $X\backslash L_2$ according to which of the H_i, $i \in \{1, \ldots, m\}$, they are contained in; thus those lying inside H_1 are given indices $H_{11}, \ldots, H_{1m_1}$. A bit more formally, we may define $H_{n_1 \cdots n_k}$ for all $k \in \mathbb{N}$, $(n_1, \ldots, n_k) \in \mathbb{N}^k$ in accordance with

(i) $H_{n_1 \cdots n_k}$ is $\emptyset$ or an unbounded component of $X\backslash L_k$,

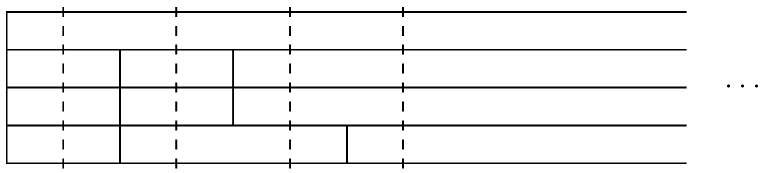
(ii) All unbounded components of $X\backslash L_k$ occur as $H_{n_1 \cdots n_k}$ for exactly one tuple $(n_1, \ldots, n_k) \in \mathbb{N}^k$,

(iii) $H_{n_1 \cdots n_k} \supseteq H_{n_1 \cdots n_k n_{k+1}}$,

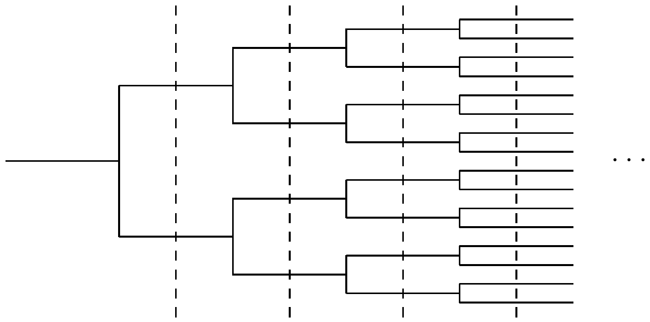
(iv) If $H_{n_1 \cdots n_k} = \emptyset$ and $n'_k > n_k$, then $H_{n_1 \cdots n_{k-1} n'_k} = \emptyset$,

because of (i)–(iii) in Proposition 10.1.9. We say that a sequence $(H_{n_1 \cdots n_k})$ is *trivial*, if $H_{n_1 \cdots n_k} = \emptyset$ for some $k \in \mathbb{N}$. It is clear that there are at most $\mathbb{N}^{\mathbb{N}} = \aleph$ non-trivial sequences. On the other hand, when X is not compact, there is at least one according to (i) and (ii) from Proposition 10.1.9.

Example 10.2.1 In the case of the simplex described by the solid lines of



then if compact sets are removed as indicated with the dashed lines, a systematic enumeration from top to bottom would give non-empty sets H_1, H_2; H_{11}, H_{21}; $H_{111}, H_{211}, H_{212}, H_{213}$; $H_{1111}, H_{2111}, H_{2121}, H_{2131}, H_{2132}$. For a 2-tree



any enumeration would give non-empty sets for $H_{n_1\cdots n_k}$ with $n_k \in \{1,2\}$, hence showing that the case occurs, where the number of non-trivial sequences is uncountable.

Lemma 10.2.2 *Let $(H_{n_1\cdots n_k})$ and $(H_{m_1\cdots m_k})$ be non-trivial sequences of the Raum X.*

(i) The sequence $(H_{n_1\cdots n_k})$ determines an end of X.

(ii) $(H_{n_1\cdots n_k}) \subseteq H_{m_1\cdots m_{k'}}$ if and only if $n_i = m_i$ for all $i \in \{1,\ldots,k'\}$.

(iii) $(H_{n_1\cdots n_k}) \approx (H_{m_1\cdots m_k})$ if and only if $n_k = m_k$ for each $k \in \mathbb{N}$.

Proof: Because each $H_{n_1\cdots n_k}$ is clopen relative to $X\backslash L_k$, $\partial H_{n_1\cdots n_k} \subseteq L_k$ and is thus compact. Consequently, $(H_{n_1\cdots n_k})$ determines an end, since

$$\bigcap_{k=1}^{\infty} \overline{H_{n_1\cdots n_k}} \subseteq \bigcap_{k=1}^{\infty} X\backslash L_k^\circ \subseteq X\backslash(\bigcup_{k=1}^{\infty} L_k^\circ) = \emptyset.$$

The second assertion follows directly from the fact that by definition, $(n_1,\ldots,n_{k'}) \neq (m_1,\ldots,m_{k'})$ yields that $H_{n_1\cdots n_{k'}}$ and $H_{m_1\cdots m_{k'}}$ are disjoint, and clearly (iii) follows from (ii). ∎

We denote by $\mathfrak{E}_\dagger$ the set of ends thus obtained, and note that by (iii) above, every $\mathfrak{e} \in \mathfrak{E}_\dagger$ has one and only one representation as $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1\cdots\mathfrak{n}_\mathfrak{k}})]$. We are ready to prove the existence of the Freudenthal compactification for non-compact Räume.

Proposition 10.2.3 *Let X be a Raum with compact nest (†). There is a topology making $X \cup \mathfrak{E}_\dagger$ a Hausdorff compactification of X.*

Proof: We specify the new topology by giving bases of the neighborhood systems for each point in $X \cup \mathfrak{E}_\dagger$. If $x \in X$, the basis is given by the set of neighborhoods of x in X. If $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1\cdots\mathfrak{n}_\mathfrak{k}})] \in \mathfrak{E}_\dagger$, the basis is

$$V_{n_1\cdots n_k} = H_{n_1\cdots n_k} \cup \{\mathfrak{e} \in \mathfrak{E}_\dagger | \mathfrak{e} \subseteq \mathfrak{H}_{\mathfrak{n}_1\cdots\mathfrak{n}_\mathfrak{k}}\}. \tag{3}$$

Since these sets are nested, this does indeed define a neighborhood basis. That $X \cup \mathfrak{E}_\dagger$ is Hausdorff follows by the fact that X is Hausdorff when the two points to be separated are both in X. When both points are in $\mathfrak{E}_\dagger$, we note that if $\mathfrak{e}_\mathfrak{o} = [(\mathfrak{H}_{\mathfrak{n}_1^\mathfrak{o}\cdots\mathfrak{n}_\mathfrak{k}^\mathfrak{o}})]$ and $\mathfrak{e}_\mathfrak{1} = [(\mathfrak{H}_{\mathfrak{n}_1^\mathfrak{1}\cdots\mathfrak{n}_\mathfrak{k}^\mathfrak{1}})]$ are given with $\mathfrak{e}_\mathfrak{o} \neq \mathfrak{e}_\mathfrak{1}$, then by Lemma 10.2.2(iii), there is a k with $n_k^0 \neq n_k^1$ so that $H_{n_1^0\cdots n_k^0}$ and $H_{n_1^1\cdots n_k^1}$ are disjoint. Then so are the corresponding basic neighborhoods $V_{n_1^0\cdots n_k^0}$ and $V_{n_1^1\cdots n_k^1}$ by Lemma 10.2.2. When

$x \in X$ and $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_\mathfrak{k}})] \in \mathfrak{E}_\dagger$ are given, we can pick a $k \in \mathbb{N}$ such that $x \in L_k^\circ$. Then there is a neighborhood V_x of x in X such that $V_x \subseteq L_k$, and then $V_x \cap V_{n_1 \cdots n_k} = V_x \cap H_{n_1 \cdots n_k} \subseteq V_x \cap (X \backslash L_k) = \emptyset$.

To see that the space is compact, let $(z_\lambda)_\Lambda$ be a universal net. If (z_λ) is eventually in X, we proceed as follows: If (z_λ) is eventually in some compact set of X, (z_λ) converges here, and hence in $X \cup \mathfrak{E}_\dagger$. We may hence assume that when K is compact, (z_λ) is eventually in $X \backslash K$. By Proposition 10.1.9(iv), (z_λ) is eventually in the union of the unbounded components of $X \backslash L_1$, and since there is only a finite number of those by Proposition 10.1.9(i), it is eventually in one of them. We can thus pick $n_1 \in \mathbb{N}$ and $\lambda_1 \in \Lambda$ such that when $\lambda \geq \lambda_1$, $z_\lambda \in H_{n_1}$. Applying this argument again, (z_λ) is eventually in $H_{n_1 n_2}$ for some n_2, and we can choose a non-trivial sequence $(H_{n_1 \cdots n_k})$ and $\lambda_k \in \Lambda$ such that

$$\lambda \geq \lambda_k \Longrightarrow z_\lambda \in H_{n_1 \cdots n_k} \subseteq V_{n_1 \cdots n_k}.$$

By definition of the neighborhood system of $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_\mathfrak{k}})]$, $z_\lambda \to \mathfrak{e}$. Now assume (z_λ) is eventually in $\mathfrak{E}_\dagger$. We use the fact that for fixed k, the finitely many sets $\mathfrak{E}_\dagger \cap \mathfrak{V}_{\mathfrak{n}_1 \cdots \mathfrak{n}_\mathfrak{k}}$ form a disjoint cover of $\mathfrak{E}_\dagger$, and get a non-trivial sequence $(H_{n_1 \cdots n_k})$ and $\lambda_1 \leq \lambda_2 \leq \cdots$ such that

$$\lambda \geq \lambda_k \Longrightarrow z_\lambda \in \mathfrak{E}_\dagger \cap \mathfrak{V}_{\mathfrak{n}_1 \cdots \mathfrak{n}_\mathfrak{k}} \subseteq \mathfrak{V}_{\mathfrak{n}_1 \cdots \mathfrak{n}_\mathfrak{k}}.$$

When $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_\mathfrak{k}})]$, $z_\lambda \to \mathfrak{e}$. As X is clearly dense in $X \cup \mathfrak{E}_\dagger$, we get a Hausdorff compactification of X this way. ■

We denote the closure of a set Y in $X \cup \mathfrak{E}_\dagger$ by $\overline{Y}^\dagger$.

Proposition 10.2.4 *Let G be an open set with compact boundary in the non-compact Raum X. When $\mathfrak{e} \in \mathfrak{E}_\dagger$, we have*

$$\mathfrak{e} \subseteq \mathfrak{G} \Longleftrightarrow \mathfrak{e} \in \overline{\mathfrak{G}}^\dagger.$$

Proof: We write $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_\mathfrak{k}})]$. If $H_{n_1 \cdots n_k} \subseteq G$ for some k, G meets all neighborhoods $V_{n_1 \cdots n_k}$ of $\mathfrak{e}$, and $\mathfrak{e} \in \overline{\mathfrak{G}}^\dagger$. If $\mathfrak{e} \in \overline{\mathfrak{G}}^\dagger$, every set $H_{n_1 \cdots n_k} = V_{n_1 \cdots n_k} \cap X$ meets G. By the definition of (†),

$$\bigcap_{k=1}^{\infty} (\partial G \cap \overline{H_{n_1 \cdots n_k}}) = \emptyset,$$

and by compactness of ∂G, $\partial G \cap \overline{H_{n_1 \cdots n_k}} = \emptyset$ for some $k \in \mathbb{N}$. By Lemma B.1.9, we conclude from the above that $X \backslash G \cap \overline{H_{n_1 \cdots n_k}} = \emptyset$, i.e. $H_{n_1 \cdots n_k} \subseteq \overline{H_{n_1 \cdots n_k}} \subseteq G$. ■

Corollary 10.2.5 $\mathfrak{E} = \mathfrak{E}_\dagger$.

Proof: Let (G_l) be any sequence determining an end of X. The sets $(\overline{G_l}^{\dagger})$ form a decreasing sequence of non-empty sets, and since $X \cup \mathfrak{E}_{\dagger}$ is compact, their intersection is not empty. Since, by the definition of sequences determining ends, cf. (1) above,

$$X \cap \bigcap_{l=1}^{\infty} \overline{G_l}^{\dagger} = \bigcap_{l=1}^{\infty} \overline{G_l} = \emptyset,$$

there is a point $\mathfrak{e}$ of $\mathfrak{E}_{\dagger}$ in $\bigcap_{l=1}^{\infty} \overline{G_l}^{\dagger}$, and hence in $\overline{G_l}^{\dagger}$ for all $l \in \mathbb{N}$. By the above proposition, $\mathfrak{e} \subseteq \mathfrak{G}_{\mathfrak{l}}$ for all l, and $\mathfrak{e} = [(\mathfrak{G}_{\mathfrak{l}})]$ by Lemma 10.1.3. ■

We have now seen that the above construction does not depend on the choice of nest, and may restate Proposition 10.2.3 as follows:

Theorem 10.2.6 *Let X be a Raum. There is a topology making $X \cup \mathfrak{E}$ a Hausdorff compactification of X such that $\mathfrak{E}$ is totally disconnected and second countable in the relative topology. Furthermore, every neighborhood V of a point $\mathfrak{e} \in \mathfrak{E}$ contains an open neighborhood U of $\mathfrak{e}$ such that $U \cap X$ is connected.*

Proof: We choose a nest (†) and do the construction of Proposition 10.2.3. Recall that a basic neighborhood of an end $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}}})]$ is denoted by $V_{n_1 \cdots n_k}$ and has the form given in (3). To see that $\mathfrak{E}$ is totally disconnected, let $\mathfrak{e}_0, \mathfrak{e}_1 \in \mathfrak{E}$ be given by $(H_{n_1^0 \cdots n_k^0})$ and $(H_{n_1^1 \cdots n_k^1})$, respectively. If $\mathfrak{e}_0 \neq \mathfrak{e}_1$, then there is a smallest $k' \in \mathbb{N}$ such that $n_{k'}^0 \neq n_{k'}^1$. By definition of the neighborhood systems of points in $\mathfrak{E}$, the sets $\mathfrak{E} \cap \mathfrak{V}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}}}$ are open relatively to $\mathfrak{E}$. Since they are disjoint, and there is only a finite number of them, they are also relatively closed, and as $\mathfrak{e}_0$ and $\mathfrak{e}_1$ lie in different such sets when $k \geq k'$, they must lie in different components of $\mathfrak{E}$. We conclude that $\mathfrak{E}$ is totally disconnected. For the second countability of $\mathfrak{E}$, note that as there are only a finite number of indices such that $H_{n_1 \cdots n_k} \neq \emptyset$ for some fixed k, the set of basic neighborhoods $V_{n_1 \cdots n_k}$ is countable.

The last statement follows directly from the definitions; when $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}}})]$, V contains $V_{n_1 \cdots n_k}$ for some $k \in \mathbb{N}$. By definition, $V_{n_1 \cdots n_k} \cap X = H_{n_1 \cdots n_k}$ is connected in X as well as in $X \cup \mathfrak{E}$. ■

We will refer to this compactification as *the Freudenthal compactification* or φX[1] in the future, and we shall see shortly that φX is uniquely determined as the maximal compactification with a totally disconnected adjoined set.

Remark 10.2.7 As X is dense in φX, we get for an open set U, that $U \subseteq \overline{U \cap X}^{\varphi}$. Applying this to the set U mentioned in Theorem 10.2.6, we get that also U itself

[1] I have taken this from the literature, but if "φ" is for "Freudenthal", is "β" for "Stone" or "Čech"? I don't get it!

is connected, and φX is locally connected, hence a Raum. The condition given is stronger that requiring a compactification to be locally connected; for instance is $\alpha\mathbb{R}$ locally connected, but it does not have this property.

We end this section by a formula for the number of ends of a Raum X.

Corollary 10.2.8 *Let X be a Raum equipped with a nest (†). We have*

$$\#_e X = \sup_{k\in\mathbb{N}} \#_{uc}(X\backslash L_k) = \lim_{k\in\mathbb{N}} \#_{uc}(X\backslash L_k).$$

Proof: By Proposition 10.1.9, $\#_{uc}(X\backslash L_k) \leq \#_{uc}(X\backslash L_{k+1})$, and the limit exists and coincides with the supremum. As the number of unbounded components of $X\backslash L_k$ is the same as the number of indices $(n_1,\ldots,n_k)$ with $H_{n_1\cdots n_k} \neq \emptyset$, the result follows from Corollary 10.2.5. ■

10.3 Further results

We now take a closer look at our new compactification.

Lemma 10.3.1 *Let X be a Raum with compact nest (†). When $(X \hookrightarrow \gamma X)$ is a compactification, we have*

$$\bigcup_{\underline{n}\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} \overline{H_{n_1\cdots n_k}}^{\gamma} = \gamma X\backslash X$$

where all sets $\bigcap_{k=1}^{\infty} \overline{H_{n_1\cdots n_k}}^{\gamma}$ are connected, and non-empty when $(H_{n_1\cdots n_k})$ is not trivial.

Proof: We always have

$$\bigcap_{k=1}^{\infty} \overline{H_{n_1\cdots n_k}}^{\gamma} \cap X = \bigcap_{k=1}^{\infty} \overline{H_{n_1\cdots n_k}} = \emptyset,$$

such that every set $\bigcap_{k=1}^{\infty} \overline{H_{n_1\cdots n_k}}^{\gamma}$ is contained in $\gamma X\backslash X$. On the other hand, let $\mathfrak{g} \in \gamma X\backslash X$ be given. Letting $\mathcal{N}$ be the set of open neighborhoods of $\mathfrak{g}$ in γX, we have that

$$\mathcal{N}_X = \{U \cap X | U \in \mathcal{N}\}$$

has the finite intersection property since X is dense, whence from compactness of φX,

$$\bigcap_{V\in\mathcal{N}_X} \overline{V}^{\varphi} \neq \emptyset,$$

whereas $\bigcap_{\mathcal{N}_X} \overline{V} = \emptyset$ by the fact that γX is Hausdorff. We conclude that for some non-trivial sequence $(H_{n_1\cdots n_k})$, $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1\cdots\mathfrak{n}_\mathfrak{k}})] \in \bigcap_{\mathcal{N}_\mathfrak{X}} \overline{\mathfrak{V}}^{\varphi}$, and by definition, every $V \in \mathcal{N}_X$ meets every $H_{n_1\cdots n_k}$, proving that $\mathfrak{g} \in \overline{H_{n_1\cdots n_k}}^{\gamma}$ for every $k \in \mathbb{N}$. Proving connectedness is trivial, and when all $H_{n_1\cdots n_k}$ are non-empty, so are all $\overline{H_{n_1\cdots n_k}}^{\gamma}$, and hence $\bigcap_{k=1}^{\infty} \overline{H_{n_1\cdots n_k}}^{\gamma}$ by compactness. ■

The point of the above lemma is that we may reach all components of any compactification of X by means of our sets $H_{n_1\cdots n_k}$. We use this result to give a characterization of φX.

Proposition 10.3.2 *Let X be a Raum with compact nest (†). When $(X \hookrightarrow \gamma X)$ is a compactification such that $\gamma X \backslash X$ is totally disconnected, then $(X \hookrightarrow \varphi X) \geq (X \hookrightarrow \gamma X)$.*

Proof: We define $\kappa : \varphi X \to \gamma X$. When $x \in X$, $\kappa(x) = x$. When $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1\cdots\mathfrak{n}_\mathfrak{k}})]$ is given, we note that for some $\mathfrak{g} \in \gamma X \backslash X$,

$$\bigcap_{k=1}^{\infty} \overline{H_{n_1\cdots n_k}}^{\gamma} = \{\mathfrak{g}\}$$

by Lemma 10.3.1 above, and put $\kappa(\mathfrak{e}) = \mathfrak{g}$.

To see that κ is continuous, let U be an open neighborhood of $\mathfrak{g} = \kappa([(H_{n_1\cdots n_k})])$. We have that $\mathfrak{g} \notin \partial_\gamma U$, and conclude from

$$\bigcap_{k=1}^{\infty} (\partial_\gamma U \cap \overline{H_{n_1\cdots n_k}}^{\gamma}) = \partial_\gamma U \cap \{\mathfrak{g}\} = \emptyset$$

by compactness that $\partial_\gamma U \cap \overline{H_{n_1\cdots n_k}}^{\gamma} = \emptyset$ for some $k \in \mathbb{N}$. As $\mathfrak{g} \in U \cap \overline{H_{n_1\cdots n_k}}^{\gamma}$, we deduce from connectedness of $\overline{H_{n_1\cdots n_k}}^{\gamma}$ that $H_{n_1\cdots n_k} \subseteq \overline{H_{n_1\cdots n_k}}^{\gamma} \subseteq U$ according to Lemma B.1.9. When $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{m}_1\cdots\mathfrak{m}_\mathfrak{k}})] \in \mathfrak{V}_{\mathfrak{n}_1\cdots\mathfrak{n}_\mathfrak{k}}$, we have $m_i = n_i$ for $i \in \{1, \ldots k\}$ by Lemma 10.2.2(ii), whence $\kappa(\mathfrak{e}) \in \overline{\mathfrak{H}_{\mathfrak{n}_1\cdots\mathfrak{n}_\mathfrak{k}}}^{\gamma}$ by definition. We conclude that $\kappa(V_{n_1\cdots n_k}) \subseteq U$, and κ is continuous. ■

Corollary 10.3.3 *Let X be a Raum. The Freudenthal compactification $(X \hookrightarrow \varphi X)$ is the maximal compactification such that $\varphi X \backslash X$ is totally disconnected.*

Remark 10.3.4 In the above proposition, assume further that γX has the same property as φX; that for every neighborhood V of $\mathfrak{g} \in \gamma X \backslash X$, there is an open neighborhood U of $\mathfrak{g}$ with $U \cap X$ connected. Then κ is 1–1, for with W_k the union of all $H_{n_1\cdots n_k}$, $X \backslash W_k$ is compact by Proposition 10.1.9(iv) and $V_k = \gamma X \backslash (X \backslash W_k) =$

$(\gamma X\backslash X)\cup W_k$ is open. Fixing $\mathfrak{g}\in\gamma X\backslash X$ and taking a neighborhood $U_k\subseteq V_k$ of $\mathfrak{g}$ as above, $U_k\cap X$ must be contained in $H_{n_1\cdots n_k}$ for some index $(n_1,\ldots,n_k)$, and $\mathfrak{g}\in\overline{H_{n_1\cdots n_k}}^{\gamma}$ for only one tuple $\underline{n}\in\mathbb{N}^{\mathbb{N}}$. We conclude that $(X\hookrightarrow\varphi X)=(X\hookrightarrow\gamma X)$, giving a different characterization of φX.

The $H_{n_1\cdots n_k}$ even capture the component structure of the maximal compactification βX.

Proposition 10.3.5 *Let X be a Raum with compact nest (†). Every component of $\beta X\backslash X$ has the form*

$$\bigcap_{k=1}^{\infty}\overline{H_{n_1\cdots n_k}}^{\beta}$$

for one and only one non-trivial sequence $(H_{n_1\cdots n_k})$.

Proof: By Lemma 10.3.1, we only need to prove that when $\underline{n}^0\neq\underline{n}^1$, then $\overline{H_{n_1^0\cdots n_{k'}^0}}^{\beta}$ and $\overline{H_{n_1^1\cdots n_{k'}^1}}^{\beta}$ can be separated for some $k'\in\mathbb{N}$, for then every connected set $\bigcap_{k=1}^{\infty}\overline{H_{n_1\cdots n_k}}^{\beta}$ is maximal commutative. We have that $H_{n_1^0\cdots n_k^0}\cap H_{n_1^1\cdots n_k^1}=\emptyset$ for some k. In $X\backslash L_{k+1}^{\circ}$ the set $G=H_{n_1^1\cdots n_k^1}\backslash L_{k+1}^{\circ}$ is clopen, such that $1_G\in C^b(X\backslash L_{k+1}^{\circ})$. We may extend 1_G to $f\in C^b(X)$ by Tietze's extension theorem, and note that

$$f(H_{n_1^0\cdots n_{k+1}^0})=\{0\},\qquad f(H_{n_1^1\cdots n_{k+1}^1})=\{1\}.$$

Letting f denote the extension of f to βX as well, $\overline{H_{n_1^0\cdots n_{k+1}^0}}^{\beta}\subseteq f^{-1}(\{0\})$ and $\overline{H_{n_1^1\cdots n_{k+1}^1}}^{\beta}\subseteq f^{-1}(\{1\})$, and we may separate by $f^{-1}(U_0)$ and $f^{-1}(U_1)$, where the U_0,U_1 are disjoint open neighborhoods of $0,1$ in $\mathbb{C}$. We take $k'=k+1$. ■

The above proposition relates the component structure of $\beta X\backslash X$ to that of $\varphi X\backslash X$. That is also the content of Proposition 10.3.7 below, but I have not been able to prove one of these propositions using the other. For reference purposes, we single out a lemma in the language of bounded $\mathfrak{A}$-valued functions, where $\mathfrak{A}$ is a C^*-algebra. In this section, we will only need this result with $\mathfrak{A}=\mathbb{C}$, in which case $\mathrm{sp}(f(x))$ below may be replaced by $\{f(x)\}$.

Lemma 10.3.6 *Let a locally compact Hausdorff space X, a C^*-algebra $\mathfrak{A}$, $1/2>\varepsilon>0$ and $f\in\mathcal{B}(X,\mathfrak{A})_{sa}$ be given.*

(i) If f^2-f is norm continuous at x_0, there is a neighborhood U of x_0 such that

$$\mathrm{sp}(f(x))\subseteq(-\varepsilon,\varepsilon)\cup(1-\varepsilon,1+\varepsilon),\qquad\forall x\in U.$$

(ii) If $f^2 - f \in C_0(X, \mathfrak{A})$, there is a compact set K such that

$$\mathrm{sp}(f(x)) \subseteq (-\varepsilon, \varepsilon) \cup (1-\varepsilon, 1+\varepsilon), \qquad \forall x \in X \backslash K.$$

Proof: We may choose $\delta > 0$ with

$$\sqrt{1+4\delta} \leq 1 + 2\varepsilon, \qquad \sqrt{1-4\delta} \geq 1 - 2\varepsilon.$$

In (i), choose U such that $\|f^2(x) - f(x)\| < \delta$ for all $x \in U$, and note that if $\lambda \in \mathrm{sp}(f(x))$ for $x \in U$, we must have $|\lambda^2 - \lambda| \leq \delta$, whence by two quadratic equations,

$$\begin{aligned} \lambda &\in \left(\frac{1-\sqrt{1+4\delta}}{2}, \frac{1-\sqrt{1-4\delta}}{2}\right) \cup \left(\frac{1+\sqrt{1-4\delta}}{2}, \frac{1+\sqrt{1+4\delta}}{2}\right) \\ &\subseteq (-\varepsilon, \varepsilon) \cup (1-\varepsilon, 1+\varepsilon). \end{aligned}$$

The second claim follows by the first by regarding the one-point compactification of X. ■

Proposition 10.3.7 *Let X be a Raum and $p \in C^b(X)/C_0(X)$ a projection. There is an $f \in C(\varphi X)$ such that $p = f\,|_X + C_0(X)$.*

Proof: Write $p = f + C_0(X)$, where we may assume that $0 \leq f \leq 1$. Also, $f^2 - f \in C_0(X)$, so there exists a compact set K such that $f(x) \in [0, 1/3) \cup (2/3, 1]$ for all $x \in X \backslash K$. Choosing a nest (†), we get a $k' \in \mathbb{N}$ with $K \subseteq L_{k'}^{\circ}$ and consequently

$$f(x) \in \left[0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right], \qquad x \in X \backslash L_{k'},$$

so we can extend f to $\tilde{f}$ on φX by means of the canonical representation of ends as follows:

$$\tilde{f}(\mathfrak{e}) = \begin{cases} 0, & \mathfrak{e} \subseteq \mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}'}} \text{ where } \mathfrak{f}(\mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}'}}) \subseteq [0, 1/3) \\ 1, & \mathfrak{e} \subseteq \mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}'}} \text{ where } \mathfrak{f}(\mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}'}}) \subseteq (2/3, 1] \end{cases}.$$

We have used Corollary 10.2.5 and the fact that every $H_{n_1 \cdots n_{k'}}$ is connected and contained in $X \backslash L_{k'}$. We need to prove that $\tilde{f}$ is continuous, which of course reduces to proving that $\tilde{f}$ is continuous at every $\mathfrak{e} = [(\mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}}})]$ in $\mathfrak{E}$. Assume that $\tilde{f}(\mathfrak{e}) = 0$ and pick for given $1/3 > \varepsilon > 0$ a $k'' \geq k'$ such that

$$f(x) \in [0, \varepsilon) \cup (1-\varepsilon, 1], \qquad x \in X \backslash L_{k''}$$

as above. There is a tuple $(n_1, \ldots, n_{k'}, \ldots, n_{k''})$ such that $\mathfrak{e} \subseteq \mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}'} \cdots \mathfrak{n}_{\mathfrak{k}''}}$, and since $\tilde{f}(\mathfrak{e}) = 0$ we get

$$f(H_{n_1 \cdots n_{k'} \cdots n_{k''}}) \subseteq f(H_{n_1 \cdots n_{k'}}) \subseteq [0, 1/3),$$

and $f(H_{n_1 \cdots n_{k''}}) \subseteq [0, \varepsilon)$. Since also $\tilde{f}(\mathfrak{e}') = 0$ for every $\mathfrak{e}' \subseteq \mathfrak{H}_{\mathfrak{n}_1 \cdots \mathfrak{n}_{\mathfrak{k}''}}$, we get that also $\tilde{f}(V_{n_1 \cdots n_{k''}}) \subseteq [0, \varepsilon)$, and $\tilde{f}$ is continuous at $\mathfrak{e}$. The argument for $\tilde{f}(\mathfrak{e}) = 1$ is similar. ■

Remark 10.3.8 Let X be a Raum. A consequence of Proposition 10.3.5 is that $\mathfrak{c}_\mathfrak{c}(\beta\mathfrak{X}\backslash\mathfrak{X}) = \mathfrak{c}_\mathfrak{c}(\varphi\mathfrak{X}\backslash\mathfrak{X})$. Proposition 10.3.7 gives another way of relating the component structure of $\beta X\backslash X$ to that of $\varphi X\backslash X$, establishing a lattice isomorphism between the lattices of projections on the two spaces, cf. Remark 7.4.4. Indeed, for every compactification $(X \hookrightarrow \gamma X)$, we can choose $\kappa : \beta X \to \gamma X$ restricting to the identity on X, and when p is a projection in $C(\gamma X\backslash X)$, it extends to $\tilde{p} \in C(\gamma X)$. The mapping $\Phi : \mathrm{Lat}(C(\gamma X\backslash X)) \to \mathrm{Lat}(C(\beta X\backslash X))$ given by

$$p \mapsto \tilde{p} \circ \kappa\,|_{\beta X\backslash X}$$

is well-defined, for if $\tilde{q}$ is another extension, $\tilde{p} - \tilde{q}$ vanishes at infinity such that $(\tilde{p} - \tilde{q}) \circ \kappa$ restricts to zero on $\beta X\backslash X$. By similar arguing, we get from the fact that $\tilde{p} - \tilde{p}^2$ vanishes at infinity that $\Phi(p)$ is in fact a projection. When $p \leq q$ we can choose extensions with $\tilde{p} \leq \tilde{q}$ (replace $\tilde{q}$ by $\tilde{p} \vee \tilde{q}$ if necessary), and $\Phi(p) \leq \Phi(q)$, proving that Φ is an order isomorphism. It is also clear that Φ is 1–1.

In the case of φX, we now have that Φ is onto, for if $r \in C(\beta X\backslash X)$ is a projection, then with $g \in C^b(X)$ extending to r, $p = g + C_0(X)$ is a projection of $C^b(X)/C_0(X)$. We choose $f \in C(\varphi X)$ as in Proposition 10.3.7 and note that $f\,|_{\varphi X\backslash X}$ is a projection. As $g - f\,|_X \in C_0(X)$, $\Phi(f\,|_{\varphi X\backslash X}) = r$.

The final result of this section is not directly related to the above results.

Proposition 10.3.9 *Let X, Y be Räume with Y not compact. Letting $Z = X \times Y$, Z is a Raum, and*

(i) If X is compact, then $\varphi Z\backslash Z$ and $\varphi X\backslash X$ are homeomorphic.

(ii) If X is not compact, then $\#_e Z = 1$.

Proof: That Z is a Raum is obvious, using Lemma B.1.2(iv) and B.2.7. In (i), let (†) be a nest for Y; $X \times L_n$ forms a nest for Z, and we may arrange the unbounded components as

$$\tilde{H}_{n_1\cdots n_k} = X \times H_{n_1\cdots n_k}.$$

according to (the proof of) Lemma B.1.5, where the axioms (i)–(iv) from Section 10.2 are satisfied for the $\tilde{H}_{n_1\cdots n_k}$, because they are satisfied for the $H_{n_1\cdots n_k}$. According to Lemma 10.2.2(iii), the mapping $\kappa : \varphi X\backslash X \to \varphi Z\backslash Z$ given by

$$\kappa([(H_{n_1\cdots n_k})]) = [(\tilde{H}_{n_1\cdots n_k})]$$

is a well-defined bijection. By compactness, we need only prove it is continuous, and that follows from

$$\kappa^{-1}(\tilde{V}_{m_1\cdots m_k}\backslash Z) = \{[(H_{n_1\cdots n_k})] | n_i = m_i, 1 \leq i \leq k\} = V_{n_1\cdots n_k}\backslash X.$$

For (ii), let K_k be a compact nest for X; clearly $K_k \times L_k$ is one for Z, and we need only prove that $Z \backslash (K_k \times L_k)$ is connected for every $k \in \mathbb{N}$ according to Corollary 10.2.8. As X and Y are not compact, we can choose $x_k \in X \backslash K_k$ and $y_k \in Y \backslash L_k$ and write

$$\begin{aligned} Z \backslash (K_k \times L_k) &= \left(\bigcup_{x \notin K_k} \{x\} \times Y \right) \cup \left(\bigcup_{y \notin L_k} X \times \{y\} \right) \\ &= \left(\bigcup_{x \notin K_k} [\{x\} \times Y \cup X \times \{y_k\}] \right) \cup \left(\bigcup_{y \notin L_k} [Y \times \{y\} \cup \{x_k\} \times Y] \right). \end{aligned}$$

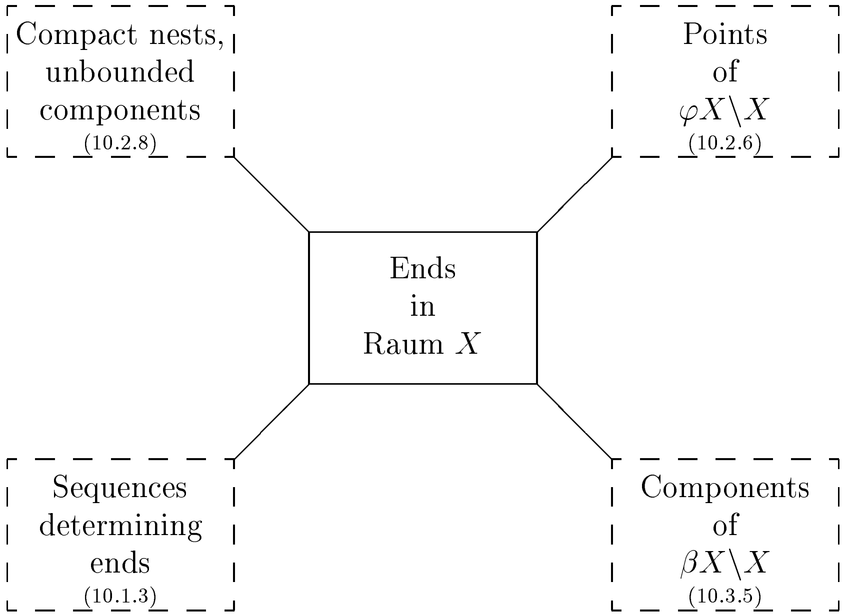
This set is connected by five applications of Lemma B.1.2(i). ■

11 On non-commutative end theory

It has been the hope of the author to end this thesis by an introduction to a non-commutative version of the theory of ends, but as deadlines draw nearer it becomes increasingly clear, that even finding good definitions may be a bit too ambitious for a project like this. In this section, we will discuss some possible ways of attacking the problem, taking into our discussion only C^*-algebras of a certain, simple form. The purpose is to show some applications of the theory developed in Part II, as well as outlining where the author hopes to take this project eventually, in collaboration with professor C.A. Akemann.

11.1 Introduction

As described in Section 10, the commutative end theory involves certain aspects as illustrated schematically below.



The first question that arises is how we generalize the notion of a Raum. Most of the axioms are easily translated, but more information is needed on the notion of local connectivity, and in what form, if any, we can find generalizations of Lemma 10.1.7. We will not address this problem here, but instead restrict ourselves to the following classes of C^*-algebras, in which no problems seem to arise:

(I) $\mathfrak{A} = C_0(X)$, X a non-compact Raum.

(II) $\mathfrak{A}$ primitive and σ-unital.

(III) $\mathfrak{A} = C_0(X) \otimes M_k(\mathbb{C})$, X a non-compact Raum.

(IV) $\mathfrak{A} = C_0(X) \otimes \mathbb{K}(\mathfrak{H})$, X a non-compact Raum, $\mathfrak{H}$ separable.

Note, however, that all such C^*-algebras are locally connected in the sense of Section 9.2, cf. Example 9.2.6 and Proposition 9.2.7.

11.2 Corona components

In a sense, the easiest way of generalizing the notion of ends to non-commutative C^*-algebras goes through Proposition 10.3.5 and the idea that an end corresponds to a component of $\beta X \backslash X$. Indeed, in this case there is a standard object we can study the component structure of; the *corona algebra* $M(\mathfrak{A})/\mathfrak{A}$, which in the commutative case is just the algebra over $\beta X \backslash X$. We may thus benefit from the fact that this is a fairly well understood object, but pay the prize that the correspondences to the component structure "at infinity" for $\mathfrak{A}$ are not manifest in any way.

Remark 11.2.1 Regarding case (II) in the previous section, there is evidence that for a reasonable class of primitive, σ-unital C^*-algebras $\mathfrak{A}$, $M(\mathfrak{A})/\mathfrak{A}$ is connected. However, we have yet to study the literature for results on the ideal structure of coronas of such C^*-algebras, and only know that in some classes of AF algebras, e.g. in the *matroid*[2] algebras, there is always a minimal non-zero ideal in the corona, so that this algebra must be connected. For this, cf. [13, 3.1,3.2], for results on another class of AF algebras, cf. [19, 2].

Examples are known of simple C^*-algebras with $M(\mathfrak{A})/\mathfrak{A}$ non-connected; indeed, every finite C^*-algebra – e.g. $\mathbb{C}\oplus\mathbb{C}$ – can be obtained as the corona of some hereditary subalgebra of a II_1 factor, as described in [24, Corollary 4]. But an $\mathfrak{A}$ with $M(\mathfrak{A})/\mathfrak{A}$ can not be σ-unital, as [6, 2.7] shows.

We now turn to the cases (III) and (IV) from Section 11.1. When X is a locally compact Hausdorff space and $\mathfrak{A}$ is an arbitrary C^*-algebra, as we have already seen, we may identify $\mathfrak{B} = C_0(X)\otimes\mathfrak{A}$ with $C_0(X,\mathfrak{A})$; the norm-continuous, $\mathfrak{A}$-valued functions vanishing at infinity. According to [6, 3.4], $M(\mathfrak{B})$ can be identified with $C^b(X, M(\mathfrak{A})_\beta)$; the *strictly* continuous, norm bounded, $M(\mathfrak{A})$-valued functions on X. The strict topology on $M(\mathfrak{A})$ is given by the family of seminorms

$$x \mapsto \|ax\| + \|xa\|, \qquad a \in \mathfrak{A}.$$

Under that identification, $\mathfrak{B}$ sits as $C_0(X,\mathfrak{A})$ in $C^b(X, M(\mathfrak{A})_\beta)$. We are interested in the component structure of $M(\mathfrak{B})/\mathfrak{B}$, and as this is a unital C^*-algebra, we need only find the central projections to determine this. Under the mentioned identifications, a central projection p has the form $f + C_0(X,\mathfrak{A})$, where $f \in C^b(X, M(\mathfrak{A})_\beta)$ satisfies

$$f^2 - f \in C_0(X,\mathfrak{A}) \tag{4}$$

and

$$\forall g \in C^b(X, M(\mathfrak{A})_\beta) : fg - gf \in C_0(X,\mathfrak{A}). \tag{5}$$

Note that although f itself might not be norm continuous, $f^2 - f$ as well as every $fg - gf$ must be. This will prove very important in the reasoning to come. First a central lemma:

Lemma 11.2.2 *Let X be a Raum, $\mathfrak{A}$ a C^*-algebra, and let sequences (x_n) in X, (a_n) in $M(\mathfrak{A})$ with $\|a_n\|$ bounded be given. Under either of the additional assertions*

(i) $x_n \to \infty$,

[2] Inductive limits of simple matrix algebras (with possibly non-unital mappings)

(ii) X metric; $x_n \to x_0 \in X$, $x_n \neq x_0$, $a_n \xrightarrow{\beta} 0$,

there exists $f \in C^b(X, M(\mathfrak{A})_\beta)$ such that for some infinite set $\{n_i\}$ in $\mathbb{N}$,

$$f(x_{n_i}) = a_{n_i}.$$

Proof: Let (L_k) be a compact nest for X. In the first case, because (x_n) tends to infinity, the sequence can not be contained in L_k for any k. We can choose $k_1 < k_2 < \cdots$ and $n_1 < n_2 < \cdots$ such that

$$x_{n_i} \in V_i = L_{k_i}^\circ \backslash L_{k_{i-1}}.$$

Applying normality of X, we get a family of functions $f_i \in C^b(X)$ such that $f_i(x_{n_i}) = 1, \|f_i\| = 1$ and $f_i\,|_{X\backslash V_i} = 0$. Notice that since the V_i are disjoint by construction, at most one f_i is non-zero at a fixed point of X. We can thus define an element of $C^b(X, M(\mathfrak{A})_\beta)$ by

$$f = \sum_{i=1}^{\infty} a_{n_i} f_i,$$

where f is continuous at a point $x' \in X$ because a compact neighborhood U of x' is contained in $L_{k'}^\circ$ for some k', so that f in this neighborhood is a finite combination of norm continuous functions, hence strictly continuous here.

In the second case, let d be a metric on X and denote by $B(x_0, \varepsilon)$ the d-ball of radius ε around x_0. There exist $\alpha_1 > \alpha_2 > \cdots \searrow 0$ and $n_1 < n_2 < \cdots$ such that

$$x_{n_i} \in V_i = B(x_0, \alpha_{i-1}) \backslash \overline{B(x_0, \alpha_i)}, \qquad i \in \mathbb{N}.$$

As above, we can pick $f_i \in C^b(X)$ such that $f_i(x_{n_i}) = 1, \|f_i\| = 1$, and $f_i\,|_{X\backslash V_i} = 0$, and define

$$f = \sum_{i=1}^{\infty} a_i f_i.$$

For $x' \in X\backslash\{x_0\}$, take i' such that $\alpha_{i'} < d(x_0, x')$. With $\varepsilon = d(x_0, x') - \alpha_{i'}$, $B(x', \varepsilon) \subseteq X\backslash B(x_0, \alpha_{i'})$, so that on this open set,

$$f(x) = \sum_{i=1}^{i'} a_i f_i(x),$$

and f is (norm, hence strictly) continuous at x'. To see that f is strictly continuous at x_0, let $b \in \mathfrak{A}$ and $\varepsilon > 0$ be given. Choose n_0 such that

$$n \geq n_0 \Longrightarrow \|a_n b\| + \|b a_n\| < \varepsilon,$$

and take i_0 with $n_{i_0} \geq n_0$. For $x \in B(X_0, \alpha_{i_0})$, $f(x) = \beta a_{n_i}$ for some $\beta \in [0,1]$ and $n_i \geq n_{i_0} \geq n_0$, so for such x,

$$\|f(x)b\| + \|bf(x)\| \leq \beta(\|a_{n_i}b\| + \|ba_{n_i}\|) < \beta\varepsilon \leq \varepsilon.$$

With continuity established, it is clear that the f have the desired properties. ∎

Proposition 11.2.3 *Let X be a Raum, $\mathfrak{A}$ a connected, unital C^*-algebra, and let $\mathfrak{B} = C_0(X) \otimes \mathfrak{A} \simeq C_0(X, \mathfrak{A})$. If $p \in M(\mathfrak{B})/\mathfrak{B}$ is a central projection, there exists $f \in C(\varphi X)$ such that $p = f\,|_X + \mathfrak{B}$.*

Proof: Let (L_k) be a compact nest for X. Writing $p = f' + \mathfrak{B}$, we may assume that $0 \leq f' \leq 1$, and by (4) combined with Lemma 10.3.6(ii), we can find $k \in \mathbb{N}$ such that

$$\operatorname{sp}(f'(x)) \subseteq [0, 1/3) \cup (2/3, 1], \qquad x \in X \backslash L_k.$$

By functional calculus, $1_{[2/3,1]}(f'(x))$ is defined for all $x \in X \backslash L_k$, and the function $g = 1_{[2/3,1]} \circ f'\,|_{X \backslash L_k}$ lies in $C^b(X \backslash L_k, \mathfrak{A})$, cf. [23, Exercise 4.4.1].

Let

$$Z = \{x \in X \backslash L_k | g(x) \notin \{0, 1\}\}$$

If $\overline{Z}$ is not compact, then by σ-compactness of X there is a sequence (x_n) in Z converging to infinity. As $g(x_n)$ is a non-central projection, there exists an irreducible representation $(\pi_n, \mathfrak{H}_\mathfrak{n})$ such that $\pi_n(g(x_n))$ is not central. We can take orthogonal unit vectors ξ_n, η_n in $\mathfrak{H}_\mathfrak{n}$ such that $\pi_n(g(x_n))\xi_n = \xi_n, \pi_n(g(x_n))\eta_n = 0$, and if we write $\mathfrak{H}_\mathfrak{n} = \mathbb{C}\xi_\mathfrak{n} \oplus \mathbb{C}\eta_\mathfrak{n} \oplus \mathfrak{H}'_\mathfrak{n}$, the operator $u_n \in \mathbb{B}(\mathfrak{H}_\mathfrak{n})$ given by the block matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is unitary. By Kadison's transitivity theorem, there is a unitary element $v_n \in \mathfrak{A}$ such that $\pi_n(v_n)$ and u_n agree on $\operatorname{span}(\xi_n, \eta_n)$, and clearly then

$$\begin{aligned} \|v_n g(x_n) - g(x_n)v_n\| &\geq \|\pi_n(v_n g(x_n) - g(x_n)v_n)\| \\ &\geq \|\pi_n(v_n)\pi_n(g(x_n))\xi_n - \pi_n(g(x_n))\pi_n(v_n)\xi_n\| \\ &= \|\eta_n - \pi_n(g(x_n))\eta_n\| \\ &= \|\eta_n\| = 1. \end{aligned}$$

By Lemma 11.2.2(i), there exist $h \in C^b(X, M(\mathfrak{A})_\beta)$ and subsequences indexed by (n_i) such that $h(x_{n_i}) = v_{n_i}$.

Letting χ be a function in $C_{00}(X)$ identically 1 on the open neighborhood L_{k+1}° of L_k, $g - \chi g$ extends with 0 to a function $f'' \in C^b(X, \mathfrak{A})$. We claim that $f' - f'' \in$

$C_0(X, \mathfrak{A})$, such that $p = f'' + \mathfrak{B}$. To see this, let $1/3 > \varepsilon > 0$ be given and pick a compact set K such that

$$\mathrm{sp}(f'(x)) \subseteq [0, \varepsilon) \cup (1 - \varepsilon, 1], \qquad \forall x \in X \backslash K$$

by Lemma 10.3.6. Letting $K' = K \cup \mathrm{supp}(\chi)$, we get for $x \in X \backslash K'$

$$\begin{aligned} \|f'(x) - f''(x)\| &= \|f'(x) - g(x)\| \\ &= \sup_{\lambda \in \mathrm{sp}(f'(x))} |1_{[2/3,1]}(\lambda) - \lambda| \\ &\leq \varepsilon. \end{aligned}$$

By (4), $f''h - hf'' \in C_0(X, \mathfrak{A})$. On the other hand, for $i \geq i'$ where $\mathrm{supp}(\chi) \subseteq L^\circ_{k_{n_{i'}}}$, $\|f''(x_{n_i})g(x_{n_i}) - g(x_{n_i})f''(x_{n_i})\| \geq 1$. Since $(x_{n_i})_{i \geq i'}$ is not contained in a compact set, we get a contradiction.

We may thus assume that $Z \subseteq L^\circ_{k'}$ for some $k' \geq k$, and by taking $\chi' \in C_{00}(X)$ identically 1 on this set and letting f be the extension with 0 of $g - g\chi'$ to X, we get $p = f + \mathfrak{B}$ where $f(x) \in \mathbb{C}1$ for all $x \in X$. Regarding f as a scalar-valued function on X, we get the result by Proposition 10.3.7. ■

Example 11.2.4 *If X is a Raum, and $\mathfrak{A} = C_0(X) \otimes M_n(\mathbb{C})$, then $\#_c(M(\mathfrak{A})/\mathfrak{A}) = \#_e X$.*

Proof: The above proposition establishes, as in Remark 11.2.1, a lattice isomorphism between $\mathrm{Lat}(C^b(\varphi X/X))$ and $\mathrm{Lat}(C^b(\mathrm{Prim}(M(\mathfrak{A})/\mathfrak{A})))$, and as noted in Remark 7.4.4, $\#_c(\varphi X \backslash X) = \#_c(\mathrm{Prim}(M(\mathfrak{A})/\mathfrak{A}))$. We have from Propositions 10.3.5 and 7.4.2, that

$$\#_e X = \#_c(\varphi X \backslash X) = \#_c(\mathrm{Prim}(M(\mathfrak{A})/\mathfrak{A})) = \#_c(M(\mathfrak{A})/\mathfrak{A}).$$

■

Lemma 11.2.5 *Let (p_k) be a sequence of non-zero projections of $\mathbb{K}(\mathfrak{H})$ converging strictly to 0 or 1 in $\mathbb{B}(\mathfrak{H})$. There exist an infinite set (n_i), $\varepsilon > 0$, and elements $a_i \in \mathbb{B}(\mathfrak{H})$, $\|\mathfrak{a}_i\| \leq 1$ such that $a_i \xrightarrow{\beta} 0$ and*

$$\|a_i p_{n_i} - p_{n_i} a_i\| \geq \varepsilon, \qquad \forall i \in \mathbb{N}.$$

Proof: First assume that $p_k \to 0$ and put $n_1 = 1$. As $p_{n_1} \neq 0$, there is a unit vector $\xi_1 \in p_{n_1}\mathfrak{H}$. Let q_1 be the rank one projection onto $\mathbb{C}\xi_1$, and choose n_2 such that $\|p_{n_2} q_1\| + \|q_1 p_{n_2}\| \leq 1/4$. Picking a unit vector $\eta_2 \in p_{n_2}\mathfrak{H}$, we have

$\|q_1\eta_2\| = \|q_1 p_{n_2}\eta_2\| \leq 1/4$, showing that $\|(1-q_1)\eta_2\| \geq 3/4$ and that with $\xi_2 = \|(1-q_1)\eta_2\|^{-1}(1-q_1)\eta_2$, we have

$$\begin{aligned}\|p_{n_2}\xi_2 - \xi_2\| &= \|(1-q_1)\eta_2\|^{-1}\|(p_{n_2}\eta_2 - \eta_2) - p_{n_2}q_1\eta_2 + q_1\eta_2\| \\ &\leq \frac{4}{3}(\|p_{n_2}q_1\| + \|q_1 p_{n_2}\|) \leq \frac{1}{3}.\end{aligned}$$

We may iterate this procedure to achieve a subsequence p_{n_i} and mutually orthogonal unit vectors (ξ_i) with

$$\|p_{n_{i+1}}\xi_i\| \leq \frac{1}{4}, \qquad \|p_{n_i}\xi_i - \xi_i\| \leq \frac{1}{3}, \tag{6}$$

for if $p_{n_1}, \ldots, p_{n_k}$ and $\xi_1, \ldots, \xi_k$ have been chosen, we let q_k be the finite rank projection onto the span of $\{\xi_i | i \in \{1, \ldots, k\}\}$, choose $n_{k+1} > n_k$ with $\|p_{n_{k+1}}q_k\| + \|q_k p_{n_{k+1}}\| \leq 1/4$ and proceed as above.

When $p_k \xrightarrow{\beta} 1$, we choose an orthonormal basis (ξ_m) for $\mathfrak{H}$ and note that for fixed m, $p_n\xi_m \to \xi_m$ when $n \to \infty$, for fixed n, $p_n\xi_m \to 0$ when $m \to \infty$. Using these facts alternately, we find subsequences (n_i) and (m_i) with

$$\|p_{n_{i+1}}\xi_{m_i}\| \leq \frac{1}{3}, \qquad \|p_{n_i}\xi_{m_i} - \xi_{m_i}\| \leq \frac{1}{3}. \tag{7}$$

We now combine (6) and (7) to see that in both cases, we can find a subsequence (p_{n_i}) and mutually orthogonal unit vectors (ζ_i) of $\mathfrak{H}$ such that

$$\|p_{n_{i+1}}\zeta_i\| \leq \frac{1}{3}, \qquad \|p_{n_i}\zeta_i - \zeta_i\| \leq \frac{1}{3}.$$

Letting, for $i \geq 2$, v_i be the finite rank operator with matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ in $\mathrm{span}\{\zeta_{i-1}, \zeta_i\}$, we get

$$\begin{aligned}\|p_{n_i}v_i - v_i p_{n_i}\| &\geq \|p_{n_i}v_i\zeta_i - v_i p_{n_i}\zeta_i\| \\ &= \|p_{n_i}\zeta_{i-1} - v_i(p_{n_i}\zeta_i - \zeta_i) + v_i\zeta_i\| \\ &\geq \|\zeta_{i-1}\| - \|p_{n_i}\zeta_{i-1}\| - \|v_i\|\|p_{n_i}\zeta_i - \zeta_i\| \\ &\geq 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}.\end{aligned}$$

As v_i lives only on $\mathrm{span}\{\zeta_{i-1}, \zeta_i\}$, we get $\|av_i\| + \|v_i a\| \to 0$ for all $a \in \mathbb{K}(\mathfrak{H})$ by compactness, showing the lemma to hold with $\varepsilon = 1/3$. ■

Example 11.2.6 *Let X be a metric Raum, $\mathfrak{H}$ a separable Hilbert space. With $\mathfrak{A} = C_0(X) \otimes \mathbb{K}(\mathfrak{H})$, $M(\mathfrak{A})/\mathfrak{A}$ is connected.*

Proof: We abbreviate $\mathbb{B} = \mathbb{B}(\mathfrak{H}), \mathbb{K} = \mathbb{K}(\mathfrak{H})$. Let p be a central projection in $M(\mathfrak{A})/\mathfrak{A}$ and write $p = f + \mathfrak{A}$, where $f \in C^b(X, \mathbb{B}_\beta)_{sa}$ satisfies (4) and (5) above. Let $\tilde{p} : X \to \mathbb{B}/\mathbb{K}$ be given by $\tilde{p}(x) = f(x) + \mathbb{K}$. Note that $\tilde{p}$ does not depend upon the choice of f, and that we do not claim any continuity of $\tilde{p}$. The proof will follow the program below:

1°: $\tilde{p} = 0$ or $\tilde{p} = 1$.

2°: f is norm continuous.

3°: $p = 0$ or $p = 1$.

1°: For fixed $x_0 \in X$ we get from the assumptions on f that $\tilde{p}(x_0)$ is a central projection of $\mathbb{B}/\mathbb{K}$, hence 0 or 1. Indeed, from (4), $\tilde{p}(x_0)$ is a projection, and if $a+\mathbb{K} \in \mathbb{B}/\mathbb{K}$ is given, we may regard a as a constant function to achieve $f(x_0)a - af(x_0) \in \mathbb{K}$, so that $\tilde{p}(x_0)$ and $a + \mathbb{K}$ commute. Let

$$Y_k = \{x \in X | \tilde{p}(x) = k + \mathbb{K}\}, \qquad k \in \{0, 1\}.$$

We prove that Y_0 and Y_1 are both open. If Y_1 is not open, take a sequence (x_n) in Y_0 converging to $x_0 \in Y_1$. Choose a function $\chi \in C_{00}(X)$ with $\chi(x_n) = 1, k \in \mathbb{N} \cup \{0\}$ and put $g = (1 - f(x_0))\chi$; obviously $g \in C_0(X, \mathbb{K})$ and $p = (f + g) + \mathfrak{A}$, so without loss of generality we may assume that $f(x_0) = 1$.

By (4) combined with Lemma 10.3.6(i), we may further assume that

$$\mathrm{sp}(f(x_n)) \subseteq (-1/3, 1/3) \cup (2/3, 4/3)$$

for all $n \in \mathbb{N}$ and let $p_n = 1_{[2/3,4/3]}(f(x_n))$. Then $p_n \in \mathbb{K}$ with $\|f(x_n) - p_n\| \to 0$, $n \to \infty$ (again by Lemma 10.3.6), and as $f(x_n) \xrightarrow{\beta} 1$, we get $p_n \xrightarrow{\beta} 1$. We may assume that for all $n \in \mathbb{N}$, $p_n \neq 0$, and get by Lemma 11.2.5 an $\varepsilon > 0$, a subsequence p_{n_i} and $a_i \in \mathbb{B}, \|a_i\| \leq 1$ such that $a_i \xrightarrow{\beta} 0$ and

$$\|a_i p_{n_i} - p_{n_i} a_i\| \geq \varepsilon.$$

By passing to a subsequence again (also indexed by n_i), Lemma 11.2.2(ii) yields a function $g \in C^b(X, \mathbb{B}_\beta)$ such that $g(x_{n_i}) = a_i$ and

$$\begin{aligned} \|f(x_{n_i})g(x_{n_i}) - g(x_{n_i})f(x_{n_i})\| &\geq \|p_{n_i}a_i - a_i p_{n_i}\| - 2\|f(x_{n_i}) - p_{n_i}\|\|a_i\| \\ &\geq \frac{\varepsilon}{2} \end{aligned}$$

when $i \geq i'$, where i' is chosen such that $n \geq n_{i'}$ implies $\|f(x_n) - p_n\| \leq \varepsilon/4$. However, $fg - gf$ vanishes at x_0, so $fg - gf$ is not norm continuous, contradicting the centrality of $p = f + \mathfrak{A}$, cf. (5)

By symmetry, this argument shows that each Y_i must be open, and by connectedness of X, $Y_k = X$ for some $k \in \{0,1\}$.

2°: We may assume that $\tilde{p} = 0$, in which case $f \in C^b(X, \mathbb{K}_\beta)$. If f is not norm continuous at x_0, say, we choose $x_n \to x_0$ such that $\|f(x_n) - f(x_0)\| \not\to 0$. Let $\chi \in C_{00}(X)$ be chosen such that $\chi(x_n) = 1, n \in \mathbb{N} \cup \{0\}$. By perturbing by $f(x_0)\chi$ as above, we may assume that $f(x_0) = 0$, and the assumptions amount to the existence of $f \in C^b(X, \mathbb{K}_\beta)$ with $f + \mathfrak{A} = p$, of $x_n \to x_0$ in X and of $\varepsilon > 0$ such that

$$\|f(x_n)\| \geq \varepsilon, \qquad f(x_0) = 0.$$

As $f^2 - f$ is norm continuous at x_0, we may assume by Lemma 10.3.6(i) that

$$\mathrm{sp}(f(x_n)) \subseteq (-1/3, 1/3) \cup (2/3, 4/3)$$

and let $p_n = 1_{[2/3,4/3]}(f(x_n))$. As in 1°, we have that $p_n \in \mathbb{K}$ and $\|p_n - f(x_n)\| \to 0$, and when n' is chosen such that

$$n \geq n' \Longrightarrow \|p_n - f(x_n)\| \leq \frac{\varepsilon}{2},$$

$\|p_n\| \geq \varepsilon/2$ for $n \geq n'$, whence $p_n \neq 0$. As $p_n \stackrel{\beta}{\to} 0$ we get from Lemma 11.2.5 in combination with Lemma 11.2.2(ii) a $g \in C^b(X, \mathbb{B}_\beta)$ such that

$$\|f(x_{n_i})g(x_{n_i}) - g(x_{n_i})f(x_{n_i})\| \geq \frac{\varepsilon}{2}$$

for some infinite set (n_i), and f can not be central, a contradiction. Hence, f must be norm continuous.

3°: We may still assume that $\tilde{p} = 0$. If f does not vanish at infinity, we can find $\varepsilon > 0$ and $x_n \to \infty$ such that

$$\|f(x_n)\| \geq \varepsilon, \qquad n \in \mathbb{N}.$$

Passing to a subsequence if necessary, we can find projections $p_n \in \mathbb{K}\backslash\{0\}$ with $\|f(x_n) - p_n\| \to 0$, and as the p_n must have finite rank, we can find $v_n \in \mathbb{K}$ such that $\|p_n v_n - v_n p_n\| \geq 1$, $\|v_n\| \leq 1$ and apply Lemma 11.2.2(i) to achieve $g \in C^b(X, \mathbb{B}_\beta)$ such that $g + \mathfrak{A}$ does not commute with $p = f + \mathfrak{A}$. With this contradiction, the proof is completed. ■

11.3 Sequences determining ends

Our second, and last, attempt at generalizing the notion of ends involves sequences "going to infinity" as in the construction leading to the Freudenthal compactification. We are, however, far from any general results allowing us to find all ends via an algorithm as in Section 10.2.

We say that a decreasing sequence (p_k) of non-zero, connected, open projections of $\mathfrak{A}^{**}$ with compact central boundaries $\partial c(p_k)$ *determines an end* of $\mathfrak{A}$ if

$$\bigwedge_{k=1}^{\infty} \overline{p_k} = 0. \tag{8}$$

Note that (8) implies that no $\overline{p_k}$ can be compact, for by the finite intersection property, Proposition 2.2.19, this would imply that some $\overline{p_k}$ were zero.

Definition 11.3.1 Let (p_k) be a sequence determining an end of X. When q is a projection of $\mathfrak{A}^{**}$ we say that $(p_k) \leq q$ if there exists a $k \in \mathbb{N}$ such that $p_k \leq q$.

Lemma 11.3.2 *Let (p_k) and (q_k) be sequences determining ends of $\mathfrak{A}$. The following conditions are equivalent.*

(i) $\forall l \in \mathbb{N} \exists \varphi_l, \psi_l \in P(\mathfrak{A}) : \begin{cases} \varphi_l \sim \psi_l \\ \varphi_l(p_l) = \psi_l(q_l) = 1 \end{cases}$

(ii) $\forall l \in \mathbb{N} : (p_k) \leq c(q_l)$.

(iii) $\forall l \in \mathbb{N} : (q_k) \leq c(p_l)$.

Definition 11.3.3 If (p_k) and (q_k) satisfy (i)–(iii) above, we say that the sequences are *equivalent* and write $(p_k) \approx (q_k)$.

Proof of Lemma 11.3.2: By symmetry, it suffices to prove (i)$\Longleftrightarrow$(ii). Given l, if $(p_k) \leq c(q_l)$ there is a $k \in \mathbb{N}$ with $p_k \leq c(q_l)$, whence by Lemma 2.4.5,

$$P(p_k) \subseteq P(c(q_l)) = \mathrm{sat}(P(q_l)).$$

As $p_k z \neq 0$ by openness, the left hand side is not empty, and we may pick $\varphi_l \in P(p_k)$. As $\varphi_l \in \mathrm{sat}(P(q_l))$, $\varphi_l \sim \psi_l$ for some $\psi_l \in P(q_l)$, showing that (i) holds. On the other hand, if (i) holds true and l is given, then since $\partial c(q_l)$ is compact and central, every $\overline{p_k}\partial c(q_l)$ is a compact projection. As

$$\bigwedge_{k=1}^{\infty} (\overline{p_k}\partial c(q_l)) = \left(\bigwedge_{k=1}^{\infty} \overline{p_k} \right) \partial c(q_l) = 0,$$

we get $\overline{p_k}\partial c(q_l) = 0$ for some $k \in \mathbb{N}$ by Proposition 2.2.19. *A fortiori*, $p_k \partial c(q_l) = 0$, and with $m = \max(k, l)$, when $\varphi_m \sim \psi_m$ are given as in (i), we get $\varphi_m \in P(p_m) \cap \mathrm{sat}(P(q_m)) \subseteq P(p_k) \cap P(c(q_l))$, such that

$$\varphi(p_k c(q_l)) = 1$$

and $p_k c(q_l) \neq 0$. By Lemma 7.1.2, $p_k \leq c(q_l)$. ■

Note that by (i) above, "$\approx$" is reflexive and symmetric, by (ii) it is transitive. It is also clear that the following definition extends that of Definition 10.1.5.

Definition 11.3.4 Let $\mathfrak{A}$ be a C^*-algebra. An *end* of $\mathfrak{A}$ is an element of

$$\mathfrak{E} = \{[(\mathfrak{p}_\mathfrak{k})] | (\mathfrak{p}_\mathfrak{k}) \text{ determines an end of } \mathfrak{A}\}.$$

The number of ends of $\mathfrak{A}$, with values in $\mathbb{N} \cup \{\infty\}$, is denoted by $\#_e\mathfrak{A}$.

Remark 11.3.5 We needed to assume that the $c(p_k)$, and not the p_k themselves, have compact boundaries in order to apply Lemma 7.1.2 in the proof above. Had we chosen the first restriction instead, "$\approx$" would not have been an equivalence relation in, e.g., our basic C^*-algebra $C_0(\mathbb{R}) \otimes \mathbb{K}(\mathfrak{H})$.

Example 11.3.6 *When $\mathfrak{A}$ is a primitive, σ-unital, non-unital C^*-algebra, $\mathfrak{A}$ has exactly one end.*

Proof: Let a be a strictly positive element, $0 \leq a \leq 1$. As the functions $1_{[0,1/k)}$ lie in $C(\mathrm{sp}(a))^m$, the projections $p_k = 1_{[0,1/k)}(a)$ are open in $\mathfrak{A}^{**}$. We have

$$\partial c(p_k) \leq 1 - c(p_k) \leq 1 - p_k,$$

and as $1 - p_k = 1_{[1/k,1]}(a)$ and $1_{[1/k,1]}$ is dominated by a norm one function $g \in C(\mathrm{sp}(a))$ with $g(0) = 0$, $1 - p_k$, and hence $\partial c(p_k)$, is compact. We also get from this that $p_k \neq 0$, and since $\overline{p_k} \leq 1_{[0,1/k]}(a)$, we have

$$\bigwedge_{k=1}^{\infty} \overline{p_k} \leq \bigwedge_{k=1}^{\infty} 1_{[0,1/k]}(a) = 0$$

by spectral theory. By Example 7.1.4, every open projection is connected, so (p_k) determines an end.

To see that this is the only one, let (q_k) be another sequence determining an end, and note that as a primitive ideal is prime ([22, 3.13.10]), $c(p_k)$ and $c(q_k)$ can not be orthogonal, as they would then support orthogonal ideals. By openness, $c(p_k)c(q_k)z \neq 0$, and $P(c(p_k)c(q_k)) \neq \emptyset$, whence by Lemma 2.4.5, $\mathrm{sat}(P(p_k)) \cap \mathrm{sat}(P(q_k)) \neq \emptyset$. We conclude that (p_k) and (q_k) satisfy (i) of Lemma 11.3.2. ■

Before we embark on finding the ends of C^*-algebras of the form (III) and (IV) from Section 11.1, the reader is asked to recall the results and the terminology to be found at the end of Section 7.1. In particular, when $\mathfrak{A} = C_0(X) \otimes \mathbb{K}(\mathfrak{H})$ for $\dim \mathfrak{H} \leq \aleph$, the identification of $\mathfrak{A}^{**}z$ with $\mathcal{B}(X, \mathbb{B}(\mathfrak{H}))$ will be employed.

Example 11.3.7 *With X a Raum and $\mathfrak{A} = C_0(X) \otimes M_n(\mathbb{C})$, there is a 1–1 correspondence between the ends of X and those of $\mathfrak{A}$.*

Proof: When (G_k) determines an end of X, we may apply Lemma 7.1.13 to achieve open central projections p_k of $\mathfrak{A}^{**}$ with $p_k z = 1_{G_k}$. By Proposition 7.1.14, the p_k are connected. With q_k the unique closed projection such that $q_k z = 1_{\overline{G_k}}$, we get from $1_{G_k} \leq 1_{\overline{G_k}}$ that $\overline{p_k} \leq q_k$ by Corollary 2.2.15. Consequently,

$$\left(\bigwedge_{k=1}^{\infty} \overline{p_k}\right) z = \bigwedge_{k=1}^{\infty} (\overline{p_k} z) \leq \bigwedge_{k=1}^{\infty} 1_{\overline{G_k}} = 0$$

whence $\bigwedge_1^{\infty} \overline{p_k} = 0$ as it is closed by Proposition 2.2.8. We also get that $r = \partial c(p_k) = \partial p_k \leq q_k - p_k$, whence $rz \leq 1_{\partial G_k}$. Taking a $\chi \in C_{00}(X)$ which is 1 on ∂G_k, $a = \chi \otimes 1 \in \mathfrak{A}$ has $\|a\| = 1$ and satisfies $arz = rz$. Also, $ar \in U(\mathfrak{A})$, for since r is closed, there are $b_\lambda \in \mathfrak{A}_+$ such that $b_\lambda \nearrow 1 - r$, whence $a^{1/2} b_\lambda a^{1/2} \nearrow a - ar$. Apply Proposition 2.2.11. We get that $ar = r$, showing r to be compact by Lemma 2.2.17. For every sequence determining an end of X we have thus defined one of $\mathfrak{A}$, and obviously two such ends of $\mathfrak{A}$ are equivalent if and only if the underlying sequences of X are.

When (p_k) is any sequence determining an end of $\mathfrak{A}$, we claim that $G_k = \mathrm{s}(p_k)$ determines an end of X. We have seen (by Proposition 7.1.14 and the discussion preceding it) that the G_k must be open and connected. Clearly $c(p_k)z = 1_{G_k}$, whence

$$G_k = \mathrm{s}(p_k) \subseteq \mathrm{s}(c(p_k)) = \mathrm{s}(1_{G_k}) = G_k.$$

We get from this that $\overline{G_k} \subseteq \mathrm{s}(\overline{c(p_k)})$ and

$$\begin{aligned} \partial G_k &= \overline{G_k} \backslash G_k \\ &\subseteq \mathrm{s}(\overline{c(p_k)}) \backslash \mathrm{s}(c(p_k)) \\ &= \{x \in X | \pi_x^{**}(\overline{c(p_k)}) \neq 0, \pi_x^{**}(c(p_k)) = 0\} \\ &= \mathrm{s}(\partial c(p_k)), \end{aligned}$$

and this last set is compact, for if $a\partial c(p_k) = \partial c(p_k)$ with $a = f(x) \in \mathfrak{A}$, it is contained in the compact set $\{x \in X | \|f(x)\| \geq 1\}$. Finally, as $\mathrm{s}(\overline{p_k})$ is closed, $\overline{G_k} \subseteq \mathrm{s}(\overline{p_k})$, and

$$\bigcap_{k=1}^{\infty} \overline{G_k} \subseteq \bigcap_{k=1}^{\infty} \mathrm{s}(\overline{p_k}) = \mathrm{s}(\bigwedge_{k=1}^{\infty} \overline{p_k}) = \emptyset.$$

Clearly (p_k) is equivalent to the sequence defined from (G_k). ■

Example 11.3.8 *With X a Raum,* $\dim \mathfrak{H} = \aleph_0$, *and $\mathfrak{A} = C_0(X) \otimes \mathbb{K}(\mathfrak{H})$, $\mathfrak{A}$ has exactly one end.*

Proof: With (ξ_n) a basis for $\mathfrak{H}$, let $p_k = [\text{span}\{\xi_n | n \geq k\}] \in \mathbb{B}(\mathfrak{H})$. By Proposition 7.1.14 and Lemma 7.1.13, there are open and connected projections $q_k \in \mathfrak{A}^{**}$ with

$$q_k z = p_k 1_X,$$

and as clearly $c(p_k)z = 1_X$, we have $c(q_k) = 1$, and $\partial c(q_k) = 0$. Also, with r_k closed such that $r_k = p_k 1_X$, $r_k z = q_k z$, and by Theorem 2.2.12, $q_k = r_k$ and q_k is closed, so that

$$\left(\bigwedge_{k=1}^{\infty} \overline{q_k} \right) z = \bigwedge_{k=1}^{\infty} (p_k 1_X) = 0$$

as $p_k \searrow 0$ in $\mathbb{B}(\mathfrak{H})$. We get that this sequence determines an end; as $c(q_k) = 1$ for all k, this is the only one by Lemma 11.3.2(ii). ■

11.4 Epilogue

We saw in the two previous sections, that in most of our test cases (I)-(IV) of Section 11.1, $\#_e\mathfrak{A} = \#_c(M(\mathfrak{A})/\mathfrak{A})$. Actually, our investigation can be summarized in the following table:

$\mathfrak{A}$	I	II	III	IV
$\#_e\mathfrak{A}$	$\#_e X$	1	$\#_e X$	1
$\#_c(M(\mathfrak{A})/\mathfrak{A})$	$\#_e X$	1?	$\#_e X$	1

It would be interesting to study this phenomenon in a more general setting, and the author feels that using the "essential centrality" built into our definitions, some results may be got from mimicking the commutative case. Another project would be to follow the idea of constructing ends as decreasing sequences of unbounded components of certain sets in the commutative case. We are pretty far from a result like that, as the key to the abelian case, Lemma 10.1.7, has no obvious generalizations. The ultimate hope would be to define a Freudenthal unitization and use it not only as a tool in this theory, but as an object worth studying in its own right for the information on the global structure of $\mathfrak{A}$ that it carries. One might hope that such an object would be substantially smaller that the multiplier algebra, as it is the case when $\mathfrak{A}$ is commutative.

Appendices

A Convex preliminaries

We recall a number of terms from convexity theory. Let Q be a convex set sitting in a linear space E. When $x \in Q$ can be written

$$x = \lambda y_0 + (1 - \lambda) y_1$$

for $\lambda \in (0,1)$ and $y_0, y_1 \in Q$, x is called *composite*. When x is not composite, it is *extreme*, and we write $\operatorname{ext} Q$ for the set of extreme points of Q.

A *face* in Q is a convex subset F with the extra property that if $x_0, x_1 \in Q$ and $0 < \lambda < 1$ are given with

$$\lambda x_0 + (1 - \lambda) x_1 \in F,$$

then also $x_0, x_1 \in F$. Note that $\{x\}$ is a face if and only if x is extreme. An important property of faces, following directly from the definitions, is that the extreme points of F are also extreme points of Q.

We write $\mathrm{co}(M)$ for the convex hull of M, i.e. the smallest convex set containing M. We write $\overline{\mathrm{co}}(M)$ (or $\overline{\overline{\mathrm{co}}}(M)$, in case of norm closure) for the closed convex hull, which is the smallest closed convex set containing M, as well as the closure of $\mathrm{co}(M)$.

A.1 The double polar theorem

Let E be a real locally convex space. For subsets $A \subseteq E$, $B \subseteq E^*$ we put

$$\begin{aligned} A^\circ &= \{\varphi \in E^* | \forall x \in A : \varphi(x) \geq -1\} \\ {}^\circ B &= \{x \in E | \forall \varphi \in B : \varphi(x) \geq -1\}. \end{aligned}$$

If we choose some locally convex topological structure on E^* then for all $B \subseteq E^*$ we can define both ${}^\circ B$ and B°, and conceivably $({}^\circ B)^\circ \neq {}^\circ(B^\circ)$. However, when E and E^* are in duality, i.e. when $E^{**} = E$, this distinction is unnecessary (and rather artificial), and we will allow ourselves to write $B^{\circ\circ}$ in this case. When A is a linear space, obviously

$$A^\circ = \{\varphi \in E^* | \forall x \in A : \varphi(x) = 0\}$$

and we will write $A^\circ = A^\perp$ in this case. Similarly, ${}^\perp B$ is defined.

The proof of the next lemma is left to the reader.

Lemma A.1.1 *Let A be a subset of E, B a subset of E^*.*

(i) A° is a weak closed convex set containing* 0.

(ii) $^\circ B$ is a weak closed convex set containing 0.

Theorem A.1.2 *For any subset A of E,*

$$^\circ(A^\circ) = \overline{\text{co}}(A \cup \{0\}).$$

Proof: By the lemma, $^\circ(A^\circ)$ is weakly closed and convex, hence closed in the original topology on E. By definition, $A \cup \{0\} \subseteq {}^\circ(A^\circ)$, so it contains the smallest closed convex set containing $A \cup \{0\}$. For the other inclusion, assume that $x \notin \overline{\text{co}}(A \cup \{0\})$. By the Hahn-Banach theorem we can find $\varphi \in E^*$ such that

$$\varphi(y) \leq \alpha_1 < \beta < \alpha_2 \leq \varphi(x), \qquad \forall y \in \overline{\text{co}}(A \cup \{0\}).$$

Since $0 \in \overline{\text{co}}(A \cup \{0\})$, $\beta > 0$ and we can define $\psi = -\beta^{-1}\varphi$. Then $\psi(y) \geq -\alpha_1/\beta \geq -1$ for all $y \in A$, showing that $\psi \in A^\circ$. But since $\psi(x) = -\alpha_2/\beta < -1$, this implies $x \notin {}^\circ(A^\circ)$, as required. ■

A.2 Using the Krein-Milman theorem

The well-known theorem of Krein and Milman states that if Q is a convex and compact subset of a locally convex space E, then $Q = \overline{\text{co}}(\text{ext}(Q))$. The following proposition can be thought of as an inverse to this result, and can often be used in conjunction with polar theory.

Proposition A.2.1 *If Q is a compact convex subset of E and $F \subseteq Q$ has the property that $Q = \overline{\text{co}}(F)$, then $\text{ext}\, Q \subseteq \overline{F}$.*

Proof: We may assume that F is closed. Suppose $x \in \text{ext}\, Q \backslash F$. Since E is locally convex, we can find a balanced, convex, and open neighborhood U of 0 such that $(x + 2U) \cap F = \emptyset$. Consequently, $(x + U) \cap (F + U) = \emptyset$, and $x \notin (F + U)^-$. Since F is compact we may pick $y_1, \ldots, y_n$ in F such that the open sets $y_1 + U, \ldots, y_n + U$ cover F. For any $i \in \{1, \ldots, n\}$, letting

$$K_i = \overline{\text{co}}(F \cap (y_i + U))$$

we have $K_i \subseteq Q \cap (y_i + \overline{U})$ since this is evidently a closed convex set containing $F \cap (y_i + U)$. Since the sets K_i are convex and compact, so is $\text{co}(K_1 \cup \cdots \cup K_n)$ and we get

$$Q = \overline{\text{co}}(F) \subseteq \overline{\text{co}}(K_1 \cup \cdots \cup K_n) = \text{co}(K_1 \cup \cdots \cup K_n) \subseteq Q.$$

In particular, x is a convex combination of points from $K_1 \cup \cdots \cup K_n$, and since it is extremal, $x \in K_i$ for some i. This yields

$$x \in y_i + \overline{U} \subseteq F + \overline{U} \subseteq (F+U)^{-},$$

a contradiction. ■

The next result is an easy consequence of the Krein-Milman theorem.

Lemma A.2.2 *Assume that E_1 and E_2 are locally convex vector spaces. If Q is a convex compact subset of E_1, and $\Psi : Q \to E_2$ is a continuous and affine map, then* $\operatorname{ext}(\Psi(Q)) \subseteq \Psi(\operatorname{ext} Q)$.

Proof: As Ψ is affine, $\Psi(Q)$ is convex. Let y be an extreme point of $\Psi(Q)$ and note that $\Psi^{-1}(\{y\})$ is a face of Q, for if for some $\lambda \in (0,1)$

$$y = \Psi(\lambda x_0 + (1-\lambda)x_1) = \lambda\Psi(x_0) + (1-\lambda)\Psi(x_1),$$

then $\Psi(x_0) = \Psi(x_1) = y$. Also, $\Psi^{-1}(\{y\})$ is compact, so by the Krein-Milman theorem, it contains an extreme point x_0. As noted above, $x_0 \in \operatorname{ext}(Q)$. ■

B A primer on connectivity

B.1 Fundamental results

As we will be concerned with a certain notion of connectivity for C^*-algebras, generalizing the usual notion for a topological space X via the algebra $C_0(X)$, some results from the classical theory will come up in the main sections of this thesis. We include the definitions and some main results, along with sketchy proofs, for reference purposes.

As we will need to apply some of the results to the spaces $\operatorname{Prim}(\mathfrak{A})$ from operator theory, and since these are not in general a Hausdorff space, we do *not* require the topological spaces in question to have any separation properties.

Definition B.1.1 Let X be a topological space. If U_0, U_1 are open sets with $U_0 \cap U_1 = \emptyset$, $U_0 \cup U_1 = X$, we call $\{U_0, U_1\}$ a *separation* of X. We say that X is *connected* when every separation $\{U_0, U_1\}$ is trivial, i.e. $\{U_0, U_1\} = \{\emptyset, X\}$. A subset Y of X is called connected, if it is connected as a topological space with the relative topology.

Proposition B.1.2 *Let X and Y be topological spaces.*

(i) If $(C_i)_I$ is a family of connected subsets of X, and the C_i have a mutual point x, then $\cup_I C_i$ is connected.

(ii) If C is connected in X, so is any set D with $C \subseteq D \subseteq \overline{C}$.

(iii) If $f : X \to Y$ is continuous and X is connected, so is $f(X)$.

(iv) If $C \subseteq X$ and $D \subseteq Y$ are connected, so is $C \times D$.

Proof: For (i), let $\{U_0, U_1\}$ be a separation of $C = \cup_I C_i$ and assume that $x \in U_1$. Since $\{U_0, U_1\}$ separates C_i for every i, we get $\{U_0 \cap C_i, U_1 \cap C_i\} = \{\emptyset, C_i\}$, and as $x \in U_1 \cap C_i$, we must have $U_0 \cap C_i = \emptyset, U_1 \cap C_i = C_i$. From this we conlude that $U_0 = \emptyset, U_1 = C$. In (ii), assume that $\{U_0, U_1\}$ separates D. As $U_0 \cap C, U_1 \cap C$ are relatively open, we may assume that $U_0 \cap C = \emptyset, U_1 \cap C = C$, and since C is dense in D, so is U_1. However, U_1 is relatively closed in D, and $U_1 = D$ as desired. Statement (iii) follows by noting that the pre-image by f of a separation of $f(X)$ is a separation of X.

The last statement is clear if any of the sets are empty. If not, fix $c_0 \in C$ and write

$$C \times D = \bigcup_{d \in D} (\{c_0\} \times D \cup C \times \{d\}).$$

For any $d \in D$, each of $\{c_0\} \times D$ and $C \times \{d\}$ are connected by (iii), and since (c_0, d) lies in both of these sets, so is their union by (i). For any fixed $d_0 \in D$, (c_0, d_0) lies in all of the sets $\{c_0\} \times D \cup C \times \{d\}$, and $C \times D$ is connected by (i). ■

Definition B.1.3 A *component* is a maximal connected subset. We denote by $\mathcal{C}(X)$ the set of components of the topological space X, by $\#_c X$ the number, in the range $\mathbb{N} \cup \{\infty\}$, of elements of $\mathcal{C}(X)$ and by $\mathfrak{c}_\mathfrak{c}\mathfrak{X}$ the cardinality of $\mathcal{C}(X)$. A space X is *totally disconnected* when all its components have only one point.

Note that by (ii) in Lemma B.1.2 above, every component is closed. Also, every connected set $C \neq \emptyset$ is contained in a component, because

$$\bigcup \{D \subseteq X | D \text{ is connected, } C \subseteq D\}$$

is connected by (i) in the lemma above. From this result we also get that all components must be disjoint.

Lemma B.1.4 *When $f : X \to Y$ is continuous, then $\mathfrak{c}_\mathfrak{c}\mathfrak{X} \geq \mathfrak{c}_\mathfrak{c}\mathfrak{f}(\mathfrak{X})$.*

Proof: We can map every element C of $\mathcal{C}(X)$ to the element of $\mathcal{C}(f(X))$ containing $f(C)$ by Lemma B.1.2(iii) and the observation above, and this mapping is onto. ■

Lemma B.1.5 *When X and Y are topological spaces, and $X \times Y$ is equipped with the product topology, $\mathfrak{c}_\mathfrak{c}(\mathfrak{X} \times \mathfrak{Y}) = \mathfrak{c}_\mathfrak{c}\mathfrak{X}\mathfrak{c}_\mathfrak{c}\mathfrak{Y}$.*

Proof: We establish a bijection from $\mathcal{C}(X \times Y)$ to $\mathcal{C}(X) \times \mathcal{C}(Y)$. Let C be a component of $X \times Y$ and note that since the projections π_X, π_Y are continuous, $\pi_X(C)$ and $\pi_Y(C)$ are connected. They are components, for if, say, $D_X \supseteq \pi_X(C)$ and D_X is connected we have that $D_X \times \pi_Y(C) \supseteq \pi_X(C) \times \pi_Y(C) \supseteq C$, whence, by Proposition B.1.2(iv), we have equality and $D_X = \pi_X(C)$. We further conclude that $C = \pi_X(C) \times \pi_Y(C)$, showing that $\Pi : C \mapsto (\pi_X(C), \pi_Y(C))$ is 1-1. Finally, if C_X and C_Y are components of X and Y, respectively, we can find a component D of $X \times Y$ containing $C_X \times C_Y$, because $C_X \times C_Y$ is connected by Lemma B.1.2(iv). Since C_X, C_Y are components, $\pi_X(D) = C_X$ and $\pi_Y(D) = C_Y$. ■

Lemma B.1.6 *When Y is dense in X, $\#_c Y \geq \#_c X$.*

Proof: If $\#_c Y = \infty$, there is nothing to prove. If $\#_c Y = n$, let $C_1, \dots, C_n$ be the components of Y and note that since

$$X = \overline{Y} = \bigcup_{i=1}^{n} \overline{C_n},$$

where $\overline{C_n}$ is connected by Lemma B.1.2(ii), $\#_c X \leq n$. ■

Remark B.1.7 We do not have $\mathfrak{c}_\mathfrak{c}\mathfrak{Y} \geq \mathfrak{c}_\mathfrak{c}\mathfrak{X}$ in the above result. Turning, as you might expect, to the Cantor set, we see that there exist separable, totally disconnected sets with uncountable many points (components).

Lemma B.1.8 *Let $Y \subseteq X$. When C is a component of X, we can write*

$$C \cap Y = \bigcup_{i \in I} D_i$$

where the D_i are components of Y.

Proof: Write $C \cap Y$ out in components $\bigcup_I D_i$. If $C_i \subseteq Y$ is connected with $D_i \subseteq C_i$, then it is connected in X, and then so is $C \cup C_i$. We conclude that $C_i \subseteq C$, and as the D_i are maximal connected sets of C, $C_i = D_i$. ■

Two more things we need to know:

Lemma B.1.9 *If C is a connected set in X and G is open, then*

$$\left.\begin{array}{l} C \cap G \neq \emptyset \\ C \cap (X \backslash G) \neq \emptyset \end{array}\right\} \Longrightarrow C \cap \partial G \neq \emptyset.$$

Proof: If $C \cap \partial G = \emptyset$, we can write

$$C \cap (X \backslash G) = C \cap (X \backslash \overline{G})$$

and C is separated by $\{C \cap G, C \cap (X \backslash G)\}$. We conclude that either $C \cap G = \emptyset$ or $C \cap (X \backslash G) = \emptyset$. ■

Lemma B.1.10 *Let X be a compact Hausdorff space, and assume that $C \subseteq G \subseteq X$, where C is a component, G an open set. Then there exists a clopen set H with $C \subseteq H \subseteq G$.*

Proof: Let

$$\mathcal{K} = \{K \subseteq X | C \subseteq K, K \text{ clopen}\}$$

and note that since $X \in \mathcal{K}$, we can define

$$K_0 = \bigcap_{K \in \mathcal{K}} K.$$

First note that if $K_0 \subseteq H$ for H open, $K \subseteq H$ for some $K \in \mathcal{K}$. To see this, we observe that since $X \backslash H$ is compact and is covered by $\{X \backslash K | K \in \mathcal{K}\}$, hence by a finite subset, and since $\mathcal{K}$ is closed under finite intersections, $X \backslash H \subseteq X \backslash K$ for some $K \in \mathcal{K}$.

We use this to see that K_0 is connected. Assume that $K_0 = U_0 \cup U_1$, a disjoint union of closed and relatively open sets. By normality of X, we can find H_0, H_1 disjoint and open with $U_i = H_i \cap K_0$. By the above argument, we can take $K \in \mathcal{K}$ with $K \subseteq H_0 \cup H_1$. As C is connected, we may assume that $C \subseteq U_0$; we claim that then $K \cap H_0 \in \mathcal{K}$. Clearly it is open and contains C. It is also closed, for since $\overline{H_0} \cap H_1 = \emptyset$, we have

$$K \cap H_0 = K \cap (H_0 \cup H_1) \cap \overline{H_0} = K \cap \overline{H_0}.$$

By definition of K_0, $K_0 \subseteq K \cap H_0 = U_0$ as desired. As C is a component, $C = K_0$, and we may apply the argument of the previous paragraph to $H = G$ to finish the proof. ■

B.2 Further notions of connectivity

We recall the following definition:

Definition B.2.1 A topological spaces X is called *arcwise connected* when it has the property that any pair of points are the endpoints of a continuous curve $\Phi : [0,1] \to X$.

Remark B.2.2 It seems safe to leave the proof that every arcwise connected space is connected to the reader. We mention one standard counterexample for the opposite statement, the *topologist's sine curve*

$$\{(x, \sin\frac{1}{x})|x \in (0,1]\}^{-}$$

in $\mathbb{R}^2$.

The next definition might not be as familiar as the previous.

Lemma B.2.3 *Let X be a topological space. The following conditions are equivalent:*

(i) If $x \in X$ and U is a neighborhood of x, there exists a connected open neighborhood V of x such that $V \subseteq U$.

(ii) If U is an open subset of X, all components of U are open.

Definition B.2.4 A topological space with these properties is called *locally connected.*

Proof of Lemma B.2.3: For (ii)$\Longrightarrow$(i), we may assume that U is open. The component of U containing x is open, and will do as V. For (i)$\Longrightarrow$(ii), let C be a component of U and take $x \in C$. By hypothesis, there is an open connected set V such that $x \in V \subseteq U$, and since C is maximal among the connected sets containing x, $V \subseteq C$. Since x was arbitrary, C is open. ■

Remark B.2.5 A set like

is clearly connected and arcwise connected, but not locally connected, as when we remove the lower left-hand point, the remaining part of the horizontal line is a component, but not open. Connected and locally connected spaces are automatically arcwise connected under the additional assumptions that they are metrizable and compact, cf. [16, 3.15].

We note two properties of local connectivity.

Lemma B.2.6 *If $f : X \to Y$ is continuous, open and onto, then Y is locally connected if X is.*

Proof: Given $y \in Y$ and a neighborhood U of y. As f is onto, we can find $x \in X$ with $f(x) = y$, and as f is continuous, $f^{-1}(U)$ is a neighborhood of x. Taking $x \in V \subseteq f^{-1}(U)$ a connected open set, $f(V)$ is connected as f is continuous, open as f is open. ■

Lemma B.2.7 *If X and Y are both locally connected, so is $X \times Y$.*

Proof: When U is a neighborhood of $(x, y) \in X \times Y$, we can find neighborhoods V_x, V_y of x, y with $V_x \times V_y \subseteq U$. Using local connectivity of X and Y, we may assume that V_x, V_y are connected, and then so is $V_x \times V_y$ by Lemma B.1.2(iv). ■

List of symbols

The symbol list is arranged in alphabetical order after letters only, including all subscripts and superscripts. Greek or Hebrew letters are treated as their Latin equivalents, e.g. $\pi = p$, $\varphi = f$. In the list, as well as in the rest of the paper, we use generic names for some mathematical objects according to the following table.

First	Alternatives	Object
a	b, c, d	Elements of a C^*-algebra
$\mathfrak{A}$	$\mathfrak{B}, \mathfrak{C}$	C^*-algebras
$\mathfrak{I}$	$\mathfrak{J}$	Ideals of C^*-algebras
φ	ψ, χ	Functionals on a C^*-algebra
$\mathfrak{H}$	$\mathfrak{K}$	Hilbert spaces
$\mathcal{M}$	$\mathcal{N}$	Von Neumann algebras
M		Miscellaneous subsets of vector spaces
Q		Convex sets
p	q, r	Projections of a C^*-algebra
X	Y, Z	Topological spaces
ξ	η, ζ	Vectors in Hilbert spaces
π	ρ	Representations of a C^*-algebra

$\mathfrak{c}_{\mathfrak{c}}\mathfrak{X}$	The cardinal of the set of components of X.	101
$\mathbb{K}(\mathfrak{H})$	The compact operators on $\mathfrak{H}$.	vi
$\mathrm{co}(M)$	The convex hull of M.	98
$\overline{\mathrm{co}}(M), \overline{\overline{\mathrm{co}}}(M)$	The closed convex hull of M.	98
$\mathrm{co}_{\mathcal{M}\otimes 1}(a)$	Convex hull of $\{(u^* \otimes 1)a(u \otimes 1) \mid u \in \mathcal{U}(\mathcal{M})\}$.	26
$c(p)$	The central cover of p in $\mathfrak{A}^{**}$.	20
$C(p)$	The hereditary cone supported by p. $[= \{\varphi \in (\mathfrak{A}^*)_+ \mid \varphi(1-p) = 0\}]$.	4
$C(X)$	$C_0(X) = C_{00}(X) = C^b(X)$ when X is compact.	vi
$C_0(X)$	The continuous functions on X vanishing at infinity.	vi
$C_{00}(X)$	The continuous functions on X with compact support.	vi
$\mathcal{C}(X)$	The set of components of X.	101
$\#_c X$	The number of components of X. Ranging in $\mathbb{N} \cup \{\infty\}$.	101
δ_{ij}	Kronecker delta.	55
$\dim X$	Covering dimension of X.	52
$\dim_{\mathcal{F}}(Q)$	Facial dimension of convex set Q.	35
$\mathfrak{e}$	An end.	72
$\mathfrak{E}$	The set of ends.	72
$\mathfrak{E}_{\dagger}$	The set of ends given by compact nest $(\dagger)$.	77
$\#_e \mathfrak{A}$	The number of ends of $\mathfrak{A}$.	94
$\#_e X$	The number of ends of X.	72
$\mathrm{ext}\, Q$	The extreme points of compact set Q.	98
$\lvert\varphi\rvert$	The absolute value of $\varphi \in \mathcal{M}_*$.	2
$F(p)$	The face supported by p. $[= \{\varphi \in Q(\mathfrak{A}) \mid \varphi(1-p) = 0\}]$.	4
$\underline{\varphi} X$	Freudenthal compactification.	79
$\overline{}^{\gamma}$	Closure within a compactification γX of X.	75
$(G_k) \approx (H_k)$	Equivalence of sequences determining ends.	72
$G(Q, v)$	Minimal face at v in convex set Q.	34
$\mathfrak{H}_{\mathfrak{a}}$	See π_a.	1
$\mathfrak{H} \hat{\otimes} \mathfrak{K}$	Hilbert space tensor product.	21
$H(p)$	Hereditary subalgebra supported by p.	6
$\mathfrak{H}_{\varphi}$	See $(\pi_{\varphi}, \mathfrak{H}_{\varphi}, \xi_{\varphi})$.	vi
$H_{n_1 \cdots n_k}$	Complex of unbounded components.	75
$\mathfrak{H}_{\mathfrak{u}}$	See π_u.	1
$\mathrm{Hull}(\mathfrak{I})$	For $\mathfrak{I}$ ideal in $\mathfrak{A}$, the set $\{J \in \mathrm{Prim}(\mathfrak{A}) \mid \mathfrak{I} \subseteq J\}$.	2
$\mathrm{Im}\ a$	Imaginary part of a. $[= -\frac{i}{2}(a - a^*)]$.	vi
$\mathrm{Ker}(F)$	For $F \subseteq \mathrm{Prim}(\mathfrak{A})$, the ideal $\bigcap \{\mathfrak{I} \mid \mathfrak{I} \in F\}$.	2
L_{φ}	Left kernel of φ. $[= \{x \in \mathfrak{A} \mid \varphi(x^* x) = 0\}]$.	vi
$L(p)$	The left invariant subspace supported by p. $[= (\mathfrak{A}^{**}(1-p))^{\perp}]$.	4
$[M]$	The orthogonal projection onto $\overline{M}$, M subspace of $\mathfrak{H}$.	20
$(M)_1$	The elements of M having norm less than or equal to 1.	vi
M_+	The positive elements of M.	vi
$^{\circ}M$	The prepolar of M. See appendix A.	98
M°	The polar of M. See appendix A.	98
$^{\perp}M$	The preannihilator of M.	98
$M^{\perp}$	The annihilator of M.	98
$\mathcal{M}_*$	The predual of $\mathcal{M}$. The set of normal functionals on $\mathcal{M}$.	vi
$\mathcal{M}'$	The commutant of $\mathcal{M}$.	vi
$M(\mathfrak{A})$	The multiplier algebra of $\mathfrak{A}$.	1
$M(\mathfrak{A})_{\beta}$	The multiplier algebra of $\mathfrak{A}$ with strict topology.	86
$\mathcal{M} \overline{\otimes} \mathcal{N}$	Von Neumann tensor product.	21
$M_n(\mathfrak{A})$	The $n \times n$ matrices over $\mathfrak{A}$, with norm from a faithful representation of $\mathfrak{A}$.	vi
$M_n(\mathbb{C})$	The $n \times n$ matrices over $\mathbb{C}$.	vi

M_{sa}	The self-adjoint elements of M.	vi
P	See $P(\mathfrak{A})$.	vi
∂p	Boundary of open projection $p[=\overline{p}-p]$.	9
$\overline{p}$	The closure of p in $\mathfrak{A}^{**}$.	9
π^{**}	The normal extension of a representation of $\mathfrak{A}$ to one of $\mathfrak{A}^{**}$.	2
$P(\mathfrak{A})$	The pure states on $\mathfrak{A}$. $[=\operatorname{ext} S(\mathfrak{A})]$. Also see $\mathfrak{A}^*$.	vi
π_a	The reduced atomic representation of $\mathfrak{A}$ into $\mathbb{B}(\mathfrak{H}_{\mathfrak{a}})$.	1
p_C	For $C\subseteq P(\mathfrak{A})$ saturated, the associated central projection.	19
π_φ	See $(\pi_\varphi,\mathfrak{H}_\varphi,\xi_\varphi)$.	vi
$(\pi_\varphi,\mathfrak{H}_\varphi,\xi_\varphi)$	The GNS triple associated with φ such that $\varphi(\cdot)=(\pi_\varphi(\cdot)\xi_\varphi\|\xi_\varphi)$.	vi
$\bigvee p_i$	See $p\vee q$.	vi
$\bigwedge p_i$	See $p\wedge q$.	vi
$(p_k)\approx(q_k)$	Equivalence of sequences determining ends.	93
$P(p)$	Set of pure states supported by p. $[=F(p)\cap P(\mathfrak{A})]$.	4
π_p	The mapping sending $a\in\mathfrak{A}$ to $ap\in\mathfrak{A}^{**}$.	14
$\varphi\otimes_\beta\psi$	Tensor product of states φ and ψ on $\mathfrak{A}\otimes_\beta\mathfrak{B}$.	21
$p\vee q$	The least upper bound of p and q.	vi
$p\wedge q$	The greatest lower bound of p and q.	vi
$\pi\hat{\otimes}\rho$	Tensor product of representations π and ρ.	21
$\pi\otimes_\beta\rho$	Tensor product of representations π and ρ on $\mathfrak{A}\otimes_\beta\mathfrak{B}$.	21
$\mathrm{Prim}(\mathfrak{A})$	The primitive ideal space of $\mathfrak{A}$ with hull-kernel topology.	2
π_u	The universal representation of $\mathfrak{A}$ into $\mathbb{B}(\mathfrak{H}_{\mathfrak{u}})$.	1
$Q;Q(\mathfrak{A})$	The quasi-states on $\mathfrak{A}$. $[=(\mathfrak{A}_+)_1]$. Also see $\mathfrak{A}^*$.	vi
$\mathrm{Re}\ a$	Real part of a. $[=\frac{1}{2}(a+a^*)]$.	vi
$S;S(\mathfrak{A})$	The states on $\mathfrak{A}$. $[=\{\varphi\in Q(\mathfrak{A})\|\|\varphi\|=1\}]$. Also see $\mathfrak{A}^*$.	vi
$\mathrm{s}(a)$	Support of $a\in C_0(X)\otimes\mathbb{B}(\mathfrak{H})$.	50
$\operatorname{sat} M$	Saturated hull of M.	20
$\mathrm{sp}(a)$	The spectrum of a, taken in $\mathfrak{A}^\sim$ if $\mathfrak{A}$ has no unit.	vi
$\#_{uc}X$	Number of unbounded components of X.	74
$U(\mathfrak{A})$	Universally measurable elements of $\mathfrak{A}^{**}$.	11
$\mathcal{U}(\mathfrak{A});\mathcal{U}(\mathcal{M})$	The unitary group of $\mathfrak{A}$;$\mathcal{M}$.	vi
$\hat{x}$	The mapping $\varphi\mapsto\varphi(x)$.	vii
ξ_φ	See $(\pi_\varphi,\mathfrak{H}_\varphi,\xi_\varphi)$.	vi
$(X\hookrightarrow\gamma X)$	A compactification of X.	75
z	The supremum of the minimal projections in $\mathfrak{A}^{**}$. The cover of π_a in $\mathfrak{A}^{**}$.	3
$\mathcal{Z}(\mathfrak{A});\mathcal{Z}(\mathcal{M})$	The center of $\mathfrak{A}$;$\mathcal{M}$.	vi

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