

Løsninger 5

1. a) If we start with initial capital $u - ch$ the survival probability becomes after time $h \wedge T_1$

$$\begin{cases} \delta(u), & \text{if } T_1 > h, \\ \delta(u - c(h - T_1) - Y_1), & \text{if } T_1 \leq h. \end{cases}$$

Using that T_1 has density $\lambda e^{-\lambda t}$ and $\mathbb{P}[T_1 > h] = e^{-\lambda h}$ we obtain the assertion.

- b) The integrand in a) is bounded by λ and therefore the integral converges to zero as $h \rightarrow 0$. Letting $h \rightarrow 0$ shows left-continuity.
- c) Reordering the terms yields

$$\begin{aligned} & c \frac{\delta(u - ch) - \delta(u)}{-ch} \\ &= \delta(u) \frac{1 - e^{-\lambda h}}{h} - \frac{1}{h} \int_0^h \int_0^{u-c(h-t)} \delta(u - c(h-t) - y) dG(y) \lambda e^{-\lambda t} dt. \end{aligned}$$

Note that the term with the integral can be written as

$$e^{-\lambda h} \frac{1}{h} \int_0^h \int_0^{u-ct} \delta(u - ct - y) dG(y) \lambda e^{\lambda t} dt.$$

Letting $h \downarrow 0$ yields the derivative from the left.

2. a) The net profit condition is $c > \lambda 2/\alpha$ or equivalently $\alpha c > 2\lambda$.
- b) From the equation obtained in the lecture we get

$$\begin{aligned} c\delta'(u) &= \lambda\delta(u) - \lambda \int_0^u \delta(u-y) \alpha^2 y e^{-\alpha y} dy \\ &= \lambda\delta(u) - \lambda \int_0^u \delta(z) \alpha^2 (u-z) e^{-\alpha(u-z)} dz \\ &= \lambda\delta(u) - \lambda u e^{-\alpha u} \int_0^u \delta(z) \alpha^2 e^{\alpha z} dz + \lambda e^{-\alpha u} \int_0^u \delta(z) z \alpha^2 e^{\alpha z} dz. \end{aligned}$$

c) Differentiating the above equation yields

$$\begin{aligned}
c\delta''(u) &= \lambda\delta'(u) - \lambda e^{-\alpha u} \int_0^u \delta(y)\alpha^2 e^{\alpha y} dy + \alpha\lambda u e^{-\alpha u} \int_0^u \delta(y)\alpha^2 e^{\alpha y} dy \\
&\quad - \lambda u \delta(u)\alpha^2 - \alpha\lambda e^{-\alpha u} \int_0^u \delta(y)y\alpha^2 e^{\alpha y} dy + \lambda\delta(u)u\alpha^2 \\
&= \lambda\delta'(u) - \lambda e^{-\alpha u} \int_0^u \delta(y)\alpha^2 e^{\alpha y} dy \\
&\quad - \alpha(-\lambda u e^{-\alpha u} \int_0^u \delta(y)\alpha^2 e^{\alpha y} dy + \lambda e^{-\alpha u} \int_0^u \delta(y)y\alpha^2 e^{\alpha y} dy) \\
&= \lambda\delta'(u) - \lambda e^{-\alpha u} \int_0^u \delta(y)\alpha^2 e^{\alpha y} dy - \alpha(c\delta'(u) - \lambda\delta(u)) \\
&= (\lambda - \alpha c)\delta'(u) + \alpha\lambda\delta(u) - \lambda e^{-\alpha u} \int_0^u \delta(y)\alpha^2 e^{\alpha y} dy.
\end{aligned}$$

Differentiating once more yields

$$\begin{aligned}
c\delta'''(u) &= (\lambda - \alpha c)\delta''(u) + \alpha\lambda\delta'(u) + \alpha\lambda e^{-\alpha u} \int_0^u \delta(y)\alpha^2 e^{\alpha y} dy - \lambda\delta(u)\alpha^2 \\
&= (\lambda - \alpha c)\delta''(u) + \alpha\lambda\delta'(u) - \alpha(\alpha\lambda\delta(u) - \lambda e^{-\alpha u} \int_0^u \delta(y)\alpha^2 e^{\alpha y} dy) \\
&= (\lambda - \alpha c)\delta''(u) + \alpha\lambda\delta'(u) - \alpha(c\delta''(u) - (\lambda - \alpha c)\delta'(u)) \\
&= (\lambda - 2\alpha c)\delta''(u) + \alpha(2\lambda - \alpha c)\delta'(u).
\end{aligned}$$

d) The general solution to the above differential equation is

$$\delta(u) = C - Ae^{r_1 u} - Be^{r_2 u}$$

where r_i are the roots of the polynomial

$$cr^2 + (2\alpha c - \lambda)r + \alpha(\alpha c - 2\lambda) = 0.$$

By the net profit condition $\alpha c - 2\lambda > 0$. Thus the two solutions are different from 0. We obtain

$$r_{1/2} = \frac{-(2\alpha c - \lambda) \pm \sqrt{(2\alpha c - \lambda)^2 - 4\alpha(\alpha c - 2\lambda)}}{2c}.$$

Since $(2\alpha c - \lambda)^2 - 4\alpha(\alpha c - 2\lambda) = \lambda^2 + 4\alpha\lambda c > 0$ it follows that $r_{1/2} \in \mathbb{R}$ and $r_1 \neq r_2$. Because $2\alpha c - \lambda > \alpha c - 2\lambda > 0$ the square root is smaller than $2\alpha c - \lambda$ and it follows that $r_{1/2} < 0$. The solutions can be written as

$$r_{1/2} = \frac{-(2\alpha c - \lambda) \pm \sqrt{\lambda^2 + 4\alpha\lambda c}}{2c}.$$

Finally, from $\delta(u) \rightarrow 1$ as $u \rightarrow \infty$, it follows that $C = 1$ and from $\delta(0) = 1 - 2\lambda/(\alpha c)$ one obtains $A + B = 2\lambda/(\alpha c)$.

3. a) The sample paths are clearly cadlag and the increments are clearly independent. To show that the increments are stationary, note that

$$\mathbf{E} \left[e^{\alpha(X_{t+s} - X_t)} \right] = \mathbf{E} \left[e^{\alpha X_s} \right].$$

The details were given in the solutions to the exercises for “SkadeStok” (see <http://www.math.ku.dk/~collamore/SkadeStok/>).

- b) In Part (c) below, it is shown that

$$\int_0^\infty z_R(x) dx = \frac{1}{cR}(c - \lambda\mu_Y),$$

where

$$z_R(u) := \frac{\lambda}{c} e^{Ru} \int_u^\infty (1 - G(y)) dy.$$

Thus, $\int_0^\infty z_R(x) dx$ exists and is finite.

Let

$$\overline{m}_k(h) := \sup \{ z_R(u) : (k-1)h \leq u \leq kh \},$$

$$\underline{m}_k(h) := \inf \{ z_R(u) : (k-1)h \leq u \leq kh \}.$$

In the definition of z_R , the function e^{Ru} is increasing in u , whereas the function $\int_u^\infty (1 - G(y)) dy$ is decreasing in u . Hence

$$\overline{m}_k(h) \leq e^{R \cdot kh} \int_{(k-1)h}^\infty (1 - G(y)) dy,$$

$$\underline{m}_k(h) \geq e^{R((k-1)h)} \int_{kh}^\infty (1 - G(y)) dy.$$

Thus, in particular,

$$\overline{m}_{k+1}(h) \leq e^{2Rh} e^{R((k-1)h)} \int_{kh}^\infty (1 - G(y)) dy.$$

What needs to be shown is that

$$\lim_{h \rightarrow 0} h \sum_{k=1}^{\infty} \underline{m}_k(h) = \lim_{h \rightarrow 0} h \sum_{k=1}^{\infty} \overline{m}_k(h).$$

The above estimates yield

$$\begin{aligned} 0 &\leq \lim_{h \rightarrow 0} h \sum_{k=1}^{\infty} \overline{m}_k(h) - \lim_{h \rightarrow 0} h \sum_{k=1}^{\infty} \underline{m}_k(h) \\ &\leq \lim_{h \rightarrow 0} \overline{m}_1(h) + \lim_{h \rightarrow 0} (e^{2h} - 1) \cdot h \sum_{k=1}^{\infty} e^{R((k-1)h)} \int_{kh}^\infty (1 - G(y)) dy = 0, \end{aligned}$$

where the last step follows since

$$\lim_{h \rightarrow 0} h \sum_{k=1}^{\infty} z_R(kh) = \int_0^{\infty} z_R(x) dx < \infty.$$

[Basically, a function will be directly Riemann integrable if its integral exists and is finite, i.e., it is Riemann integrable, *and* if it does not oscillate wildly at infinity. Here, z_R is a product of two functions; the first is increasing and the second is decreasing and their product is thus “well-behaved.” Hence one expects that this product should be d.R.i. For an example of a function which is *not* d.R.i., let

$$z(n) = a_n, \quad n = 1, 2, \dots;$$

let $z(x) = 0$ for all x which lies *outside* the intervals $|x - n| < h_n < 1/2$; and let $z(\cdot)$ vary linearly between n and $n \pm h_n$. If $\sum a_n h_n < \infty$, then the integral of $z(\cdot)$ will be finite, so it will be Riemann integrable, but in any case it will not be d.R.i. as long as $h_n \rightarrow 0$.]

c) By definition,

$$\mu_R = \int_0^{\infty} x e^{Rx} \cdot \frac{\lambda}{c} (1 - G(x)) dx = \frac{\lambda}{c} \int_0^{\infty} x e^{Rx} \left(\int_x^{\infty} dG(y) \right) dx.$$

Interchanging the order of integration in the last integral yields

$$\begin{aligned} \mu_R &= \frac{\lambda}{c} \int_0^{\infty} \left(\int_0^y x e^{Rx} dx \right) dG(y) \\ &= \frac{\lambda}{cR^2} \int_0^{\infty} \{ Rye^{Ry} - e^{Ry} + 1 \} dG(y) \\ &= \frac{\lambda}{cR^2} \{ RM'_Y(R) - M_Y(R) + 1 \} \\ &= \frac{1}{cR} \{ \lambda M'_Y(R) - c \}. \end{aligned} \tag{1}$$

The last step follows, since R is the adjustment coefficient and consequently satisfies the basic equation

$$R = \frac{\lambda}{c} (M_Y(R) - 1). \tag{2}$$

Next observe that, by definition,

$$\int_0^{\infty} z_R(x) dx = \int_0^{\infty} \frac{\lambda}{c} e^{Rx} \left(\int_x^{\infty} (1 - G(y)) dy \right) dx.$$

Interchanging the order of integration gives

$$\begin{aligned}
\int_0^\infty z_R(x)dx &= \frac{\lambda}{c} \int_0^\infty \left(\int_0^y e^{Rx} dx \right) (1 - G(y)) dy \\
&= \frac{\lambda}{cR} \int_0^\infty (e^{Ry} - 1)(1 - G(y)) dy \\
&= \frac{\lambda}{cR} \left\{ \int_0^\infty e^{Ry} \left(\int_y^\infty dG(z) \right) dy - \mu_Y \right\} \\
&= \frac{\lambda}{cR} \left\{ \int_0^\infty \left(\int_0^z e^{Ry} dy \right) dG(z) - \mu_Y \right\} \\
&= \frac{\lambda}{cR^2} \{M_Y(R) - 1 - \mu_Y R\} \\
&= \frac{1}{cR} (c - \lambda\mu_Y), \tag{3}
\end{aligned}$$

where the last step was obtained by another application of (2). The required result then follows from (1) and (3).

4. a) Just integrate left and right hand sides in the definition of $\tilde{\mu}$. Since $\Lambda(-R) = 0$, where Λ is the cumulant generating function of Z_i , it follows that the integral must equal one. Since $\Lambda'(R) < 0$, it follows upon differentiating this integral expression that $\tilde{\mathbf{E}}[Z_i] < 0$.

b) Note

$$\begin{aligned}
\Psi(u) &= \sum_k \int_{\mathfrak{P}_k} d\mu(z_1) \cdots d\mu(z_k) \\
&= \sum_k \int_{\mathfrak{P}_k} e^{R(z_1 + \cdots + z_k)} d\tilde{\mu}(z_1) \cdots d\tilde{\mu}(z_k) \\
&= \tilde{\mathbf{E}}[e^{RS_\tau}].
\end{aligned}$$

Note that since $\tilde{\mathbf{E}}[Z_i] < 0$, we have that ruin occurs with probability one under $\tilde{\mu}$, and so we don't need to condition on the event $\{\tau < \infty\}$ in the final expectation. Finally, note that $S_\tau < -u$ and Lundberg's inequality consequently holds. Roughly speaking, in the above approach we compare the probability of ruin under μ with this probability under $\tilde{\mu}$. The ruin probability under $\tilde{\mu}$ is one—which we know—and when comparing the two probabilities, we introduce the factor “ e^{RS_τ} ,” and this gives us a means for estimating the ruin probability in the original measure μ .

c) Elementary.

- d) Simulate random variables Z_1, Z_2, \dots having law $\tilde{\mu}$, and use this to generate sample paths S_1, S_2, \dots . Stop the process S_1, S_2, \dots when it falls below the level $-u$. (This

process will terminate with probability one, since the process has a negative drift, as discussed above.) Generate numerous paths, then compute the expectation on the r.h.s. of (1), and finally compute the sample mean and sample variance. Use these to obtain an estimate and corresponding confidence bounds; thus an approximation of the form

$$\bar{p} \pm z_{\beta} \bar{\sigma} / \sqrt{n},$$

where $\beta = .95$, say, and is a percentile obtained from the standard normal distribution function.

[*Note:* The solutions to Problems 1-2 are by H. Schmidli and correspond to the notation of his notes, which may be slightly different from the notation of the lectures.]