Homework Problems

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1. Let L have a standard Student's t-distribution with $\nu > 1$ degrees of freedom. Then L has a density function given by

$$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}.$$

Show that

$$ES_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha} \left(\frac{\nu + (t_\nu^{-1}(\alpha))^2}{\nu - 1}\right),$$

where t_ν denotes the distribution function of L.

2. (a) Consider a company which invests equal amounts in three stocks, which (as in the Black-Scholes model) are assumed to have independent normally-distributed logarithmic returns. It is estimated that

$$X_{n+1}^{(j)} = \log S_{n+1}^{(j)} - \log S_n^{(j)} \sim N(\boldsymbol{\mu}, S),$$

where

$$1000\boldsymbol{\mu} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} \text{ and } 1000S = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & 1 & -\frac{1}{5} \\ \frac{1}{3} & -\frac{1}{5} & 1 \end{pmatrix}.$$

We are interested in studying the one-period loss from these investments, given that a capital of 1000 is invested in each stock at the current time. By simulating the r.v. $\mathbf{X}$ repeatedly, compute $\mathrm{VaR}_{0.9999}$ with a confidence level of 99%.

(b) Next suppose that we would like to estimate

$$\mathbf{P}\left\{L_{n+1} \geq \mathrm{VaR}_{0.9999}\right\} \qquad (1)$$

more accurately, so we use importance sampling. Instead of simulating $\mathbf{X}_{n+1}$ directly, we simulate according to the probability distribution

$$dF_{\boldsymbol{\xi}}(x_1, x_2, x_3) = Ce^{\boldsymbol{\xi}\cdot\mathbf{x}} dF(x_1, x_2, x_3),$$

where F is the distribution of $\mathbf{X}_{n+1}$ and C is a normalizing constant. By experimenting with some possible choices, find a choice of $\boldsymbol{\xi}$ for which the event $\{L_{n+1} \geq \mathrm{VaR}_{0.9999}\}$ occurs relatively frequently, say with probability 1/2 (where L_{n+1} denotes the one-period loss at time $n+1$). Now compute (1) with a confidence level of 99%.

[Note that the methods of this problem are quite general and do not require that one assume normally-distributed log. returns.]

3. In a well-known article, Artzner-Delbaen-Eber-Heath suggested a collection of axioms which should be satisfied by a "coherent risk measure" ρ, namely:

   (a) *Translation invariance*: $\rho(X + a) = \rho(X) - a$, for any constant a.
   (b) *Subadditivity*: $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.
   (c) *Positive homogeneity*: $\rho(\lambda X) = \lambda\rho(X)$, for any positive constant λ.
   (d) *Monotonicity*: $X_1 \leq X_2 \Longrightarrow \rho(X_2) \leq \rho(X_1)$.

(Here the r.v. X is interpreted as the capital gain, i.e., $X = -L$, where L is the loss.)

Find an example which shows that the subadditivity axiom may *fail* when the risk measure is taken to be value-at-risk. Show that this axiom *is* satisfied in your example when the risk measure is taken to be expected shortfall.

4. Download the data set DAX-returns from the course web site, which consists of daily log-returns of the DAX stock index over the period from January 1990 until July 1996. Assume that today's price of the stock index is $S_n = 100$.

   (a) Estimate the empirical $\text{VaR}_{0.99}$ and $ES_{0.99}$ for the one-day loss of a portfolio consisting of one share of the DAX index. Construct an exact confidence bound for $\text{VaR}_{0.99}$ with confidence level 95%.

   (b) Now compute the same quantities for the 10-day loss. This can, of course, be done in more than one way. Suggest a second approach, and discuss the advantages and disadvantages of your alternative approach.

5. Using the data set from the previous example:
   (a) Make a QQ-plot of the log. return data against the normal distribution and against $t-$distributions with different degrees of freedom. Which distribution seems to give a good fit?

   (b) Multiply the data by -1 so that the negative returns appear along the positive axis. Construct a mean-excess plot. Does it seem to be linear? If so, then above which level u (and corresponding exceedence number N_u) is the mean-excess linear?

   (c) Extract the excesses over your chosen level u and estimate the parameters (γ, β) of a fitted GPD. (A useful $R-$function is optim.)

   (d) Determine asymptotic 95%-confidence intervals for the parameters γ and β, respectively.

   (e) Plot the fitted GPD approximation of $1 - F(u+x)$, $x > 0$, as a function of x, and compare it with its empirical counterpart, i.e., plot both distribution functions on the same figure.

   (f) Use your estimated values for γ and β to estimate the one-day $\text{VaR}_{0.99}$ for the loss of a portfolio consisting of one share of the DAX index, assuming that today's price is 100.

6. Now download the set BMW-returns from the course web site, which consists of daily log-returns of the BMW stock over the same period as the DAX index of the previous exercises. Consider a bivariate vector of risk-factor changes $\mathbf{X}_k$ consisting of the log. returns of the DAX index and the BMW stock, respectively. Assume that today's index and today's stock price are 100.

   (a) Propose an elliptical (or other) distribution which is appropriate for this data, and compute the one-day $\text{VaR}_{0.99}$ and $\text{ES}_{0.99}$.

   (b) Now use the standard mean-covariance method to compute the one-day $\text{VaR}_{0.99}$. How do your two estimates compare?

7. Suppose $(X_1, \ldots, X_{50})$ are the daily log-returns for 50 different stocks, and suppose that today's stock prices are given by $(S_1, \ldots, S_{50}) = (100, \ldots, 100)$ and that we hold one share of each stock. Hence the portfolio loss over one day is given by

$$L = -\sum_{i=1}^{50} S_i \left(e^{X_i} - 1\right).$$

Let $100\sqrt{3}(X_1, \ldots, X_{50})$ be a random variable with standard t_3 marginal distributions, and suppose that $\rho_\tau(X_i, X_j) = 0.4$ for all $i \neq j$. Thus, if Z has a standard t_3 distribution, then

$$X_i \stackrel{d}{=} \frac{1}{100\sqrt{3}} Z.$$

Estimate $VaR_{0.99}(L)$ and $ES_{0.99}(L)$ by simulation under the assumption that:
(a) $(X_1, \ldots, X_{50})$ has a Gaussian copula.
(b) $(X_1, \ldots, X_{50})$ has a Clayton copula.
Simulate samples of size at least 10000. Do you see a difference in your estimates? If so, explain why.