Ordinary differential equations

The following exercises must be handed in in a hand-written format. *Deadline:* Tuesday, 23rd of October, 4pm.

1. *(2 points).* Find the 2nd order Taylor approximation at $x = 0$ of the functions:

$$\begin{aligned} f(x) &= xe^x, \\ f(x) &= \frac{1}{x-1}. \end{aligned}$$

2. *(2 points).* Consider the following differential equations:

$$\begin{array}{llll} \textit{(i)} & y^2 + y = y', & \textit{(iv)} & y^{(2)} - y = y', \\ \textit{(ii)} & y^{(4)} - y^{(3)} = 0, & \textit{(v)} & (t+2)y' + y = 0, \\ \textit{(iii)} & y' = 2y + 1, & \textit{(vi)} & (y')^3 = y^2. \end{array}$$

Give the order of each of them. Decide which of them are linear with constant coefficients, linear with non-constant coefficients and non-linear. Decide also which of them are autonomous and which of them are not.

3. *(3 points).* Consider the differential equation

$$y' = f(y), \quad \text{with } f(y) = y^3 - 4y.$$

Determine the equilibrium points and their stability, both in a visual way and in an analytical way.

To visualize the graphic of the function $f(y)$, you can use the Octave commands:

$$\begin{aligned} &> x = linspace(-3, 3, 100) \\ &> y = x.^3 - 4 * x \\ &> z = 0 * x \\ &> plot(x, y, x, z) \end{aligned}$$

Draw a sketch of the obtained graphic in the hand-written paper and use it to solve the exercise.

4. *(1 point).* A spherical water drop loses volume by evaporation at a rate proportional to its surface area. If k is this constant of proportionality, show that the differential equation

$$\frac{dr}{dt} = 3k$$

describes the phenomena. Solve it in order to express the radius at time t in terms of the constant of proportionality and its radius r_0 at $t = 0$. [*Hints: the surface of a sphere is given by the formula* $4\pi r^2$*, if* r *is the radius; the volume of the sphere is given by the formula* $4/3\pi r^3$.]