Løsninger 4

1. a) By definition,

$$\kappa_t(\alpha) = \mathbf{E}\left[e^{\alpha Y_t}\right] = \mathbf{E}\left[\mathbf{E}\left[e^{\alpha(X_1+\cdots+X_{N_t})}\,\middle|\, N_t\right]\right]. \tag{1}$$

Since $\{X_i\}$ is i.i.d.,

$$\mathbf{E}\left[e^{\alpha(X_1+\cdots+X_n)}\right] = \mathbf{E}\left[e^{\alpha X_1}\right]\cdots\mathbf{E}\left[e^{\alpha X_n}\right] = \hat{\kappa}(\alpha)^n$$

for any positive integer n. Hence (1) becomes

$$\kappa_t(\alpha) = \mathbf{E}\left[\hat{\kappa}(\alpha)^{N_t}\right] = \mathbf{E}\left[e^{(\log\hat{\kappa}(\alpha))N_t}\right]. \tag{2}$$

Since N_t has a Poisson(λt) distribution,

$$\mathbf{E}\left[e^{\alpha N_t}\right] = e^{\lambda t(e^{\alpha}-1)}.$$

Hence (2) yields

$$\kappa_t(\alpha) = e^{\lambda t(\hat{\kappa}(\alpha)-1)}.$$

b) The sample paths are clearly cadlag (just draw a the graph of a possible path). Also, the increments are independent, since $\{N_t\}$ has independent increments and the sequence $\{X_i\}$ is i.i.d.

What needs to be shown is that $\{Y_t\}$ has *stationary* increments, i.e.,

$$Y_{t+s} - Y_t \stackrel{d}{=} Y_s.$$

To this end, note that

$$\begin{aligned}\exp\{\lambda(t+s)(\hat{\kappa}(\alpha)-1)\} &= \mathbf{E}\left[e^{\alpha Y_{t+s}}\right] \\ &= \mathbf{E}\left[e^{\alpha(Y_{t+s}-Y_t)}\right]\mathbf{E}\left[e^{\alpha Y_t}\right] \text{ (by independence)} \\ &= \mathbf{E}\left[e^{\alpha(Y_{t+s}-Y_t)}\right]\cdot e^{\lambda t(\hat{\kappa}(\alpha)-1)},\end{aligned}$$

and hence

$$\mathbf{E}\left[e^{\alpha(X_{t+s}-X_t)}\right] = e^{\lambda s(\hat{\kappa}(\alpha)-1)} = \mathbf{E}\left[e^{\alpha X_s}\right],$$

which establishes stationarity (due to the fact that the moment generating function uniquely characterizes a distribution).

2. **a)** If a r.v. Y has a Normal$(0,1)$ distribution, then its moment generating function is given by

$$\mathbf{E}\left[e^{\alpha Y}\right] = e^{\alpha^2/2}.$$

Hence $X_1 - 1$ ∼Normal$(0,1)$ implies

$$\Lambda(\alpha) = \log \mathbf{E}\left[e^{\alpha(X_1-1)+\alpha}\right] = \log(e^{\alpha} \cdot e^{\alpha^2/2}) = \alpha(\frac{\alpha}{2}+1).$$

Letting α^* denote the nonzero solution to the equation $\Lambda(\alpha) = 0$, we then obtain that $(\alpha^*/2 + 1) = 0$, or $\alpha^* = -2$.

b) Since $\Lambda(\alpha^*) = 0$, $M_t \stackrel{\text{def}}{=} e^{\alpha^* X_t}$ is a martingale. Hence by the optional sampling theorem,

$$\begin{aligned} 1 &= \mathbf{E}\big[e^{\alpha^* X_T}\big] \\ &= e^{\alpha^* a}\mathbf{P}\{X_T \geq a\} + e^{-\alpha^* a}\mathbf{P}\{X_T \leq -a\} \\ &= e^{-2a}\left(1 - \mathbf{P}\{X_T \leq -a\}\right) + e^{2a}\mathbf{P}\{X_T \leq -a\}, \end{aligned}$$

which yields

$$\mathbf{P}\{X_T \leq -a\} = \frac{1-e^{-2a}}{e^{2a} - e^{-2a}}.$$

3. **a)** The argument is the same for the processes $\{N_t^{(1)}\}$ and $\{N_t^{(0)}\}$, so it suffices to establish the result for $\{N_t^{(1)}\}$.

Since $\{N_t\}$ and $\{X_i\}$ are independent,

$$\mathbf{P}\left\{N_h^{(1)}\right\} = \lambda h \mathbf{P}\{X_1 = 1\} + o(h) = \lambda p h + o(h),$$

and

$$\mathbf{P}\left\{N_h^{(1)} \geq 2\right\} \leq \mathbf{P}\{N_h \geq 2\} = o(h).$$

Furthermore, the increments of $N_t^{(1)}$ are independent, since $\{N_t\}$ has independent increments and the sequence $\{X_i\}$ is i.i.d. (implying that the random jumps $\{X_N, \ldots, X_{N'}\}$ which occur in some time interval $[t,s]$ are independent of those jumps which occur at some subsequent time interval $[\tilde{t}, \tilde{s}]$, where $\tilde{t} \geq s$). Also, the increments of $N_t^{(1)}$ are stationary, since

$$\begin{aligned} &\mathbf{P}\big\{N_{t+s} - N_t = k; X_{N+1} = x_1, \ldots, X_{N+k} = x_k\big\} \\ &\qquad = \mathbf{P}\{N_s = k;\ X_1 = x_1, \ldots, X_k = x_k\}. \end{aligned}$$

(Alternatively, you could also proceed by computing the moment generating function for the corresponding intervals, as in Problem 2(b) below.)

b) Recall that the probability of m "heads" in $m+n$ trials is given by

$$\frac{(m+n)!}{m!n!}p^m(1-p)^n,$$

where p is the probability of "heads." [This fact is used in the standard derivation of the binomial distribution.] Hence

$$\begin{aligned}\mathbf{P}\left\{N_t^{(1)}=m,\, N_t^{(0)}=n\right\} &= \mathbf{P}\left\{N_t=m+n \text{ and } X_1+\cdots+X_{k+l}=m\right\}\\ &= \frac{(\lambda t)^{m+n}}{(m+n)!}e^{-\lambda t}\cdot\frac{(m+n)!}{m!n!}p^m q^n\\ &= \frac{(\lambda p t)^m}{m!}e^{-\lambda p t}\cdot\frac{(\lambda q t)^n}{n!}e^{-\lambda q t}\\ &= \mathbf{P}\left\{N_t^{(1)}=m\right\}\mathbf{P}\left\{N_t^{(0)}=n\right\},\end{aligned}$$

which establishes independence.

4. (af H. Schmidli)

a) i) We condition on the first event T_1. If $T_1=s$ there are to different cases. If $s\le t$ a renewal process starts at s and the quantity of interest is $Z(t-s)$. If $s>t$ we have $N_t=0$ and therefore $A_t=s-t$. The event of interest is $\mathbb{1}_{s-t>x}$. This yields the renewal equation

$$Z(t)=1-F(t+x)+\int_0^t Z(t-s)\,\mathrm{d}F(s)\,.$$

ii) The function $1-F(t+x)$ is decreasing. Therefore it is directly Riemann integrable if it is integrable. The integral is

$$\int_0^\infty 1-F(t+x)\,\mathrm{d}t=\int_x^\infty 1-F(t)\,\mathrm{d}t\le\int_0^\infty 1-F(t)\,\mathrm{d}t=\lambda^{-1}<\infty$$

and is therefore finite.

iii) The renewal theorem yields the limit

$$\lambda\int_0^\infty 1-F(t+x)\,\mathrm{d}t=\lambda\int_x^\infty 1-F(t)\,\mathrm{d}t\,.$$

b) i) If $T_1=s$ and $s\le t$ the event of interest is $Z(t-s)$. If $s>t$ we have $B_t=t$ and the event of interest is $\mathbb{1}_{t\le x}$. This yields the renewal equation

$$Z(t)=\mathbb{1}_{t\le x}(1-F(t))+\int_0^t Z(t-s)\,\mathrm{d}F(s)\,.$$

ii) The function $\mathbb{1}_{t \leq x}(1 - F(t))$ is decreasing. Therefore it is directly Riemann integrable if it is integrable. The integral is

$$\int_0^\infty \mathbb{1}_{t \leq x}(1 - F(t))\,\mathrm{d}t = \int_0^x 1 - F(t)\,\mathrm{d}t \leq x < \infty$$

and is therefore finite.

iii) The renewal theorem yields the limit

$$\lambda \int_0^\infty \mathbb{1}_{t \leq x}(1 - F(t))\,\mathrm{d}t = \lambda \int_0^x 1 - F(t)\,\mathrm{d}t\ .$$

Note that the asymptotic distributions are the same as the initial distribution in the stationary case.