Homework

Monte Carlo Methods in Insurance and Finance, J. F. Collamore, Block 1, Autumn 2010
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1. The discounted payoff of a "up-and-put" option is

$$e^{-rT}\left(K - S(T)\right)^{+} \mathbf{1}\left\{S(t) \le H(t), \text{ for all } t \in [0,T]\right\}.$$

Here, $S(t)$ is assumed to be geometric Brownian motion with parameters given by the stardard Black-Scholes model; and we will assume that $H(t)$ is a time-dependent barrier with $H(t) = H_i$, $t_i \le t < t_{i+1}$ ($i = 0, \ldots, m-1$, $t_m = T$). Set

$$p_i = \mathbf{P}\left\{\max_{t_i \le t \le t_{i+1}} S(t) \le H_i \mid S(t_i),\ S(t_{i+1})\right\}. \tag{1}$$

If $\xi_i = \mathbf{1}\{U_i \le p_i\}$,where U_i ~Unif(0,1), then unbiased estimates for the option price are given by either of the following:

$$e^{-rT}\left(K - S_T\right)^{+} \prod_{i=0}^{m-1} \xi_i; \tag{2}$$

$$e^{-rT}\left(K - S_T\right)^{+} \prod_{i=0}^{m-1} p_i. \tag{3}$$

(a) Give an explicit expression for the p_i's in (1) in terms of S_{t_i} and $S_{t_{i+1}}$.

(b) Show that (2) and (3) have the same expectation. How do their variances compare?

(c) Using the more efficient of these two possible etimators, estimate the expectation using simulation methods for the special case that interest rate in the Black-Scholes model is given by $r = 0.04$, the volatility is given by $\sigma = 0.3$, $S(0) = 100$, $T = 5$, $K = 120$ and $H(t) = 105 + 3[t]$, where $[t]$ denotes the greatest integer less than or equal to t. Also give confidence bounds for your estimate.

(Hint: The following property of Brownian bridge is useful: If $W^{(a)}(t)$ is Brownian bridge which starts from 0 at time 0 and ends at a at time T, then

$$\mathbf{P}\left\{\max_{0 \le t \le T} W^{(a)}(t) \le a + x\right\} = 1 - e^{-2x(a+x)/T}.)$$

2. As an alternative to the usual "inverse transform method," one can observe that if X is a discrete random variable taking values in $\mathbb{Z}_+ = \{1, 2, \ldots\}$, and if ξ is Uniform(0,1), then

$$\eta = \sum_{k=1}^{\infty} \lfloor \xi + 1 - \sum_{j=1}^{k-1} f(j) \rfloor,$$

where $f(\cdot)$ is the density function of X and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(a) What are the possible values of $\eta_k \stackrel{\text{def}}{=} \lfloor \xi + 1 - \sum_{j=1}^{k-1} f(j) \rfloor$?

(b) Prove that the distribution of X is the same as that of η.

3. Suppose that $\{X_i\}$ is a sequence of empirical samples with an (possibly unknown) density function f, where $f(x) = 0$ for $|x| > B$. We are interested in studying the probability that $\{X_i\}$ takes on unusually large values. To do so, we employ a variant of the "acceptance-rejection" technique. Namely, simulate samples $U_1, U_2, \ldots$ having a Uniform(0,1) distribution; accept a given sample X_i if

$$U_i \leq \exp\{\theta(X_i - B)\},$$

and reject the sample X_i otherwise, where $\theta \in \mathbb{R}$. Let $\{Y_i\}$ denote the sequence of accepted samples, and let g denote the density function of Y_i.

(a) Determine the density function g. (Your answer will depend on the density function f and on the parameter θ.)

(b) Now suppose that $\{X_i\}$ is an i.i.d. sequence having a Unif(-1,1) distribution. With $\theta = 1$ and $\theta = 10$, perform the acceptance-rejection procedure described above. Produce a histogram of the accepted values $\{Y_i\}$. What proportion of samples are accepted for the two different values of θ? Roughly speaking, how do the sizes of the accepted values compare with the sizes of the rejected values?

4. The double exponential distribution function has the form

$$F(x) = \exp\left\{-e^{-x}\right\}, \quad x \in \mathbb{R}.$$

The expected value μ of this distribution cannot be calculated analytically, and one way to obtain an approximate numerical value is to use stochastic simulation. In this exercise, the objective is to compute μ using the method of control variates. Specifically, take the control variate to be a r.v. having an exponential distribution with parameter one. Using this method, provide a numerical estimate (and an appropriate confidence interval) for μ. (Note: You should use the same r.v. to generate both the double exponential sample and the exponential sample.)
Compare your results with those that you obtain without using the method of control variates.

5. Consider the problem of estimating the probability that the sample mean is far away from the true mean. Namely, given a statistical sample $X_1, \ldots, X_k$, we would like to estimate

$$\mathbf{P}\left\{\hat{X}_k > a\right\}, \quad \text{where } \hat{X}_k := \frac{1}{k}(X_1 + \cdots + X_k) \text{ and } a > \mathbf{E}[X_1].$$

In this problem, we assume that $\{X_i\}$ is an i.i.d. sequence having a Gamma(2,3) distribution, that is, a density function on the positive axis given by

$$f(x) = \frac{9}{\Gamma(2)} x e^{-3x}, \quad x \geq 0,$$

and we will take $a = 1$.

The objective of this problem is to estimate $\mathbf{P}\{\hat{X}_{100} > 1\}$ using importance sampling. Give error bounds for your estimate, and comment on the statistical properties of your sample; in particular, the samples should be unbiased, but are they symmetric about their mean, or is there skewness or are there other irregularities?

6. Consider the following variant of the classical ruin problem. Suppose that $\{X_i\}$ is an i.i.d. sequence of Normal$(1, \sigma^2)$ random variables in $\mathbb{R}$, let $S_k = X_1 + \cdots + X_k$, and suppose that the boundary corresponding to ruin is time-dependent; in particular, the probability of ruin is now interpreted as

$$\psi(u) := \mathbf{P}\{S_k < -uf(u,k), \text{ some } k \in \mathbb{Z}_+\},$$

where

$$f(u,k) := 1 + \left(\frac{k}{u}\right)^2.$$

Then $\psi(u) \to 0$ as $u \to \infty$, and the objective of the present problem is to estimate this probability.

This time-dependent barrier could arise, for example, if the insurance company invests its initial capital and earns deterministic returns on its investments.

(a) Compute the "rate function" $\tilde{I}(x) := \sup\{\langle \alpha, x\rangle : \Lambda(\alpha) = 0\}$, which describes the exponential decay of $\psi(u)$ as $u \to \infty$. Where is this function minimized over the set A, where $A = \{(t,y) : y \le -(1+t^2)\} \subset \mathbb{R}^2$?

(b) Find the "correct" exponential shift parameter α which is optimal for importance sampling.

(c) Run a simulation experiment to determine the probability of ruin for the case that $u = 100$ and $\sigma^2 = 2$. Compare your results with those you obtain without using importance sampling. As in the previous problem, also comment on the statistical properties of the sample you obtain using importance sampling. (Note: When you simulate without using importance sampling, you will need to truncate at some "large" time, e.g., $T = 100$.)

7. (a) In earthquake modeling, one lets $S(t)$ denote the stress potential at time t, and let $s = 20$ denote a theshold value. At time $\tau = \inf\{t : S(t) > s\}$, an earthquake occurs, and the potential is then reset to zero, i.e., $\lim_{t\to 0} S(\tau + t) = 0$. In between earthquakes, assume that the stress potential builds up according to a pure jump Lévy process with Lévy measure on $[1,\infty)$ given by

$$d\nu(x) = \left(\frac{1}{1+x}\right)^3 dx, \quad x \ge 1.$$

Produce a histogram of the severity of an earthquake, as defined by the random variable $S(\tau) - s$.

(b) Now consider the standard European call option, where the objective is to estimate

$$\mathbf{E}\left[e^{-rT}\left(S(T) - K\right)^+\right], \tag{4}$$

where $S(t)$ denotes the stock price at time t and is usually modeled as a geometric Brownian motion. However, suppose that instead we model

$$\log S(t) - \log S(0) = \left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t) + J(t),$$

where $W(t)$ denotes standard Brownian motion, and where $J(t)$ denotes a pure jump process with a Lévy measure which is symmetric about the x-axis and for which, along the positive axis, we have:

$$d\nu(x) = \frac{1}{5}\frac{x^{1/2}+3}{x^{7/4}}dx, \quad x > 0.$$

Now assume $r = 0.04$, $\sigma = 0.3$, $S(0) = 100$, $T = 1$, and $K = 105$.

Describe a procedure for estimating the expectation in (4) and implement your procedure to obtain a numerical estimate for this quantity.

8. In its simplest setting, a basic problem in life insurance mathematics studies the following model:

p

$$\boxed{\text{0: ALIVE}} \qquad \xrightarrow{\mu(t),\kappa} \qquad \boxed{\text{1: DEAD}}$$

A premium of p is paid until the random time τ :=the time of death, or until an end time, T, whichever comes first. For simplicity, take $T = 10$ and $p = 5$. If a transition from state 0 to state 1 occurs, then a payment of $\kappa = 1000$ is made, and this occurs with an intensity

$$\mu(t) := \lim_{h\to 0} \mathbf{P}\{X(t+h) = 1 \,|\, X(t) = 0\},$$

where $X(t)$ denotes the state at time t, and we assume that

$$\mu(30+t) = a + bc^{30+t}, \quad 0 \le t \le 10,$$

where $a = 10^{-3}$, $b = 10^{-4}$, and $c = 1.1$. The objective is to evaluate the prospective reserves,

$$V(t) := \mathbf{E}\left[R(t) \,|\, X(t) = 0\right],$$

where the stochastic process $\{R(t)\}$ is given by

$$R(t) = \int_t^T e^{-\int_t^s r(u)du} \left(p\mathbf{1}_{\{X(t)=0\}}dt - \kappa dX(t)\right). \tag{5}$$

As a departure from the standard model, we now assume that the rate of interest is *random*, governed by the Vasicek model:

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW(t)$$

(see Glasserman, p. 108); take $\alpha = 0.2$, $\sigma = 0.1$, $\beta = 0.05$, and $r(0) = 0.44$.

Using Euler's method to handle the integral arising in (5), evaluate $V(0)$.

Please include a written program (as on p. 85 of Glasserman) in your answers to exercises where programming is required. You are free to choose any programming method you like.