# Løsninger 6

**1.** **a)** The self insurance function is $s(x) = (1-a)x$ and hence the MGF for the claims under reinsurance is

$$M_{s(Y)}(\alpha) = \mathbf{E}\left[e^{\alpha(1-a)Y_i}\right] = M_Y((1-a)\alpha).$$

Now for an $\exp(\theta)$ distribution,

$$\mu_Y = \frac{1}{\theta} \quad \text{and} \quad M_Y(\alpha) = \frac{\theta}{\theta-\alpha}.$$

Hence

$$\Lambda(\alpha) = \alpha c_a + \lambda\left\{\frac{\theta}{\theta+(1-a)\alpha} - 1\right\},$$

where

$$c_a = \frac{\lambda}{\theta}\left\{(\xi-\xi_r) + (1-a)(1+\xi_r)\right\}.$$

Setting $\Lambda(-R) = 0$ then yields

$$Rc_a = \lambda\left(\frac{R(1-a)}{\theta - R(1-a)}\right),$$

or

$$R = \frac{\theta}{1-a} - \frac{\lambda}{c_a} = \theta\left\{\frac{1}{1-a} - \frac{1}{-(\xi_r-\xi)+(1-a)(1+\xi_r)}\right\}.$$

**b)** Let $b = 1-a$, and set

$$\frac{dR(b)}{db} = 0$$

and solve. This yields

$$-\frac{1}{b^2} + \frac{1+\xi_r}{(-(\xi_r-\xi)+b(1+\xi_r))^2} = 0$$

and hence

$$b = \left(1-\frac{\xi}{\xi_r}\right)\left(1+\sqrt{1-\frac{\xi_r}{1+\xi_r}}\right).$$

(Here we have used the net profit condition to determine that we should take the positive root on the right hand side.)

**c)** In this case we get $b = 0$, or $a = 1$. In other words, we choose full insurance, meaning that all claims are covered, the premiums income is zero, and the insurance company is never ruined but also earns no additional capital. Ruin is minimized but profits are not, and so minimizing ruin is obviously not the correct criteria for optimizing the insurance business in any meaningful sense.

**2.** **a)** The mean value of the claims is $2/1.5 = 4/3$. Thus $\lambda\mu = 8/3$ which is smaller than $c = 5 = 15/3$.

**b)** $c - \lambda\mu = 7/3$ and the moment generating function is

$$M_Y(r) = \big(\frac{1.5}{1.5 - r}\big)^2 = \big(\frac{3}{3 - 2r}\big)^2 .$$

Therefore

$$\begin{aligned}\frac{c - \lambda\mu}{cs - \lambda(1 - M_Y(-s))} &= \frac{7}{15s - 6(1 - 9/(3+2s)^2)} \\ &= \frac{7(3+2s)^2}{15s(3+2s)^2 - 6((3+2s)^2 - 9)} \\ &= \frac{63 + 84s + 28s^2}{s(135 + 180s + 60s^2 - 72 - 24s)} \\ &= \frac{28s^2 + 84s + 63}{s(60s^2 + 156s + 63)}\end{aligned}$$

**c)** For $\hat{\psi}(s)$ we obtain

$$\begin{aligned}\frac{1}{s} - \frac{28s^2 + 84s + 63}{s(60s^2 + 156s + 63)} &= \frac{60s^2 + 156s + 63 - 28s^2 - 84s - 63}{s(60s^2 + 156s + 63)} \\ &= \frac{32s + 72}{60s^2 + 156s + 63} .\end{aligned}$$

**d)** The denominator has the roots

$$\frac{-78 \pm \sqrt{78^2 - 60 \cdot 63}}{60} = \frac{-39 \pm \sqrt{39^2 - 15 \cdot 63}}{30} = \begin{cases} -0.5 \\ -2.1 \end{cases}$$

and thus

$$\hat{\psi}(s) = \frac{32s + 72}{3(2s+1)(10s+21)} .$$

We want to write this as

$$\frac{32s + 72}{3(2s+1)(10s+21)} = \frac{A}{2s+1} + \frac{B}{10s+21} .$$

We multiply the above equation by $2s + 1$

$$\frac{32s + 72}{3(10s+21)} = A + \frac{B(2s+1)}{10s+21}$$

and let then $s = -0.5$. Thus $A = 7/6$. Analogously $B = -1/2$. Thus

$$\hat{\psi}(s) = \frac{7}{12}\frac{1}{s+0.5} - \frac{1}{20}\frac{1}{s+2.1}$$

which has the inverse

$$\psi(u) = \frac{7}{12}\mathrm{e}^{-0.5u} - \frac{1}{20}\mathrm{e}^{-2.1u}\,.$$

**3.** **a)** Given $\tau_i < \infty$ we find $\mathbb{P}[\tau_{i+1} < \infty \mid \tau_i < \infty] = \lambda/(\alpha c)$. It follows by induction that

$$\mathbb{P}[K \ge n] = \mathbb{P}[\tau_n < \infty] = \mathbb{P}[\tau_n < \infty \mid \tau_{n-1} < \infty]\mathbb{P}[\tau_{n-1} < \infty] = \Bigl(\frac{\lambda}{\alpha c}\Bigr)^n\,.$$

Thus $K$ has a geometric distribution (NB(1,$1 - \lambda/(\alpha c)$))

$$\mathbb{P}[K = n] = \Bigl(1 - \frac{\lambda}{\alpha c}\Bigr)\Bigl(\frac{\lambda}{\alpha c}\Bigr)^n\,.$$

**b)** Note that $\mu = \alpha^{-1}$. $L_1$ has density $\alpha(1 - (1 - \mathrm{e}^{-\alpha x})) = \alpha\mathrm{e}^{-\alpha x}$ and is therefore $\mathrm{Exp}(\alpha)$ distributed.

**c)** $S$ has a compound negative binomial distribution. Its moment generating function is

$$\begin{aligned}
M_S(r) &= \mathbb{E}\bigl[\mathbb{E}\bigl[\exp\{r\sum_{n=1}^{K} L_n\} \mid K\bigr]\bigr] = \mathbb{E}\bigl[(M_L(r))^K\bigr] = M_K(\log(M_L(r)))\\
&= \frac{1 - \lambda/(\alpha c)}{1 - \lambda/(\alpha c)\,M_L(r)} = \frac{\alpha c - \lambda}{\alpha c - \lambda\alpha/(\alpha - r)} = \frac{(\alpha c - \lambda)(\alpha - r)}{\alpha c(\alpha - r) - \alpha\lambda}\\
&= \frac{\alpha c - \lambda}{\alpha c} + \frac{\lambda(\alpha c - \lambda)}{\alpha c(\alpha c - \lambda - cr)} = 1 - \frac{\lambda}{\alpha c} + \frac{\lambda}{\alpha c}\frac{\alpha - \lambda/c}{\alpha - \lambda/c - r}\,.
\end{aligned}$$

**d)** The moment generating function has the form $pM_X(r) + (1-p)M_Z(r)$ and thus $\mathbb{P}[S \le x] = p\mathbb{P}[X \le x] + (1-p)\mathbb{P}[Z \le x]$. $M_X(r) = 1$ for all $r$ and therefore $X = 0$ is deterministic. $M_Z(r)$ is easily seen to be the moment generating function of an $\mathrm{Exp}(\alpha - \lambda/c)$ random variable. It follows therefore that $\mathbb{P}[S \le x] = 0$ for $x < 0$ and for $x \ge 0$

$$\mathbb{P}[S \le x] = 1 - \frac{\lambda}{\alpha c} + \frac{\lambda}{\alpha c}(1 - \mathrm{e}^{-(\alpha - \lambda/c)x}) = 1 - \frac{\lambda}{\alpha c}\mathrm{e}^{-(\alpha-\lambda/c)x}\,.$$

**e)** We have

$$\psi(u) = 1 - \mathbb{P}[S \le u] = \frac{\lambda}{\alpha c}\mathrm{e}^{-(\alpha - \lambda/c)u}\,.$$

**4.** **a)** Note that $\log Y_1 \sim \mathrm{N}(m, \sigma^2)$. Thus

$$\begin{aligned}
\int_u^\infty \mathbb{P}[Y_1 > x]\,\mathrm{d}x &= \int_u^\infty \mathbb{P}[\log Y_1 > \log x]\,\mathrm{d}x\\
&= \int_u^\infty\int_{\log x}^\infty \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-(y-m)^2/(2\sigma^2)}\,\mathrm{d}y\,\mathrm{d}x\,.
\end{aligned}$$

**b)** Note that $\{(x,y) : u < x < \infty, \log x < y < \infty\} = \{(x,y) : \log u < y < \infty, u < x < \mathrm{e}^y\}$. Thus

$$\begin{aligned}\int_u^\infty \int_{\log x}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-(y-m)^2/(2\sigma^2)} \,\mathrm{d}y\,\mathrm{d}x &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\log u}^\infty \int_u^{\mathrm{e}^y} \mathrm{e}^{-(y-m)^2/(2\sigma^2)} \,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\log u}^\infty (\mathrm{e}^y - u)\mathrm{e}^{-(y-m)^2/(2\sigma^2)} \,\mathrm{d}y\,.\end{aligned}$$

**c)** The two functions coincide at $y = m$. The equation is therefore correct if the derivatives coincide. This is the case because

$$\frac{y-m}{\sigma^2} - 1 = \frac{y-m-\sigma^2}{\sigma^2}\,.$$

**d)** We get

$$\begin{aligned}\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\log u}^\infty \mathrm{e}^y \mathrm{e}^{-(y-m)^2/(2\sigma^2)} \,\mathrm{d}y &= \mathrm{e}^{m+\sigma^2/2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\log u}^\infty \mathrm{e}^{-(y-m-\sigma^2)^2/(2\sigma^2)} \,\mathrm{d}y \\ &= \mathrm{e}^{m+\sigma^2/2} \Phi\bigl(-\frac{\log u - (m+\sigma^2)}{\sigma}\bigr).\end{aligned}$$

That

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\log u}^\infty u\,\mathrm{e}^{-(y-m)^2/(2\sigma^2)} \,\mathrm{d}y = u\Phi\bigl(\frac{m - \log u}{\sigma}\bigr)$$

follows readily.

**e)** Because $B(u)$ is a subexponential distribution we have

$$\begin{aligned}\psi(u) &\sim \frac{\lambda}{c - \lambda\,\mathrm{e}^{m+\sigma^2/2}} \int_u^\infty \mathbb{P}[Y_1 > x]\,\mathrm{d}x \\ &= \frac{\lambda\,\mathrm{e}^{m+\sigma^2/2}}{c - \lambda\,\mathrm{e}^{m+\sigma^2/2}} \Phi\bigl(\frac{m+\sigma^2-\log u}{\sigma}\bigr) - \frac{\lambda u}{c - \lambda\,\mathrm{e}^{m+\sigma^2/2}} \Phi\bigl(\frac{m - \log u}{\sigma}\bigr)\,.\end{aligned}$$

Using the asymptotic formula $\Phi(x) \sim -(2\pi)^{-\frac{1}{2}}\mathrm{e}^{-x^2/2}/x$ as $x \to -\infty$ the approximation can further be simplified to

$$\begin{aligned}&\frac{\lambda\sigma}{\sqrt{2\pi}(c - \lambda\,\mathrm{e}^{m+\sigma^2/2})} \\ &\qquad \times \bigl(\frac{\mathrm{e}^{m+\sigma^2/2}}{\log u - m - \sigma^2}\mathrm{e}^{-(\log u - m - \sigma^2)^2/(2\sigma^2)} - \frac{u}{\log u - m}\mathrm{e}^{-(\log u - m)^2/(2\sigma^2)}\bigr) \\ &\quad = \frac{\lambda\sigma\mathrm{e}^{-((\log u)^2 + m^2)/(2\sigma^2)} u^{m/\sigma^2+1}}{\sqrt{2\pi}(c - \lambda\,\mathrm{e}^{m+\sigma^2/2})}\bigl(\frac{1}{\log u - m - \sigma^2} - \frac{1}{\log u - m}\bigr) \\ &\quad = \frac{\lambda\sigma^3\mathrm{e}^{-((\log u)^2 + m^2)/(2\sigma^2)} u^{m/\sigma^2+1}}{\sqrt{2\pi}(c - \lambda\,\mathrm{e}^{m+\sigma^2/2})(\log u - m - \sigma^2)(\log u - m)} \\ &\quad \sim \frac{\lambda\sigma^3\mathrm{e}^{-((\log u)^2 + m^2)/(2\sigma^2)} u^{m/\sigma^2+1}}{\sqrt{2\pi}(c - \lambda\,\mathrm{e}^{m+\sigma^2/2})(\log u)^2}\,.\end{aligned}$$

[*Note*: The solutions to Problems 2-4 are by H. Schmidli and correspond to the notation of his notes, which may be slightly different from the notation of the lectures.]