Øvelse 3

1. Let $\bar{u}(y)$ be the utility function of an insurer and $\overline{w}$ its initial capital. Suppose that the premium depends on the pure net premium only. Let $0 \leq r(x) \leq x$ be an arbitrary insurance form such that $\mathbb{P}[r(X) = X] < 1$ and let $r_b(x) = x \wedge b$ be the first risk deductible.

a) Show that there exists a b such that $\mathbb{E}[r_b(X)] = \mathbb{E}[r(X)]$.

b) Show that if there were $b < b'$ such that $\mathbb{E}[r_b(X)] = \mathbb{E}[r_{b'}(X)]$ then $\mathbb{P}[X > b] = 0$. If, moreover, $\mathbb{E}[r_b(X)] = \mathbb{E}[r(X)]$ show that $r(x)$ would be full insurance (which we had excluded). Thus b is unique.

c) Show that $r_b(x)$ with b chosen as above yields a higher utility for the insurer if $\mathbb{P}[r(X) \neq r_b(X)] > 0$.

2. We adopt the usual assumptions in utility theory. A risk X is to be insured, and the customer is offered a policy with self-insurance function

$$s(x) = \begin{cases} x & \text{if } x \leq b, \\ b + k(x-b)\,, & \text{if } x > b, \end{cases}$$

where $b \geq 0$ and $0 \leq k \leq 1$.

a) How should b and k be chosen optimally from the point of view of the insured? (Comparisons are for fixed expected self-insurance amount and fixed premium as usual).

b) Suppose the insurer requires $k \geq k_0$ for some strictly positive k_0 and that $\mathbb{E}[s(X)]$ should be a fixed amount $\eta > k_0\mathbb{E}[X]$. What is then the optimal solution?

c) How should b and k be chosen optimally from the point of view of the insurer?

3. Consider a market with two agents (insured, insurer) and a risk $\boldsymbol{X} = (X, 0)^\top$, utility functions $\{u(x), \bar{u}(x)\}$ and initial wealths $\{w, \overline{w}\}$. We consider the risk exchange $\boldsymbol{f}(\boldsymbol{X}) = (X - r(X) + \pi, r(X) - \pi)^\top$.

a) Explain why stop-loss insurance cannot be Pareto-optimal, unless there is $x_1 < b < x_2$ such that $\mathbb{P}[X \in \{x_1, x_2\}] = 1$, or $X = \mathbb{E}[X]$.

b) Give an example of utility functions $\{u(x), \bar{u}(x)\}$ for which proportional insurance is Pareto-optimal.

c) Show that the insurer will not sign a contract if the free parameter θ_2 for a Pareto-optimal risk exchange fulfils $\theta_2 \geq \bar{u}'(\overline{w})/u'(w)$.
Hint: Have a look at the premium.

4. Determine for each of the following specifications of the utility functions $u_i(x)$ the general form of the Pareto-optimal risk exchange $\boldsymbol{f}(\boldsymbol{X})$. Discuss the dependence on w_i, on the parameters of $u_i(x)$ and on the free parameters θ_i. Look for solutions of well-known forms, e.g. quota share exchanges. Observe in particular the rôle of the θ_i, for instance, whether they appear in the premia $f_i(0,\ldots,0)$ only. In the latter case, the premia are the only issue to negotiate. Note that some of the utility functions place restrictions on the range of $\boldsymbol{X}$ and the admissible $\boldsymbol{f}(\boldsymbol{X})$.

a) Quadratic utility functions $u_i(x) = a_i + b_i x(1 - x/(2c_i))$, $b_i > 0$, $x \leq c_i < \infty$.

b) Exponential utility functions $u_i(x) = a_i - b_i \mathrm{e}^{-c_i x}$, $b_i, c_i > 0$, $x \in \mathbb{R}$.

c) Logarithmic utility functions $u_i(x) = a_i + b_i \log(c_i + x)$, $b_i > 0$, $x > -c_i$.