Forsikringsvidenskabelig kandidateksamen
Ruinteori

Københavns Universitet, Blok 2, 2006-7 (J. F. Collamore)

Skriftlig prøve den 24.01.07 kl. 9:00-12:00

The exam consists of four problems worth a total of 100 points. You may use the lecture notes, texbooks, the assignments and their solutions, etc. In the solution to each problem, you must *justify your answer* by providing a proof or by referring to a proved result from the lecture notes, a book, a written handout given out in lecture, etc. Simplify your answers as much as possible.

Use your time wisely. Do not copy the text of the problems or reprove results which have been completely proved in lecture. You are allowed to write your answers in pencil, and you may answer the questions in English or Danish or any mixture thereof.

Problem 1. [30 pts.] Consider an insurance company with reinsurance, such that:

- The insurance company begins with an initial capital of $u > 0$, and the reinsurance company begins with an initial capital of $u_r > 0$.
- The insurance company receives premiums at rate $c = 1$ from its policy holders, and pays a premium at rate $c_r = 1/5$ to the reinsurance company.
- The claims $\{Y_i\}$ incurred by the insurance company are i.i.d. with Pareto tails, more precisely, the probability density function of Y_i is given by $f(y) = (4/5)y^{-5}$, $1 \le y < \infty$, and $f(y) = 4/5$, $0 \le y < 1$ (and $f(y) = 0$ for all negative y). Assume that the number of claims occuring by time t is N_t, where $\{N_t\}$ is a Poisson(1) process.
- The reinsurance contract is excess of loss reinsurance, namely, for a claim of size y, the reinsurance company covers the amount $r(y) = (y - 1)^+$.

We will make the (obviously unrealistic) assumption that the reinsurance company insures only claims from the single insurance company described above.

(a) Write down an equation for the adjustment coefficient characterizing the asymptotic probability of ruin for the insurance company (*given* that the insurance company has the reinsurance contract described above).

(b) Give an asymptotic estimate (as $u_r \to \infty$) for the probability of ruin for the reinsurance company.

(c) Give a diffusion approximation for the finite-time probability of ruin for the insurance company, i.e. for $\mathbf{P}\{\text{ruin before time } K\}$, where $0 < K < \infty$.

Problem 2. [25 pts.] Consider an insurance company with *two* lines of business. The initial capital of the company is denoted by $u > 0$, and the total capital of the company at time t is denoted by C_t, for all $t \ge 0$.

In the jth line of business ($j = 1, 2$), the premium payments occur at a constant rate $c_j > 1/2$, the claims losses occur according to a Poisson(j) process (denoted $\{N_t^{(j)}\}$), and the claims sizes (denoted $\{Y_i^{(j)}\}$) are i.i.d. with a exp($2j$) distribution. It is assumed that the two lines of business are independent of one another and that the process $\{N_t^{(j)}\}$ is independent of the sequence $\{Y_i^{(j)}\}$ for each j.

(a) Write down an equation for $C_t \stackrel{\text{def}}{=}$ the total capital of the company at time t.

(b) Let $\delta(u) \stackrel{\text{def}}{=} \mathbf{P}\{C_t > 0, \text{ for all } t \ge 0\}$ denote the survival probability. Find an equation for δ', and use this to determine the Laplace transform of δ.

(c) Suppose that we actually would like to study $F(u) \stackrel{\text{def}}{=} \int_0^u \delta(v)dv$. Using (b), determine the Laplace transform of the function F.

Problem 3. [25 pts.] Consider a "perturbed" Cramér-Lundberg model given by

$$C_t = u + ct + W_t + \sum_{i=1}^{N_t} Y_i,$$

where $\{W_t\}_{t\geq 0}$ is standard Brownian motion (BM(0,1)), $\{Y_i\}$ is an i.i.d. sequence of random variables having an $\exp(\theta)$ distribution (where $\theta > 0$), $\{N_t\}$ is a Poisson(2) process, and each of these processes is assumed to be independent of the others. As usual, C_t denotes the total capital of the insurance company at time t, $u > 0$ the initial capital, $c > 0$ the premium rate, $\{Y_i\}$ the claim sizes, and $\{N_t\}$ the arrival rate of the claims, while $\{W_t\}$ denotes a perturbation term describing some additional random effects. It is assumed that c is sufficiently large so that the net profit condition is satisfied.

(a) Let $X_t \stackrel{\text{def}}{=} ct + W_t + \sum_{i=1}^{N_t} Y_i$. Compute the cumulant generating function of X_1, that is, $\Lambda(\alpha) \stackrel{\text{def}}{=} \log \mathbf{E}\left[e^{\alpha X_1}\right]$.

(b) Using martingale techniques, *derive* an exponential upper bound for

$$\mathbf{P}\left\{C_t < 0, \text{ for some } 0 \leq t \leq Ku\right\}, \tag{1}$$

where $0 < K < \infty$, and where you may take $\theta = 1$ and $c = 5/2$. Simplify your expression as much as possible. (In particular, is your bound the same as the usual Lundberg upper bound?)

Problem 4. [20 pts.] (a) Consider an insurance company which invests its excess capital, so that the total capital at time n (denoted S_n) satisfies the system of equations $S_0 = u$, and for $n = 1, 2, \ldots,$

$$S_n = R_n S_{n-1} - B_n.$$

As usual, B_n denotes the claims losses minus premiums income during the time interval n, i.e.,

$$B_n = -c + \left(\sum_{i=1}^{N_n} Y_i - \sum_{i=1}^{N_{n-1}} Y_i\right),$$

where we take the premium rate $c = 2$, $\{N_t\}$ to be a Poisson(1) process, and the claim sizes $\{Y_i\}$ to have an $\exp(\theta)$ distribution. Also assume that the interest process satisfies $R_n = 1/A_n$, where $\{\log A_n : n = 1, 2, \ldots\}$ is an i.i.d. sequence of Normal(-1,1) random variables.

Give an asymptotic expression for the probability of ruin in this case, and identify the constant which characterizes the rate of decay (as $u \to \infty$). Does your asymptotic expression depend of θ?

(b) Next we concentrate on the interest process but, in contrast to (a), we assume now that the interest process is continuous, given by

$$R_t = \sum_{i=1}^{N_t} X_i,$$

where $\{N_t\}$ is a Poisson(1) process, and where $\{X_i\}$ is an i.i.d. sequence of Normal(1, 1) random variables. We are interested in studying the following "large exceedence" probability:

$$\gamma(u) \stackrel{\text{def}}{=} \mathbf{P}\left\{R_t < -u, \text{ for some } t \geq 0\right\}.$$

Find a renewal equation satisfied by the function γ.

THE END