Forsikringsvidenskabelig kandidateksamen
Ruinteori

Københavns Universitet, Blok 2, 2005-6 (J. F. Collamore)

Skriftlig prøve den 26.01.06 kl. 10:00 – 13:00

The exam consists of three problems worth a total of 100 points. You may use the lecture notes, texbooks, the assignments and their solutions, etc. In the solution to each problem, you must *justify your answer* by providing a proof or by referring to a proved result from the lecture notes, a book, a written handout given out in lecture, etc. Simplify your answers as much as possible.

Use your time wisely. Do not copy the text of the problems or reprove results which have already been proved in lecture. You are allowed to write your answers in pencil, and you may answer the questions in English or Danish or some mixture thereof.

Problem 1. [30 pts.] In this problem, we consider life insurance company (with annuity insurance), where the capital growth is modelled similarly to a Cramér-Lundberg process[1], and therefore may be analyzed using the ruin-theoretic methods of this course. Specifically, we assume the following:

- The insurance company *pays out* a premium to its policy holders. It is assumed that the number of policy holders at any given time is constant, as is the premium rate. Thus, the insurance company pays out a total capital of ct to its policyholders during the time interval $[0, t]$, where $c > 0$.
- The insurance company *receives* capital when a policy holder dies. It is assumed that the deaths occur at a Poisson rate (so that the total number of deaths by time t is described by a Poisson(λ) process, for some $\lambda > 0$), and the i^{th} death (or "claim") leads to a capital gain of amount Y_i for the insurance company. It is assumed that $\{Y_i\}$ is i.i.d. and that $Y_i - K$ has an Exp(θ) distribution[2], for some "threshhold" $K > 0$, and is independent of the process $\{N_t\}$.
- The insurance company begins with an initial capital of $u > 0$.

This process is the same as the usual Cramér-Lundberg process, except that the insurance company *loses* capital from the premiums and *gains* capital from the claims. It is assumed that $\lambda \mathbf{E}[Y_i] > c$.

(a) [4 pts.] Write down an expression for the total capital C_t of the insurance company at time t in terms of the parameters c, u, and the random quantities N_t and $\{Y_i\}$.

(b) [14 pts.] Derive an expression for the adjustment coefficient "R." [You may express your final answer as an equation involving the known parameters θ, λ, c and K—you do not need to solve this equation explicitly for R.] Now write down the exponential martingale associated with R, and derive an *exact* expression for the probability of ruin. Justify that your estimate is, in fact, exact.

(c) [12 pts.] Next suppose that the claims are modelled as a *mixture* of two types; namely, with probability $1/2$, Y_i now has an Exp(1) distribution, and with probability $1/2$ it has an Exp(2) distribution. Let $\delta(u)$ denote the company's survival probability, i.e., $\delta(u) := \mathbf{P}\{C_t \geq 0, \text{ for all } t \geq 0\}$. Using a perturbation argument, derive an equation for $\delta'(u)$.

Problem 2. [34 pts.] Conisider the standard Cramér-Lundberg process,

$$C_t = u + X_t, \text{ where } X_t = ct - \sum_{i=1}^{N_t} Y_i,$$

for $c = 3$, $\{N_t\}$ a Poisson process with parameter $\lambda = 2$, and $\{Y_i\}$ an i.i.d. sequence of random variables with mean $\mu_Y = 11/12$, variance $\sigma_Y^2 = 47/144$, and distribution function

$$\mathbf{P}\{Y_i \leq x\} = \begin{cases} x/2 & , \quad 0 \leq x \leq 1, \\ 1 - (1/2)x^{-4} & , \quad x > 1. \end{cases}$$

[1] This formulation has been taken (with different assumptions on $\{Y_i\}$) from: H. Cramér (1954), On some questions connected with mathematical risk. Univ. California Publ. Stat. **2** (5), 99-123.

[2] By definition, an Exp(θ) distribution has the density function $\theta e^{-\theta x}$ for $x > 0$ (and zero for $x \leq 0$). Its mean value is $1/\theta$; its variance is $1/\theta^2$; and its moment generating function is $M(\alpha) = \theta/(\theta - \alpha)$, $\alpha < \theta$, and $M(\alpha) = \infty$, $\alpha \geq \theta$.

(a) [10 pts.] Find the asymptotic behavior of the probability of ruin, namely,

$$\psi(u) := \mathbf{P}\{X_t < -u, \text{ for some } t \geq 0\} \text{ as } u \to \infty.$$

(b) [10 pts.] Compute the diffusion approximation for $\psi(u)$ when $u = 10$.

(c) [14 pts.] Suppose that the process $\{X_t\}$ is modified to

$$\hat{X}_t := X_t + \hat{N}_t,$$

where $\hat{N}_t$ is a Poisson($\hat{\lambda}$) process and $\hat{\lambda} > 0$. It is assumed that $\{\hat{N}_t\}$ is independent of $\{X_t\}$. For the process $\{\hat{X}_t\}$, compute the diffusion approximation for the probability of ruin,

$$\hat{\psi}(u) := \mathbf{P}\{\hat{X}_t < -u, \text{ for some } t \geq 0\}.$$

[Your final answer will depend on the parameters u and $\hat{\lambda}$.] Your estimate, of course, will only be an approximation. Also explain—as precisely as possible—in what sense your approximation actually holds.

Problem 3. [36 pts.] Consider the standard Cramér-Lundberg process,

$$C_t = u + X_t, \text{ where } X_t = ct - \sum_{i=1}^{N_t} Y_i,$$

for $c = 3$, $\{N_t\}$ a Poisson(λ) process with $\lambda = 2$, and $\{Y_i\}$ an i.i.d. sequence of random variables with a Uniform(0,2) distribution[3].

(a) [14 pts.] Suppose that the insurance company takes out reinsurance, paying a premium of c_r to the reinsurance company. In return, the insurance company only pays a part of each claim amount, namely $s(Y_i)$, where s is the following self-insurance function:

$$s(x) = \begin{cases} x & , \quad 0 < x \leq 1, \\ 1 + \beta(x-1) & , \quad 1 < x < 2, \end{cases}$$

where $\beta \in (0,1)$. Let $r(x) = x - s(x)$ and assume that $c_r = 2\mathbf{E}[r(Y_i)]$.

Compute the adjustment coefficient as a function of β (or, in other words, the asymptotic decay rate of the ruin probability, for any given β). *Note:* You do not need to simplify your answer—it is enough to find an equation involving the adjustment coefficient and the known parameter β.

For the remainder of the problem, suppose now that the insurance company does *not* have the reinsurance described in (a), but instead receives a *payment* at unit time intervals, as follows. At time $i \in \{1, 2, \ldots\}$ there is a payment $\{Z_i\}$, where the sequence $\{Z_i\}$ is i.i.d. with an Exp(10) distribution, and where the sequence $\{Z_i\}$ is independent of the process $\{X_t\}$. Then the total capital of the company at the unit time intervals is given by

$$W_n = u + X_n + (Z_1 + \cdots + Z_n), \quad n = 1, 2, \ldots.$$

(b) [8 pts.] Explain why the sequence $\{W_n - W_{n-1} : n = 1, 2, \ldots\}$ is i.i.d., and compute the moment generating function of the random variable $(W_1 - W_0)$.

(c) [14 pts.] Now let

$$\bar{\psi}(u, Tu) := \mathbf{P}\{W_n < 0, \text{ for some } n \geq Tu\}.$$

[Here we only consider the possibility of negative total capital at *unit* times, i.e., $n \in \{1, 2, \ldots\}$ in the previous formula.]

Determine the "critical value" or, equivalently, the "likely" time of ruin (as in part (a), you do not need to simplify your answer). For which values of T will we have

$$\lim_{u \to \infty} \frac{\bar{\psi}(u, Tu)}{\psi(u)} \not\to 1,$$

where $\psi(u) := \mathbf{P}\{W_n < 0, \text{ for some } n \in 0, 1, \ldots\}$ is the ruin probability? Explain. *THE END*

[3] By definition, an Uniform(a,b) distribution has the density function $f(x) = 1/(b-a)$ for $x \in (a,b)$, zero for $x \notin (a,b)$. Its mean value is $(a+b)/2$; its variance is $(b-a)^2/12$; and its moment generating function is $M(\alpha) = (e^{\alpha b} - e^{\alpha a})/\alpha(b-a)$.