Flow equivalence of shift spaces (and their C^*-algebras), IV

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Content

Outline

Theorem

When an $n \times n$-matrix A and an $n' \times n'$-matrix A' define irreducible and infinite SFTs the following are equivalent

1. $\mathsf{X}_A \sim \mathsf{X}_{A'}$
2. $\mathbb{Z}^n/(I-A)\mathbb{Z}^n \simeq \mathbb{Z}^{n'}/(I-A')\mathbb{Z}^{n'}$ *and* $\operatorname{sgn}\det(I-A) = \operatorname{sgn}\det(I-A')$

Basic operation

Lemma

When $A \geq 0$ *with* $a_{ij} > 0$ *we have that* $\mathsf{X}_A \sim \mathsf{X}_{A^{(ij)}}$ *where*

$$A^{(ij)} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} + a_{j1} & \dots & a_{ij} + a_{jj} - 1 & \dots & a_{in} + a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

Step 1

Outsplit to go

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ a_{11} & a_{11} & a_{12}-1 \\ a_{21} & a_{21} & a_{22} \end{bmatrix}$$

Step 2

Insplit to go

$$\begin{bmatrix} 0 & 0 & 1 \\ a_{11} & a_{11} & a_{12}-1 \\ a_{21} & a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ a_{11} & a_{11} & 0 & a_{12}-1 \\ a_{21} & a_{21} & 0 & a_{22} \\ a_{21} & a_{21} & 0 & a_{22} \end{bmatrix}$$

Step 3

Symbol reduce to go

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ a_{11} & a_{11} & 0 & a_{12}-1 \\ a_{21} & a_{21} & 0 & a_{22} \\ a_{21} & a_{21} & 0 & a_{22} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{11} & a_{12}-1 \\ a_{21} & a_{21} & a_{22} \\ a_{21} & a_{21} & a_{22} \end{bmatrix}$$

Step 4

Out-amalgamate to go

$$\begin{bmatrix} a_{11} & a_{11} & a_{12}-1 \\ a_{21} & a_{21} & a_{22} \\ a_{21} & a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} a_{11}+a_{21} & a_{12}+a_{22}-1 \\ a_{21} & a_{22} \end{bmatrix}$$

Proposition

For any $A \geq 0$ there is a $B \geq 0$ such that

$$\mathsf{X}_A \sim \mathsf{X}_{I+B}$$

Proof

If all $a_{jj} > 0$ we are done. If not, employ that

$$A^{(ij)} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} + a_{j1} & \dots & a_{ij} - 1 & \dots & a_{in} + a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

to create a zero column, which may then be deleted.

Proposition

If a row or column addition takes an irreducible matrix $B \geq 0$ to $B' \geq 0$, we have

$$\mathsf{X}_{I+B} \sim \mathsf{X}_{I+B'}$$

Proof

Suppose row 2 of B is added to row 1 to create B'. The first row of $I + B'$ is

$$\begin{bmatrix} 1 + b_{11} + b_{21} & b_{12} + b_{22} & b_{13} + b_{23} & \ldots \end{bmatrix}$$

and the first two rows of $I + B$ are

$$\begin{bmatrix} 1 + b_{11} & b_{12} & b_{13} & \ldots \\ b_{21} & 1 + b_{22} & b_{23} & \ldots \end{bmatrix}$$

Note how this coincides with "basic move" when $b_{12} > 0$. In general, use irreducibility.

Proposition

Let an irreducible matrix $B \geq 0$ be of size $n \times n$ with $n > 1$. Then

$$\mathsf{X}_{I+B} \sim \mathsf{X}_{I+C}$$

where we may assume that $C > 0$ of any size $m \geq n$.

Proof

We may keep adding rows until all entries are $\geq N$ for any $N > 0$. New rows may be added as required by state splitting as soon as the entries are sufficiently large.

Proposition

When $C > 0$ we have $\mathsf{X}_{I+C} \sim \mathsf{X}_{I+D}$ where the first column of D is identically d, with

$$d = \gcd\{c_{ij}\} = \gcd\{d_{ij}\}$$

Proof

Subsequent "column prepared row subtractions" and "row prepared column subtractions". See example.

Standard form 1

When $C > 0$ is a given $n \times n$-matrix with $\mathbb{Z}^n / C\mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z}/d_i\mathbb{Z}$ where

$$d_1 \mid d_2 \mid \cdots \mid d_n$$

and $\det(-C) = (-1)^n \det(C) < 0$ we have that $\mathsf{X}_{I+C} \sim \mathsf{X}_{I+D}$ where

$$D = \begin{bmatrix} 0 & & \dots & 0 & d_n \\ d_1 & 0 & & 0 & 0 \\ 0 & d_2 & & 0 & 0 \\ & & \ddots & & \vdots \\ 0 & & \dots & d_{n-1} & 0 \end{bmatrix}$$

Standard form 2

When $C > 0$ is a given $n \times n$-matrix with $\mathbb{Z}^n / C\mathbb{Z}^n = \sum_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}$ where

$$d_1 \mid d_2 \mid \cdots \mid d_n$$

and $\det(-C) = (-1)^n \det(C) > 0$ we have that $\mathsf{X}_{I+C} \sim \mathsf{X}_{I+D}$ where

$$D = \begin{bmatrix} 0 & & \dots & d_{n-1} & d_{n-1} \\ d_1 & 0 & & 0 & 0 \\ 0 & d_2 & & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ 0 & & \dots & d_{n-1} & d_{n-1} + d_n \end{bmatrix}$$

Standard form 3

When $C > 0$ is a given $n \times n$-matrix with $\mathbb{Z}^n / C\mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z}/d_i\mathbb{Z}$ where

$$d_1 \mid d_2 \mid \cdots \mid d_n$$

and $\operatorname{rank}(C) = k < n$ we have that $\mathsf{X}_{I+C} \sim \mathsf{X}_{I+D}$ where

$$D = \begin{bmatrix} 0 & & \dots & 0 & d_k & \dots & d_k \\ d_1 & 0 & & 0 & 0 & \dots & 0 \\ 0 & d_2 & & & 0 & & 0 \\ & & \ddots & \vdots & \vdots & & \\ & & & d_{k-1} & 0 & \dots & 0 \\ 0 & & \dots & 0 & d_k & \dots & d_k \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & & \dots & 0 & d_k & \dots & d_k \end{bmatrix}$$

Outline

Reducible case

Theorem

Suppose A is in $\mathfrak{M}^{\circ}_{\mathcal{P},+}(\mathbb{Z})$ and A' is in $\mathfrak{M}^{\circ}_{\mathcal{P}',+}(\mathbb{Z})$ The following are equivalent.

1. *The SFTs X_A and $\mathsf{X}_{A'}$ are flow equivalent.*
2. *For some permutation matrix implementing an isomorphism from $\mathcal{P}$ to $\mathcal{P}'$, there exists a positive $SL_{\mathcal{P}}(\mathbb{Z})$ equivalence from $I - A$ to $I - P^{-1}A'P$*
3. *For some permutation matrix implementing an isomorphism from $\mathcal{P}$ to $\mathcal{P}'$, and sending cycle components to cycle components, there exists an $SL_{\mathcal{P}}(\mathbb{Z})$ equivalence from $I - A$ to $I - P^{-1}A'P$ which is positive on cycle components.*

Equivariant case

Theorem

Let G be a finite group, and let A and B be square matrices over $\mathbb{Z}^+G$. Then X_A and X_B are G-flow equivalent precisely when $I - A$ and $I - B$ are G-positively equivalent.

Theorem

Let G be a finite group, and let A and B be essentially irreducible nontrivial matrices over $\mathbb{Z}_+G$. For X_A and X_B to be G-flow equivalent, it is necessary that $W(A) = W(B)$.
Suppose $W(A) = G$. Then the following are equivalent:

1. *X_A and X_B are G-flow equivalent.*
2. *There exists $\gamma \in G$ and an $E(\mathbb{Z}G)$ equivalence from $(I - A)$ to $I - \gamma B \gamma^{-1}$.*