Diagonalization and linear maps

The following exercises must be handed over before the class of the 9th of October starts.

Part 1: Theoretical questions.

1. *(1.5 points).* Decide whether the following statements are true or false. Give a detailed answer.

(i) All matrices are diagonalizable.

(ii) The eigenvectors of a matrix A are the vectors v satisfying $Av = \lambda v$, for some λ.

(iii) If the eigenvectors of a matrix of order $n \times n$ generate $\mathbb{R}^n$, then the matrix is diagonalizable.

2. *(0.5 points).* Explain why the following statement is true: "If a matrix is diagonalizable, then its determinant is equal to the product of the eigenvalues (with the exponent corresponding to their multiplicity in the characteristic polynomial)".

Part 2: Hand-written exercises. The following exercises must be solved hand-written. All the computations must be detailed.

3. *(2 points).* Consider the following matrices:

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 4 \end{pmatrix}.$$

(i) Find the characteristic polynomial for each of the matrices.

(ii) Find the zeroes of the characteristic polynomial, that is, the eigenvalues.

(iii) Compute the spaces of eigenvectors for each eigenvalue.

(iv) Which matrices are diagonalizable?

4. *(0.5 points).* Decide which of the following maps $f : \mathbb{R}^2 \to \mathbb{R}$ are linear:

$$\begin{array}{rlrl} \text{(i)} & f(x,y) = x + y - 1, & \text{(iv)} & f(x,y) = \frac{x+2y}{3}, \\ \text{(ii)} & f(x,y) = \frac{2x+y}{x-y}, & \text{(v)} & f(x,y) = 4x - y, \\ \text{(iii)} & f(x,y) = x + y^2, & \text{(vi)} & f(x,y) = \log(x). \end{array}$$

5. *(0.5 points).* Consider the linear map

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{f} & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (2x + y, -z + x, 2x) \end{array}$$

Give the matrix associated.

Part 3: Octave. To be solved with Octave. The file must be uploaded in the assignment prepared for that.

6. *(0.5 points).* Check the results you obtained in exercise 3, using the command $[V, x] = eig(matrix)$.