Solutions to:

Forsikringsvidenskabelig kandidateksamen
FM2B Skadesforsikringsmatematik

Københavns Universitet, Forår 2005 (J. F. Collamore)

Skriftlig prøve den 23.06.05 kl. 10:00 – 12:00

Problem 1. (a) We have:

$$X_t = \tilde{c}t + \sum_{i=1}^{N_t} \tilde{Y}_t,$$

where $\tilde{c} = c - b = 3/2$, $\tilde{Y}_i = Y_i/2$, and N_t is a Poisson(λ) process. The Lundberg bound is

$$\mathbf{P}\left\{X_t < -u, \text{ some } t \geq 0\right\} < e^{-Ru},$$

where $R > 0$ is determined from the equation

$$0 = \Lambda(-R) = -R\tilde{c} - \lambda\left(1 - M_{\tilde{Y}}(R)\right).$$

To solve, recall that $Y_i \sim \text{Exp}(1)$ and hence

$$M_{\tilde{Y}}(\alpha) \equiv \mathbf{E}\left[e^{\alpha Y_i/2}\right] = M_Y(\alpha/2) = \frac{1}{1-\alpha/2}.$$

Therefore

$$1 = \frac{\lambda}{R\tilde{c}}\left(M_{\tilde{Y}} - 1\right) = \frac{\lambda}{R\tilde{c}}\frac{R}{2-R} = \frac{4/3}{2-R},$$

and hence $R = 2/3$.

(b) Denote the desired probability by $\psi(u, uT)$. Then

$$\psi(u, uT) \leq \exp\left\{-Tu \sup_{\alpha \leq -R}\left\{-\frac{\alpha}{T} - \Lambda(\alpha)\right\}\right\}.$$

Let $g(\alpha) = -\alpha/T - \Lambda(\alpha)$. Then supremum is maximized at the point α_0 where $g'(\alpha_0) = 0$, provided that $\alpha_0 \leq -R$, and at the point $-R$ otherwise. Note that

$$0 = g'(\alpha_0) = -\frac{1}{T} - \Lambda'(\alpha_0), \text{ or } \Lambda'(\alpha_0) = -\frac{1}{T}.$$

Case 1: $T < \rho$.

Note that ρ satisfies the equation

$$\rho = -\frac{1}{\Lambda'(-R)}(= \frac{4}{3}),$$

and hence

$$\Lambda'(\alpha_0) = -\frac{1}{T} < -\frac{1}{\rho} = \Lambda'(-R).$$

Since Λ' is increasing on $(-\infty, -R]$ (draw the graph), this implies that $-R > \alpha_0$, and hence

$$Tu \sup_{\alpha \leq -R} g(\alpha) = Tug(\alpha_0) > Tug(-R) = Tu(R/T) = Ru,$$

i.e., the decay in this case is *faster* than that obtained from the classical Lundberg inequality.

Case 2: $T > \rho$.

In this case, the same argument gives that $\Lambda'(\alpha_0) > \Lambda'(-R)$, and hence $-R < \alpha_0$. Consequently, the maximum over the specified interval occurs at the endpoint, that is at the point $-R$, and so we have

$$Tu \sup_{\alpha \le -R} g(\alpha) = Tug(-R) = Ru.$$

Problem 2. (a) Note that

$$\Phi(u) = \mathbf{P}\left\{S_n < -u, \text{ some } n \ge 1\right\},$$

where $S_n = \log Y_n$. Moreover,

$$S_n = U_1 + \cdots + U_n, \text{ where } U_i = \log W_i + \log Z_i,$$

$\log Z_i$ is *negative* and heavy-tailed, and $\log W_i$ is *positive* and has an Exp(2/3) distribution. By comparison, consider the Cramér-Lundberg process

$$X_t = t - \sum_{i=1}^{N_t} \tilde{Y}_i,$$

where N_t is a Poisson(2/3) process. According to the random walk representation, the probability of ruin for this process is

$$\mathbf{P}\left\{\tilde{S}_n < -u, \text{ some } n \ge 1\right\},$$

where

$$\tilde{S}_n = \tilde{Z}_1 + \cdots + \tilde{Z}_n \text{ and } \tilde{Z}_i = \tau_i - \tilde{Y}_i.$$

Here $\{\tau_i\}$ is Exp(2/3) and $\tilde{Y}_i$ has the same distribution as $\log Z_i$. Hence $\{S_n\}$ has the same probability law as $\{\tilde{S}_n\}$.

(b) Since $\log Z_i$ is subexponential, we obtain that the probability of ruin for $\{\tilde{S}_n\}$, hence $\{S_n\}$, decays as follows:

$$\Phi(u) \sim \frac{\lambda}{c - \lambda\mu_{\tilde{Y}}} \int_u^\infty \frac{1}{(y+1)^2} dy = \frac{2/3}{1-2/3} \frac{1}{u+1},$$

since $c = 1$, $\lambda = 2/3$, and

$$\mu_{\tilde{Y}} = \int_0^\infty \frac{1}{(y+1)^2} dy = -\frac{1}{y+1}\Big|_0^\infty = 1.$$

Problem 3. (a) We need to find the solution π_r to the equation

$$\mathbf{E}\left[u(w - \pi_r - s(X) - s(Y)\right] = \mathbf{E}\left[u(w - X - Y)\right].$$

In our setting, $u(y) = -e^{-y}$, $s(X) = \max\{X, 5\}$, and $s(Y) = 0$, and so we obtain

$$e^{\pi_r}\mathbf{E}\left[e^{\min\{X,5\}}\right] = M_X(1)M_Y(1) = \frac{2}{2-1} \cdot \frac{3}{3-1} = 3.$$

Moreover,

$$\mathbf{E}\left[e^{\min\{X,5\}}\right] = \int_0^5 2e^{-2x}e^x dx + e^5\mathbf{P}\left\{X > 5\right\} = -2e^{-x}\Big|_0^5 + e^{-5} = 2(1 - e^{-5}) + e^{-5}.$$

A premium below this value is advantageous to the insured, above this value is advantageous to the insurer.

(b) Apply Ohlin's lemma to conclude that

$$\int_{\mathbb{R}} v(y) dF_1(y) \ge \int_{\mathbb{R}} v(y) dF_2(y),$$

where $v(y) = u(w - y)$ is a concave function (since u is), and F_1 and F_2 are the distribution functions of the Exp(2) distribution and Pareto r.v., respectively. It remains to show that the hypotheses of Ohlin's lemma are satisfied. To this end, note that $F_2(x) \ge F_1(x)$ on $[-1/2, 0]$, and for large values of x we evidently have

$F_1(x) \geq F_2(x)$, since F_1 has a faster decay at infinity than F_2. Finally, we claim that there is only one point where the two distributions are the same on $[0, \infty)$. For them to be equal, we would need the tails to be equal, i.e.,

$$\frac{1}{(x + 3/2)^2} = e^{-2x},$$

and solving yields that either $x + 3/2 = e^x$ or $-x - 3/2 = e^x$. The first has one solution on $[0, \infty)$ and the second has none on this range.