Øvelse 1

1. Let $u(y)$ be a function on an interval I.

a) Suppose $u(y)$ is concave.

i) Show that there exists a function $k : I \to \mathbb{R}$ such that for any $y, x \in I$

$$u(x) \leq u(y) + k(y)(x - y)\,. \tag{1}$$

ii) Show that the function $k(y)$ is decreasing.

iii) Show that if $u(y)$ is strictly concave then the above inequality is strict for $x \neq y$ and $k(y)$ is strictly decreasing.

b) Suppose that a function $k(y)$ exists such that (1) is fulfilled.

i) Show that $u(y)$ is concave.

ii) Suppose, moreover, that the strict inequality holds in (1) for $x \neq y$. Show that $u(y)$ is strictly concave.

2. Let $u(y)$ be a strictly increasing concave function. Then the inverse function $u^{-1}(t)$ exists. Show that $u^{-1}(t)$ is convex. Show that $u^{-1}(t)$ is strictly convex if $u(y)$ is strictly concave.

3. Let $\{u_n(y) : n \in \mathbb{N}\}$ be a family of concave functions such that $u_n(y)$ converges pointwise to a function $u(y)$ and let $c > 0$.

a) Show that $u_1(y) + u_2(y)$ is concave.

b) Show that $cu_1(y)$ is concave.

c) Suppose $u_1(y)$ is increasing. Show that $u_1(u_2(y))$ is concave.

d) Show that $u(y)$ is concave.

e) Find an example, for which $u_1(y)u_2(y)$ is not concave.

f) Suppose $u_1(y)$ and $u_2(y)$ are strictly concave. Show that also $u_1(y) + u_2(y)$, $cu_1(y)$ and $u_1(u_2(y))$ (provided $u_1(y)$ is strictly increasing) are strictly concave.

g) Find an example where all the $u_n(y)$ are strictly concave, but $u(y)$ is not.

4. Suppose that $u(y)$ is a strictly concave function. Show that for all $h > 0$ the function

$$y \mapsto u(y + h) - u(y)$$

is strictly decreasing.