Cultivating Operads in String Topology

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English Abstract

The present master's thesis circles around the cacti operad, originally introduced by Voronov.

A new construction presented is an explicit 'gravity' map from the space of framed little disks, to the space of cacti. Based on this map it is shown how to obtain an explicit locally equivalent 'gravity' morphism from Boardmann-Vogts cofibrant replacement of the framed little disk operad to the cacti operad.

The basic theory of operads is introduced as well. Furthermore, we pursue an argument for how a Chas-Sullivan loop-product on the shifted homology of a free loop space on a smooth orientable manifold arises as an action of the homology over the cacti operad.

Dansk resumé

Dette speciale drejer sig om kakti operaden, introduceret af Voronov.

En ny konstruktion der gives er en explicit 'gravitations'-afbildning fra rummet af indrammede små diske til rummet af kakti. Baseret på denne afbildning konstrueres en explicit morfi af operader fra Boardmann-Vogts kofibrerende resolution for operaden af indrammede små diske, til kakti operaden. Det vises eksplicit at dette er en svag ækvivalens af operader.

Vi introducerer den grundlæggende teori om operader. Ydermere forfølges et argument for hvordan Chas-Sullivan løkke-produktet på den forskubbede homologi af et frit løkkerum over en glat orienterbar mangfoldighed, kan ses som en virkning af homologien over kakti operaden.

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1 Introduction

1.1 String Topology

String topology was boosted forth in the article [CS99]. Basically, Chas and Sullivan show that for the *free loop space* $LM := \mathrm{Map}(S^1, M)$ – where M is an oriented smooth d-manifold – the shifted homology $\mathbb{H}_*(LM) := \mathrm{H}_{*+d}(LM)$ has the structure of a so-called *Batalin-Vilkovisky algebra*. This structure on $\mathbb{H}_*(LM)$ is commonly referred to as the *Chas-Sullivan loop product*.

The conception of the Chas-Sullivan loop product has produced a lot of activity in the mathematical community, and indeed the result is striking in many ways.

First of all, free loop spaces are relatively weird topological spaces. Computations of invariants of them is an active area of research, and having a general structure theorem on the homology of a wide class of free loop spaces is indeed very important in the understanding of free loop spaces.

Widening the perspective, physicist are interested in free loop spaces, since they resemble a (toy-)model of what goes on in the field of string theory, where the world – instead of being modelled as having particles as basic building blocks – is modelled by its basic building blocks made out of strings (open or closed). Indeed, LM is the space of all closed strings in the manifold M. And having the structure of an algebra on $\mathbb{H}_*(LM)$ gives some idea of how two strings interact with each other.

Note however that indeed $\mathbb{H}_*(LM)$ has non-zero elements living in negative degrees (for instance in degree $-d$), which tells us that the Chas-Sullivan loop product cannot be induced from a space-level composition of loops.

Furthermore, one can show (see 2.58) that indeed the standard composition of based loops in based – twice iterated – loop spaces, $\Omega^2 X$, induces the structure of a Batalin-Vilkovisky algebra on $\mathrm{H}_*(\Omega^2 X)$. Indeed, this tells us that from the homological point of view – up to a shift in degree – twice iterated based loop spaces and free loop spaces carries the same type of structure.

This thesis centers around the Chas-Sullivan loop product. However we follow an approach at a slightly higher level of abstraction than was done in [CS99]; we shall namely give an approach to the Chas-Sullivan loop product via *operads* and their actions.

1.2 Operads

Operads are all about control. Equivalating path homotopies in the based loop space ΩX, we get something sensible: the group $\pi_1(X)$. In this sense, ΩX contains all the algebraic information of $\pi_1(X)$ – we can compose loops in ΩX, but instead of a single choice of composite, we obtain a continuum of different parametrizations of the composed loop. The algebraic structure on ΩX therefore appears as a bit to much of the same thing: Choices of different reparametrizations indeed yield different 'multiplications' in ΩX, but whatever homotopy invariant we apply, the result remains constant.

Operads provide control over the situation: All multiplications on ΩX arise from an action of a single operad (in this case, the little intervals operad).

From an algebraic point of view, action of operads provide control over operations on algebraic gadgets we meet in our day-to-day lives: Products, Lie-algebras, etc. We shall see how an action of a single operad classify wide classes of algebraic structure, such as for instance associative products. The point being that given some – potentially weird – operation, we can (in many cases) easily find an operad whose action does the trick and classifies the structure.

As indicated, operads is a terminology that makes sense in a wide range of categories. It is quite easy to – via so-called *monoidal functors* – transfer operads and their actions from one category to another – our prime example of a monoidal functor is $\mathrm{H}_*(-)$.

Naturally – as an algebraic topologist – I find the real beauty of operads in the strong interplay one gets between topology and algebra. A main point of the present thesis will be that finding some non-trivial operadic action on a space (or a near-space) level, will induce a rich algebraic structure on the level of homology.

A quite recent example of such an action of an operad is the *cacti operad*, initially introduced in [Vor05] as an alternative method for obtaining the Chas-Sullivan loop product. The true core of the work we present is an attempt of understanding the cacti-operad.

The 'jewel' of this thesis is 3.44, that states that the cacti operad is weakly equivalent to the framed little disk operad. As we shall see, this implies that an action on a graded module M of the homology of the cactus operad is precisely the same as giving M the structure of a Batalin-Vilkovisky algebra.

3.44 is a well-known result, due to Voronov – and it is usually prooved via a rather abstract-nonsense argument called the Fiedorowich recognition principle. We shall explain more about it in 3.29, however we do deviate quickly from this path. Instead, we give an attempt at devising a more concrete – and admittedly – more intricate approach – via what seems reasonable to call *gravity morphism*. The motivation of the approach is that the Fiedorowich recognition breaks down in higher dimensional generalizations, so indeed one should look for alternatives to the Fiedorowich recognition principle.

In the string topology community, this is an active part of the ongoing research. Different attempts flourish; the one that resembles the approach given here most is one of Salvatores, who likewise claims to have constructed concrete maps (that differs from ours though). To my knowledge, no one has generalized the local equivalence to higher dimensions. Neither have I, yet – at least – there is hope that the construction generalizes, although the details are not fully worked out.

As noted in [CV06, p.75-78], higher dimensional analogues are indeed desired, as they would provide higher-dimensional analougues of the Chas-Sullivan loop product on $\mathrm{Map}(S^n, M)$.

1.3 The structure of the thesis

The thesis is divided into three chapters.

In the first we start out by introducing the basic language needed of operads. We then turn to give the core examples of operads, that concern string topologists, such as little disk operads and Batalin-Vilkovisky operads. We finish the chapter off by giving the cofibrant replacement of a topological operad, known as Boardmann-Vogts $\mathcal{W}$-resolution. The point will be that it is easier to map out of the $\mathcal{W}$-resolution, in a structure-preserving way, as it – in analogy – is easier to map out of a mapping cylinder.

In the second chapter, we introduce the cacti operad as a topological operad. We then progress on to the construction of a gravity morphism from the $\mathcal{W}$-resolution of the framed little disk operad to the cacti operad, which indeed turns out to be a local equivalence of operads.

Finally in chapter three, we go through the argument presented in [CJ02] on how – via a Thom collapse argument – the homology of the cacti operad has an action on $\mathbb{H}_*(LM)$, giving the Chas-Sullivan loop product. In doing this, a problem – concerning existence of tubular neighborhoods – that I weren't able to solve, arose. Therefore the picture of the action isn't quite complete in the presentation. We present the encountered problem and give a possible strategy – posed by Veronique Godin – that might solve the problem.

As for the notation, we have a caveat: We try to keep things clean, and shall therefore not write coefficients in $\mathrm{H}_*(-)$. Similarly, we write for instance tensors as $-\otimes-$ without explicit mentioning of any rings. We allow ourselves to do this, as we are not performing any change of coefficients/rings arguments.

From time to time, we shall however restrict the choice of coefficients/rings. We shall say when we do so. If in doubt, it should be safe to assume that we are working over a principal ideal domain.

1.4 Appreciation

The present text marks the achievement of a goal I set out as a teenager, namely to get some idea of what goes on at the frontier of mathematics.

I have however not arrived all alone. Indeed, during the last years – topology has started to bloom at Copenhagen; it is inspiring – and fun – to have 'been at the right place at the right time'. Therefore it seems suitable to thank the topology group for putting such great effort into building a new center of research in Copenhagen, at least I can feel a difference!

In particular, I feel that Craig Westerland and my advisor Nathalie Wahl deserves extra credit. Truly they have spent numerous hours of discussing ideas present in this thesis with me; even believing in me, whenever there was the slightest chance that I might not be wrong. I have learned a lot, and even had a pleasant time doing so.

During interrogation, it is a classical method of police departments to use a good cop and bad cop. In analogy, it would probably not have been possible to undergo the hermeneutical process of crystallizing the ideas, had I not had support from Nathalie – forcing me to talk in a proper mathematical language – and Craig – telling me that the math all seemed cool.

Finally, I thank the beauty for surrounding me.

Tarje Bargheer, Copenhagen, Feb. '08

2 Operads

2.1 The Language needed to define Operads

Before we can actually define what we mean by an operad, we will need some framework. First, we introduce the type of categories we will be working with:

Definition 2.1 By a monoidal category, we will understand a pair $(\mathcal{C}, \boxtimes)$, where $\mathcal{C}$ is a category, and $- \boxtimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bi-endo-functor, called the *tensor product*. We require that the following conditions should be satisfied:

1. There should exist a natural associativity isomorphism

$$\alpha : (- \boxtimes -) \boxtimes - \cong - \boxtimes (- \boxtimes -)$$

   on $\mathcal{C}$, with no further "twists" in applying α to more parentheses, i.e., the following diagram is commutative:

$$\begin{tikzcd}
 & (- \boxtimes -) \boxtimes (- \boxtimes -) \arrow[dl, "\alpha"'] & \\
- \boxtimes (- \boxtimes (- \boxtimes -)) & & ((- \boxtimes -) \boxtimes -) \boxtimes - \arrow[ul, "\alpha"'] \arrow[d, "\alpha \boxtimes \mathbb{1}"] \\
- \boxtimes ((- \boxtimes -) \boxtimes -) \arrow[u, "\mathbb{1} \boxtimes \alpha"] & & (- \boxtimes (- \boxtimes -)) \boxtimes - \arrow[ll, "\alpha"']
\end{tikzcd}$$

2. A specified *identity* object e in $\mathcal{C}$, along with two natural identity isomorphisms $\lambda\colon e \boxtimes - \cong -$ and $\rho\colon\ - \boxtimes e \cong -$, that fit together with the associativity isomorphism, i.e. we require the following diagram to commute:

$$\begin{tikzcd}
(- \boxtimes e) \boxtimes - \arrow[rr, "\alpha"] \arrow[dr, "\rho \boxtimes \mathbb{1}"'] & & - \boxtimes (e \boxtimes -) \arrow[dl, "\mathbb{1} \boxtimes \lambda"] \\
 & - \boxtimes - &
\end{tikzcd}$$

Definition 2.2 We say that a functor $F\colon (A, \boxtimes) \to (B, \odot)$ between monoidal categories is a *monoidal functor* if there is a natural transformation

$$\Theta\colon F(-) \odot F(-) \to F(- \boxtimes -)$$

taking identity object to identity object, and commutes with application of α, ρ and λ.

Example 2.3 for categories having all finite (including empty) products $- \times -$, we get a monoidal structure by setting $\boxtimes = \times$.

By the universal property of the product, we get the following diagram:

$$\begin{tikzcd}
 & - \times (- \times -) \arrow[dl] \arrow[dr] \arrow[dd, bend left, "\exists!"] & \\
(-) & & (- \times -) \\
 & (- \times -) \times - \arrow[ul] \arrow[ur] \arrow[uu, bend left, "\exists!"] &
\end{tikzcd}$$

As the two middle arrows are unique, composing them must yield the identity, we therefore define the upward arrow as our desired α.

Again universality gives the diagram

$$\begin{tikzcd}
 & ((- \times -) \times -) \times - \arrow[dl] \arrow[dr] \arrow[d, dashed, "\exists!"] & \\
(-) & - \times ((- \times -) \times -) \arrow[l] \arrow[r] & (- \times -) \times -
\end{tikzcd}$$

The middle morphism being unique forces that after composing with $\mathbb{1} \times \alpha$, the two ways of travelling the diagram in 2.1(1.) are the same.

The empty product gives us by definition of products a terminal object e; we obtain ρ through the diagram, again from universality (and existence of morphisms into e), making ρ unique.

$$\begin{tikzcd}
 & (-) \arrow[dl, "\mathbb{1}"'] \arrow[d, dashed, "\rho^{-1}"] \arrow[dr] & \\
(-) & - \times e \arrow[l, "\rho"'] \arrow[r] & e
\end{tikzcd}$$

The morphism λ is obtained similarly, namely by flipping the above diagram along the middle arrow. Uniqueness of α, λ and ρ yields commutativity of the diagram in 2.1(2.)

The example above verifies that especially topological spaces, with cartesian product as tensor product is a monoidal category.

However, the point of generalizing to monoidal categories, is that it encapsulate more than just categories with finite products. For instance in the category Mod of R-modules, we have the tensor product $- \otimes -$, which – as a basic fact of homological algebra – is a monoidal category, with $\boxtimes = \otimes$. This also applies to the category of $(\mathrm{dgMod}, \otimes)$ of differential-graded modules.

We shall use many different monoidal categories, throughout the thesis. Note that in all the mentioned examples, we actually have a natural isomorphism $A \boxtimes B \cong B \boxtimes A$, that fits with the following definition:

Definition 2.4 A monoidal category $(\mathcal{C}, \boxtimes, \alpha, e)$ is said to be *symmetric*, if there is a natural isomorphism $\gamma : A \boxtimes B \to B \boxtimes A$, playing along with the monoidal structure of $\mathcal{C}$; meaning that given objects A, B, C of $\mathcal{C}$, we have:

1. $\begin{tikzcd} A \boxtimes B \arrow[r, "\gamma"] \arrow[rr, bend right, "\mathbb{1}"'] & B \boxtimes A \arrow[r, "\gamma"] & A \boxtimes B \end{tikzcd}$ is commutative.

2. $\begin{tikzcd} e \boxtimes A \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & A \boxtimes e \arrow[dl, "\lambda"] \\ & A & \end{tikzcd}$ is a commutative diagram

3.
$$\begin{tikzcd} & (B \boxtimes A) \boxtimes C \arrow[r, "\alpha"'] & B \boxtimes (A \boxtimes C) \arrow[dr, "\mathbb{1}\boxtimes\gamma"] & \\ (A \boxtimes B) \boxtimes C \arrow[ur, "\gamma\boxtimes\mathbb{1}"] \arrow[dr, "\alpha"'] & & & B \boxtimes (C \boxtimes A) \\ & A \boxtimes (B \boxtimes C) \arrow[r, "\gamma"] & (B \boxtimes C) \boxtimes A \arrow[ur, "\alpha"] & \end{tikzcd}$$

and

$$\begin{tikzcd} & A \boxtimes (C \boxtimes B) \arrow[r, "\alpha^{-1}"'] & (A \boxtimes C) \boxtimes B \arrow[dr, "\gamma\boxtimes\mathbb{1}"] & \\ A \boxtimes (B \boxtimes C) \arrow[ur, "\mathbb{1}\boxtimes\gamma"] \arrow[dr, "\alpha^{-1}"'] & & & (C \boxtimes A) \boxtimes B \\ & (A \boxtimes B) \boxtimes C) \arrow[r, "\gamma"] & (B \boxtimes C) \boxtimes A \arrow[ur, "\alpha^{-1}"] & \end{tikzcd}$$

are commutative diagrams.

Remark 2.5 As shown in [ML98], one can prove "abstract nonsense" coherence statements, saying that for symmetric monoidal categories, diagrams built out of the morphisms α, γ, λ and ρ all commute. First of all, this will allow us to omit parentheses.

Furthermore, we get via γ a method for shuffling entries in a monoidal product.

Any permutation of the entries in a (large) monoidal product will therefore yield an isomorphism. We shall allow ourselves – from time to time – to be sloppy and instead of specifying explicitly how γ permutes large monoidal product, instead simply write the isomorphism as

$$\mathrm{shuffle}_\sigma \colon A_1 \boxtimes \cdots \boxtimes A_n \to A_{\sigma(1)} \boxtimes \cdots \boxtimes A_{\sigma(n)}$$

where σ is an element in the symmetric group on k letters, Σ_k.

From time to time, we will allow ourselves to omit – in notation – what particular permutation we are using, and simply write shuffle

Remark 2.6 For the category $(\mathrm{dgMod}, \otimes)$ – which is important to us as $\mathrm{H}_*(-)$ lives in this category – the natural isomorphism γ is – by necessity of the differential – given a sign:

$$\gamma(a \otimes b) = (-1)^{|a||b|} b \otimes a,$$

where $|\cdot|$ denotes the degree of a homogenous element. This will imply that for these categories, the shuffle isomorphism involves a sign.

For other important categories, such as Top, Set or Mod, which we shall also consider, γ is simply given without any signs as

$$\gamma(a \boxtimes b) = b \boxtimes a,$$

so in these categories, the shuffle-isomorphism will cause us less concern.

Definition 2.7 We say that a symmetric monoidal category is *closed* if there is a natural adjunction-isomorphism

$$\mathrm{Hom}(A, \mathrm{Hom}(B, C)) \cong \mathrm{Hom}(A \boxtimes B, C)$$

It is a fact that the symmetrical monoidal categories, we mainly are interested in, such as $(\mathrm{Top}, \times)$ and $(\mathrm{dgMod}, \otimes)$, are closed.

A last notion we shall need, before turning to defining operads, concerns symmetric groups:

Definition 2.8 Fix k numbers $n_1, \dots, n_k \in \mathbb{N}$.

We can embed $\sigma_1 \oplus \cdots \oplus \sigma_k \colon \bigoplus_{i=1}^k \Sigma_{n_i} \to \Sigma_{n_1+\cdots+n_k}$ by mapping $\sigma_1, \dots, \sigma_k$ to

$$\begin{pmatrix} \overbrace{1 \cdots n_1}^{n_1-\text{block}} & \overbrace{\cdots}^{n_i-\text{block}} & \overbrace{(\sum_{i=1}^{k-1} n_i) + 1 \cdots (\sum_{i=1}^{k-1} n_i) + n_k}^{n_k-\text{block}} \\ \sigma_1(1) \cdots \sigma_1(n_1) & \cdots & (\sum_{i=1}^{k-1} n_i) + \sigma_k(1) \cdots (\sum_{i=1}^{k-1} n_i) + \sigma_k(n_k) \end{pmatrix}$$

I.e. we divide the number $1, \dots, n_1 + \cdots + n_k$ into k blocks, called n_1-$, \dots ,n_k-$*block* – the i'th of length n_i. Each σ_i permutes the elements in the n_i-block.

Assume furthermore that we are given $\gamma \in \Sigma_k$. Given $\gamma, \sigma_1, \dots, \sigma_k$, we define the *block-permutation* $\gamma.(\sigma_1 \oplus \cdots \oplus \sigma_k)$ via the diagram

$$\begin{tikzcd}
\Sigma_{n_1} \oplus \cdots \oplus \Sigma_{n_k} \arrow[r, "(\sigma_1,\dots,\sigma_k) \mapsto (\sigma_{\gamma(1)},\dots,\sigma_{\gamma(k)})"] \arrow[ddr, dashed, "\gamma.(\sigma_1 \oplus \cdots \oplus \sigma_k)"'] & \Sigma_{n_{\gamma(1)}} \oplus \cdots \oplus \Sigma_{n_{\gamma(k)}} \arrow[d, hook] \\
& \Sigma_{n_1+\cdots+n_k} \arrow[d, "\sigma \mapsto \gamma_{n_1,\dots,n_k}\sigma"] \\
& \Sigma_{n_1+\cdots+n_k}
\end{tikzcd}$$

Where $\gamma_{n_1,\dots,n_k}$ is the permutation

$$\begin{pmatrix} \overbrace{1 \cdots n_1}^{n_1-\text{block}} & \overbrace{\cdots}^{n_i-\text{block}} & \overbrace{\left(\sum_{i=1}^{k-1} n_i\right) + 1 \cdots \left(\sum_{i=1}^{k-1} n_i\right) + n_k}^{n_k-\text{block}} \\ \underbrace{\left(\sum_{i=1}^{\gamma^{-1}(1)-1} n_i\right) + 1 \cdots \left(\sum_{i=1}^{\gamma^{-1}(1)} n_i\right)}_{n_{\gamma^{-1}(1)}-\text{block}} & \underbrace{\cdots}_{n_{\gamma^{-1}(i)}-\text{block}} & \underbrace{\left(\sum_{i=1}^{\gamma^{-1}(k)-1} n_i\right) + 1 \cdots \left(\sum_{i=1}^{\gamma^{-1}(k)} n_i\right)}_{n_{\gamma^{-1}(k)}-\text{block}} \end{pmatrix}$$

In this sense, $\gamma.(\sigma_1 \oplus \cdots \oplus \sigma_k)$ permutes first among the σ_i-blocks, and then permutes via γ the individual blocks themselves.

An image of $\gamma.(\sigma_1 \oplus \cdots \oplus \sigma_k)$ we would like to promote for the sake of the definition of an operad, is the following

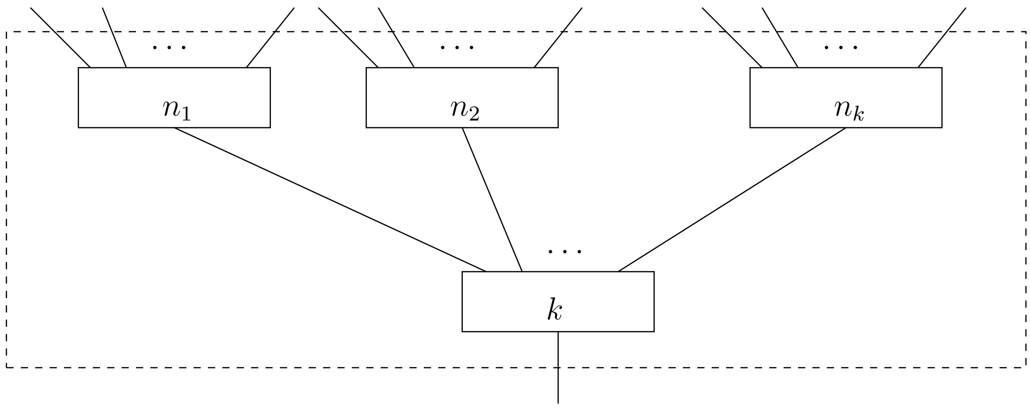

Figure 1: Image of an operad

Where we have n_i '*leafs*' sticking out of the boxes labelled n_i, and k k-'leafs' sticking out of the box labelled k. The dotted box therefore has $n_1 + \cdots + n_k$ 'leafs' sticking out.

We can obtain $\gamma.(\sigma_1 \oplus \cdots \oplus \sigma_k)$ as the permutation of all the leafs of the surrounding dotted box, obtained by first permuting among the leafs of the n_i-boxes by σ_i for all i, and then permute the leafs of the box labelled k via γ, and hence permuting the corresponding n_i-boxes that the k-leafs are attached to.

2.2 Operads: The definition

We define the notion of an operad, and give basic properties, that will be essential in working with operads.

Conceptually figure 1 shows what essentially goes on in an operad – putting things on top of other things – and we shall motivate the definition of operads via the picture.

Definition 2.9 Let $(\mathcal{C}, \boxtimes)$ be a symmetric monoidal category. By an *operad*, $\mathcal{O}$, we shall mean a sequence $\{\mathcal{O}(k)\}_{k\in\mathbb{N}}$ of objects in $\mathcal{C}$, such that there is a unit morphism $\varepsilon\colon e \to \mathcal{O}(1)$, and for each n an action of Σ_n on $\mathcal{O}(n)$, and is equipped with a *k-ary operation* – or *operadic composition* (we shall throughout use both notions as we like):

$$\omega : \mathcal{O}(k) \boxtimes \mathcal{O}(n_1) \boxtimes \cdots \boxtimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k),$$

In terms of figure 1, we can think of $\mathcal{O}(k)$ as a box with k leafs sticking out, the action of Σ_k on $\mathcal{O}(k)$ permutes the leafs, and ω takes all the data given in figure 1 and forgets most of it – except the dotted box and the leafs outside of the box. There are $n_1 + \cdots + n_k$ leafs sticking out.

We require the following axioms to be satisfied for an operad:

- *operational associativity*; that is the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{O}(m) \boxtimes \left(\mathcal{O}(k_1) \boxtimes \mathcal{O}(n_{1,1}) \boxtimes \cdots \boxtimes \mathcal{O}(n_{k_1,1}) \right) \boxtimes \cdots \boxtimes \left(\mathcal{O}(k_m) \boxtimes \mathcal{O}(n_{1,m}) \boxtimes \cdots \boxtimes \mathcal{O}(n_{k_m,m}) \right) \arrow[r, "\omega \boxtimes \mathbb{1} \boxtimes \cdots \boxtimes \mathbb{1}"] \arrow[d, "\mathbb{1} \boxtimes \omega \boxtimes \cdots \boxtimes \omega"'] & \mathcal{O}\left(\sum_{j=1}^{m} k_j\right) \boxtimes \mathcal{O}(n_{1,1}) \boxtimes \cdots \boxtimes \mathcal{O}(n_{k,1}) \boxtimes \cdots \boxtimes \mathcal{O}(n_{1,m}) \boxtimes \cdots \boxtimes \mathcal{O}(n_{k_m,m}) \arrow[d, "\omega"] \\
\mathcal{O}(m) \boxtimes \mathcal{O}\left(\sum_{i=1}^{k_1} n_{i,1}\right) \boxtimes \cdots \boxtimes \mathcal{O}\left(\sum_{i=1}^{k_m} n_{i,m}\right) \arrow[r, "\omega"] & \mathcal{O}\left(\sum_{j=1}^{m} \sum_{i=1}^{k_j} n_{i,j}\right)
\end{tikzcd}
$$

  saying that the order of operation is not an issue.

  In terms of figure 1, operadic associativity means that if we continue adding boxes to the leafs sticking out in the upwards direction, no matter what order we place the dotted boxes in – and apply ω to forget the data they contain – the result will be the same.

- *unit identities*, meaning that the diagrams

$$
\begin{tikzcd}
\mathcal{O}(n) \boxtimes e^{\boxtimes n} \arrow[r, "\cong"] \arrow[d, "\mathbb{1} \boxtimes \varepsilon^{\boxtimes n}"'] & \mathcal{O}(n) \\
\mathcal{O}(n) \boxtimes \mathcal{O}(1)^{\boxtimes n} \arrow[ur, "\omega"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
e \boxtimes \mathcal{O}(n) \arrow[r, "\lambda"] \arrow[d, "\varepsilon \boxtimes \mathbb{1}"'] & \mathcal{O}(n) \\
\mathcal{O}(1) \boxtimes \mathcal{O}(n) \arrow[ur, "\omega"'] &
\end{tikzcd}
$$

  commutes. Here the upper isomorphism in the left-most diagram is given by iterated application of ρ from 2.1.

- Σ_k-*equivariance* – that is for permutations $\sigma_1, \ldots \sigma_k$ acting on $\mathcal{O}(n_1), \ldots, \mathcal{O}(n_k)$

respectively, and γ acting on $\mathcal{O}(k)$, the following diagram should commute:

$$\begin{tikzcd}
\mathcal{O}(k) \boxtimes \mathcal{O}(n_1) \boxtimes \cdots \boxtimes \mathcal{O}(n_k) \arrow[r, "\sigma\boxtimes\mathrm{shuffle}_\sigma"] \arrow[dd, "\omega"] & \mathcal{O}(k) \boxtimes \mathcal{O}(n_{\sigma(1)}) \boxtimes \cdots \boxtimes \mathcal{O}(n_{\sigma(k)}) \arrow[d, "\mathbb{1}\boxtimes\sigma_{\sigma(1)}\boxtimes\cdots\boxtimes\sigma_{\sigma(k)}"] \\
 & \mathcal{O}(k) \boxtimes \mathcal{O}(n_{\sigma(1)}) \boxtimes \cdots \boxtimes \mathcal{O}(n_{\sigma(k)}) \arrow[d, "\omega"] \\
\mathcal{O}(n_1 + \cdots + n_k) \arrow[r, "\gamma.(\sigma_1\oplus\cdots\oplus\sigma_k)"] & \mathcal{O}(n_1 + \cdots + n_k)
\end{tikzcd}$$

To understand this diagram in terms of figure 1, consider our motivation for introducing $\gamma.(\sigma_1 \oplus \cdots \oplus \sigma_k)$ in the first place – written below the image.

Remark 2.10 Note that we have not yet specified what convention of $\mathbb{N}$ we are using, i.e. whether $0 \in \mathbb{N}$ or not. That is, we have not specified if the operad involves a $\mathcal{O}(0)$ term or not. Both definitions of operads do make perfect sense, and we shall use both definition as we please.

Whether $\mathcal{O}(0)$ is included or not, will give slight differences in some computations with operads – if we want to specify precisely which definition of operads we are using, we shall call $\mathcal{O}$ a 0-*operad* if $\mathcal{O}(0)$ is included and call it a 1-*operad* otherwise.

Definition 2.11 A *morphism of operads* $f : \mathcal{O} \to \mathcal{P}$ is a collection of morphisms

$$\{f_n \colon \mathcal{O}(n) \to \mathcal{P}(n)\}_{n\in\mathbb{N}}$$

that sends unit to unit: $f_1 \circ \varepsilon_{\mathcal{O}} = \varepsilon_{\mathcal{P}}$, commutes with the k-ary operation:

$$\begin{tikzcd}
\mathcal{O}(k) \boxtimes \mathcal{O}(n_1) \boxtimes \cdots \boxtimes \mathcal{O}(n_k) \arrow[r, "f_k\boxtimes f_{n_1}\boxtimes\cdots\boxtimes f_{n_k}"] \arrow[d, "\omega"] & \mathcal{P}(k) \boxtimes \mathcal{P}(n_1) \boxtimes \cdots \boxtimes \mathcal{P}(n_k) \arrow[d, "\omega"] \\
\mathcal{O}(n_1 + \cdots + n_k) \arrow[r, "f_{n_1+\ldots+n_k}"] & \mathcal{P}(n_1 + \cdots + n_k)
\end{tikzcd}$$

and commutes with the action of $\sigma \in \Sigma_k$ on $\mathcal{O}(k)$:

$$\begin{tikzcd}
\mathcal{O}(k) \arrow[r, "\sigma"] \arrow[d, "f_k"] & \mathcal{O}(k) \arrow[d, "f_k"] \\
\mathcal{P}(k) \arrow[r, "\sigma"] & \mathcal{P}(k)
\end{tikzcd}$$

We call this last condition Σ_k-*equivariance* of f_k or Σ-*equivariance* of f.

Remark 2.12 A morphism of operads $f \colon \mathcal{O} \to \mathcal{P}$ actually commutes with the entire structure of the operad. Mapping via f between the large diagrams of the definition of operads, would consist of natural diagrams for the k-ary operations and the action of Σ_k (as well as natural diagrams involving morphisms defining the symmetric monoidal category). As these diagrams all commute by definition, the larger diagrams commute as well.

This leads us to naturally say that an *isomorphism of operads* is a morphism of operads f, where each f_n is an isomorphism.

Definition 2.13 Let $f\colon \mathcal{O} \to \mathcal{P}$ be a morphism of operads in the symmetrical monoidal category of topological spaces. We say that f is a *local equivalence* if all the induced maps of $f_n\colon \mathcal{O}(n) \to \mathcal{P}(n)$ are homotopy equivalences.

Two operads $\mathcal{O}$ and $\mathcal{P}$ are *locally equivalent* if there is a string of local equivalences of operads:

$$\mathcal{O} \longleftarrow \mathcal{O}_1 \longrightarrow \cdots \longleftarrow \mathcal{O}_n \longrightarrow \mathcal{P}.$$

Proposition 2.14 Let $F\colon (A,\boxtimes) \to (B,\odot)$ be a monoidal functor. To an operad $\mathcal{O}$ in $(A,\boxtimes)$, we have that $\{F(\mathcal{O}(n))\}$ gives an operad $F(\mathcal{O})$ in $(B,\odot)$ with structure induced via F. The construction is functorial, in the sense that for a morphism of operads $f\colon \mathcal{O} \to \mathcal{P}$, $F(f)$ is a morphisms of operads.

Proof. Since F involves a natural transformation $\Theta\colon F(-) \odot F(-) \to F(- \boxtimes -)$, we indeed can use Θ to form the commutative diagram

$$\begin{tikzcd}
F(\mathcal{O}(k)) \odot F(\mathcal{O}(n_1)) \odot \cdots \odot F(\mathcal{O}(n_k)) \arrow[r] \arrow[dr, dashed, "\omega_F"] & F(\mathcal{O}(k) \boxtimes \mathcal{O}(n_1) \boxtimes \cdots \boxtimes \mathcal{O}(n_k)) \arrow[d, "F(\omega)"] \\
 & F(\mathcal{O}(n_1 + \cdots + n_k))
\end{tikzcd}$$

and define the k-ary operations of $F(\mathcal{O})$ as ω_F.

The group action of Σ_n on $F(\mathcal{O}(n))$ is given via the factorization $\Sigma_n \to \mathrm{Aut}(\mathcal{O}(n)) \to \mathrm{Aut}(F(\mathcal{O}(n)))$ where the first arrow is the group action given as a homomorphism into the automorphism group of $\mathcal{O}(n)$, and the last arrow is the group homomorphism $g \mapsto F(g)$.

$F(\varepsilon)$ is again a unit morphism of $F(\mathcal{O})$, as F maps the identity object in $(A,\boxtimes)$ to the identity in $(B,\odot)$.

It follows that the diagrams of 2.9 commute for $F(\mathcal{O})$ – and that $F(f)$ is a morphism of operads for $F(\mathcal{O})$ – as the diagrams commute for $\mathcal{O}$ together with the fact that Θ is a natural transformation. □

Corollary 2.15 Let $\mathrm{H}_*(-)$ denote homology with coefficients in a principal ideal domain. Suppose that we are given an operad $\mathcal{O}$ in the symmetric monoidal category of topological spaces. Then $\mathrm{H}_*(\mathcal{O})$ is an operad.

Proof. We have the Künneth natural short exact sequence (cf. [Hat02, 3B.6]). The left morphism in the short exact sequence is given as $\Theta\colon \mathrm{H}_*(X) \otimes \mathrm{H}_*(Y) \to \mathrm{H}_*(X \times Y)$. As the Künneth short exact sequence is natural, we have that Θ is the desired natural transformation. □

Remark 2.16 Because of 2.15, we shall from hereon by $\mathrm{H}_*(-)$ mean homology with coefficients in a principal ideal domain. Whenever we need to strengthen this assumption, we shall explicitly say so.

Corollary 2.17 If $\mathcal{O}$ and $\mathcal{P}$ are locally equivalent operads in the symmetric monoidal category of topological spaces, then $\mathrm{H}_*(\mathcal{O})$ and $\mathrm{H}_*(\mathcal{P})$ are isomorphic operads.

Proof. $\mathrm{H}_*(-)$ is a homotopy invariance. $\square$

Example 2.18 To an object X in a symmetric monoidal category, $(\mathcal{C}, \boxtimes)$, we form the endomorphism operad $\mathcal{E}nd_X$. The n'th component is defined to be the given as

$$\mathcal{E}nd_X(n) := \mathrm{Hom}(X^{\boxtimes n}, X)$$

and the k-ary operations $\omega\colon \mathcal{E}nd_X(k) \boxtimes \mathcal{E}nd_X(n_1) \boxtimes \cdots \boxtimes \mathcal{E}nd_X(n_k) \to \mathcal{E}nd_X(n_1 + \cdots n_k)$ are given as the composition

$$\omega(f_k; f_{n_1}, \cdots f_{n_k}) = f_k \circ (f_{n_1} \boxtimes \cdots \boxtimes f_{n_k})$$

The action of Σ_n on $\mathcal{E}nd_X(n)$ permutes the entries in the domain of the morphisms, and the identity is given by $\mathbb{1}_X \in \mathcal{E}nd_X(1)$.

Verifying that this actually is an operad, is straightforward from the definitions. For instance, associativity of composition of morphisms in categories, together with the natural associativity morphism in monoidal categories, ensures operational associativity of 2.9 for ω.

Definition 2.19 Given an object, X in $\mathcal{C}$, an *action* of an operad $\mathcal{O}$ on X, is a morphism of operads

$$\mathcal{O} \to \mathcal{E}nd_X .$$

As we shall see, having an action of an operad on some object in a category is a quite strong condition. We shall see examples of how a lot of algebraic structure is being controlled by having an action of an operad.

Although the above definition is more clean, we shall need in forthcoming examples to have the action somewhat more explicitly written out.

Proposition 2.20 Assume that $(\mathcal{C}, \boxtimes)$ is closed. An algebra over $\mathcal{O}$ is equivalent to having morphisms

$$\alpha_n\colon \mathcal{O}(n) \boxtimes X^{\boxtimes n} \to X$$

for all n such that $\alpha_1(\varepsilon \boxtimes -) = \mathbb{1}_X$, satisfying that the following diagrams is commutative:

$$\begin{tikzcd}
\mathcal{O}(k) \boxtimes \left(\mathop{\boxtimes}_{i=1}^{k} \mathcal{O}(n_i) \times X^{\boxtimes n_i} \right) \arrow[rr, "\mathbb{1} \boxtimes \left(\mathop{\boxtimes}_{i=1}^{k} \alpha_{n_i} \right)"] \arrow[d, "\mathrm{shuffle} \circ (\omega \times \mathbb{1})"'] & & \mathcal{O}(k) \boxtimes X^{\boxtimes k} \arrow[d, "\alpha_k"] \\
\mathcal{O}(n_1 + \cdots + n_k) \boxtimes X^{\boxtimes (n_1 + \cdots n_k)} \arrow[rr, "\alpha_{(n_1 + \cdots + n_k)}"] & & X
\end{tikzcd}$$

(here the shuffle isomorphism moves the $\mathcal{O}(n_i)$ factor out to the left, so ω can be applied). Furthermore, given $\sigma \in \Sigma_k$, the diagram

$$\begin{tikzcd}
\mathcal{O}(k) \boxtimes X^{\boxtimes k} \arrow[d, "\sigma\boxtimes\mathrm{shuffle}_\sigma"'] \arrow[drr, "\alpha_k"] & & \\
\mathcal{O}(k) \boxtimes X^{\boxtimes k} \arrow[rr, "\alpha_k"] & & X
\end{tikzcd}$$

commute

Proof. By the adjunction isomorphism, we have

$$\mathrm{Hom}(\mathcal{O}(n), \mathcal{E}nd_X(n)) = \mathrm{Hom}(\mathcal{O}(n), \mathrm{Hom}(X^{\boxtimes n}, X)) \cong \mathrm{Hom}(\mathcal{O}(n) \boxtimes X^{\boxtimes n}, X)$$

So to each $f_n \colon \mathcal{O}(n) \to \mathcal{E}nd_X(n)$, the above isomorphism gives a one-to-one correspondence to $\alpha_n \colon \mathcal{O}(k) \boxtimes X^k \to X$.

Using this adjunction, it is a matter of checking the involved diagrams – keeping the operadic structure on $\mathcal{E}nd_X$ in mind – and noting that indeed the first diagram above corresponds to the first diagram of 2.11 and the second to the second diagram of 2.11. $\square$

If the symmetrical monoidal category is not closed, we shall mean what is stated in 2.20 by an action of an operad.

2.3 Operads from the Algebraic Viewpoint

In the following we shall mainly give examples of how to classify various algebraic structures using the language of operads.

Definition 2.21 Consider for a commutative ring the category of R-modules. By the *commutative operad*, $\mathcal{C}omm$, we shall understand the operad living in $(\mathrm{Mod}, \otimes)$ where we set $\mathcal{C}omm(n) = R$.

The k-ary operations are given by the identity $R \otimes \cdots \otimes R \cong R \to R$. The unit is given by the generator $1 \in R$, and the action of Σ_k on $\mathcal{C}omm(k)$ is trivial.

Trivially, $\mathcal{C}omm$ is an operad. However actions of it classifies a quite important class of structures of modules:

Proposition 2.22 Giving an action of $\mathcal{C}omm$ on a module M is precisely the same as giving M the structure of a commutative algebra.

Proof. Assume that we are given an action α of $\mathcal{C}omm$ on M. We give a multiplication μ via $\alpha_2 \colon \mathcal{C}omm(2) \otimes M \otimes M \to M$, and define $\mu \colon M \otimes M \to M$ as $\alpha_2(1 \otimes - \otimes -)$.

We first of all need to check associativity – namely that $\mu(\mu(a\otimes b)\otimes c) = \mu(a\otimes\mu(b\otimes c))$. However, we get from 2.20, the diagram

$$\begin{tikzcd}
\mathcal{C}omm(2)\otimes(\mathcal{C}omm(2)\otimes M\otimes M)\otimes(\mathcal{C}omm(1)\otimes M) \arrow[d, "\mathrm{shuffle}\circ(\omega\otimes\mathbb{1})"'] \arrow[rr, "\mathbb{1}\otimes\alpha_2\otimes\alpha_1"] & & \mathcal{C}omm(2)\otimes M\otimes M \arrow[dl, "\alpha_2"'] \\
\mathcal{C}omm(3)\otimes M\otimes M\otimes M \arrow[r, "\alpha_3"] & M & \\
\mathcal{C}omm(2)\otimes(\mathcal{C}omm(1)\otimes M)\otimes(\mathcal{C}omm(2)\otimes M\otimes M) \arrow[u, "\mathrm{shuffle}\circ(\omega\otimes\mathbb{1})"] \arrow[uurr, bend right, "\mathbb{1}\otimes\alpha_1\otimes\alpha_2"'] & &
\end{tikzcd} \quad (1)$$

hence $\mu(\mu(a\otimes b)\otimes c) = \alpha_3(1\otimes a\otimes b\otimes c) = \mu(a\otimes\mu(b\otimes c))$.

Commutativity $\mu(a\otimes b) = \mu(b\otimes a)$, follows as we have chosen a trivial action of Σ_2 on $\mathcal{C}omm(2)$, so from 2.20, we get the diagram

$$\begin{tikzcd}
\mathcal{C}omm(2)\otimes M\otimes M \arrow[r, "\alpha_2"] \arrow[d, "\mathbb{1}\otimes\mathrm{shuffle}"'] & V \\
\mathcal{C}omm(2)\otimes M\otimes M \arrow[ur, "\alpha_2"'] &
\end{tikzcd} \quad (2)$$

from wich commutativity obviously follows.

On the other hand, by defining $\alpha_n\colon \mathcal{C}omm(n)\otimes M^{\otimes n}\cong M^{\otimes n}\to V$ by $\mu(\mu(\cdots\mu(a_1\otimes a_2)\cdots\otimes a_{n-1})\otimes a_n)$, we get that the diagrams of 2.20 commute by associativity and commutativity of μ. $\square$

Remark 2.23 In 2.22 we have not specified whether a commutative algebra has a unit or not, and in 2.21 we have not specified whether $\mathcal{C}omm$ is a 0- or 1-operad.

There is a reason; namely if we specify $\mathcal{C}omm$ as a 0-operad, we get the 2-ary composition $\omega\colon \mathcal{C}omm(2)\otimes\mathcal{C}omm(2)\otimes\mathcal{C}omm(0)\to\mathcal{C}omm(2)$, so using similar commutative diagrams as above, one checks that having $\alpha_0\colon \mathcal{C}omm(0)\to M$ specifies a unit in M as $\alpha_0(1)$, and vice versa.

In the following, we shall give a lot of similar examples, where action of some operads gives rise to some algebraic structure. We shall not mention it any further, but whether we are giving an action of a 0-operad or not will give similar algebraic statements about the unit.

Operads are intimately tied together to trees. We therefore need to introduce a language of trees.

Definition 2.24 By a *tree*, we will mean an acyclic graph; that is an abstract graph, consisting of a vertex set $V = \{v_0,\dots,v_n\}$ together with an edge set $E = \{e_1,\dots,e_m\}$ with edge maps such that the geometric realization as a CW-complex is connected and has trivial fundamental group.

The univalent vertices of a tree will be called *leaves*, and the other vertices, *knots* or *internal vertices*.

Definition 2.25 By an *n-tree*, T, we shall understand a tree with precisely $n+1$ leaves. Furthermore, let the subset $V_l \subseteq V$ denote the set $\{v_{i_0}, \ldots, v_{i_n}\}$ of leafs. We equip T with a *labelling map* $\tau_T \colon \{0, \ldots, n\} \to V_l$ such that τ_T is a bijection, and $\tau(0) = v_{i_0}$. A *k-corolla* is a k-tree with precisely one knot.

We say that a leave v of T satisfying $\tau_T(j) = v$ is labelled j. The leave labelled 0 is special, and called the *root*.

We consider two n-trees T_1, T_2 equivalent if there is a graph isomorphism $\Gamma \colon T_1 \to T_2$ between them that preserves the labellings of the leafs. I.e. satisfies $\Gamma\tau_{T_1} = \tau_{T_2}$

To each vertex v of an n-tree, we associate the *level*, $l(v)$, to be the number of vertices on the shortest path from v to the root. We say that an edge e between two vertices v, w, is *ingoing* to v, if it is on the shortest path from v to the root. If an edge between v and w is not ingoing to v, it is *outgoing*.

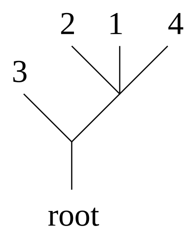

Figure 2: A 4-tree with one level 0 vertex (as is always the case), two level 1 vertices (one leave, one knot), and three level 1 vertices

Definition 2.26 We say that we *graft* a k-tree T_k onto the i'th leave of an n-tree, T_n, if we identify the root of T_k with the leave labelled i of T_n, and thereby obtain an $n+k-1$-tree. In the graph thus obtained, we remove the 2-valent vertex arising from the identification. We call the resulting tree $T_n \circ_i T_k$. The leaves of this *grafted tree* will be relabelled order-preserving w.r.t. both labellings of T_n and T_k, such that the leaf labelled 1 of T_k is labelled i in $T_n \circ_i T_k$; that is:

- The root is the leave originating from the root of T_n
- The leaves originating from T_n labelled by j, will be labelled by j, if $j < i$ and $j+k$ if $j > i$.
- The leaves originating from T_k labelled by j will be labelled by $j+i-1$.

Later on, we shall need a sort of 'inverse' to the process of grafting: Assume we are given an edge e of a k-tree T_k. by the *branch* of e, we will understand the sub-tree $T_k|e$ of T_k, spanned by all vertices v – and the edges attached to these – of T_k that contains e in the shortest path from v to the root of T_k. Say that there are n leaves in $T_k|e$. In order to make $T_k|e$ into an n-tree, we append an edge and a root, at the position of e in T_k. We relabel the leafs of $T_k|e$ by numbers 1 through k, in the unique order-preserving way.

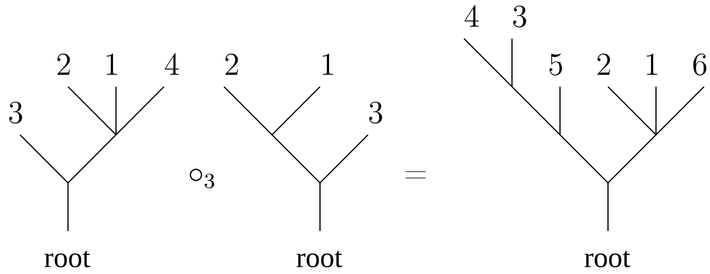

Figure 3: Grafting a 3-tree onto a 4-tree yielding a $(4 + 3 - 1 = 6)$-tree.

Definition 2.27 Let $\mathcal{O}$ be an operad. A family $I = \{I(n)\}_{n\in\mathbb{N}}$ with $I(n) \subseteq \mathcal{O}(n)$[1] is called an operadic ideal if it is closed under operadic composition in $\mathcal{O}$ and action of the symmetric group Σ_n on $\mathcal{O}(n)$.

That is; assume we are given $o_k \in \mathcal{O}(k)$ and $o_{n_i} \in \mathcal{O}(n_i)$ where $i \in \{1, \ldots, k\}$, and assume that either $o_k \in I(k)$ or one of $o_{n_i} \in I(n_i)$. We require that the operadic composition $\omega(o_k; o_{n_1}, \ldots, o_{n_k}) \in I(n_1 + \cdots + n_k)$.

Furthermore, if $o_k \in I(k)$, then for $\sigma \in \Sigma_k$, we require $\sigma.o_k \in I(k)$ as well.

Remark 2.28 Note that given an ideal, for the object $\mathcal{O}/I$ given by the sequence of quotients $\{\mathcal{O}(n)/I(n)\}_{n\in\mathbb{N}}$, we obtain a structure of an operad as well.

Under the equivalence relation defining the quotient, we have obviously by the requirements on I, that the induced of the maps defining an operad structure on $\mathcal{O}/I$ are well-defined.

We call $\mathcal{O}/I$ the *quotient operad* (w.r.t I).

Construction 2.29 Note that if we let $\mathcal{T}ree(n)$ denote the set of all n-trees, then if we let the operadic composition in $\mathcal{T}ree$

$$\omega\colon \mathcal{T}ree(k) \times \mathcal{T}ree(n_1) \times \cdots \times \mathcal{T}ree(n_k) \to \mathcal{T}ree(n_1 + \cdots + n_k)$$

be given by grafting of trees: $\omega(T_k, T_{n_1}, \ldots T_{n_k}) = ((\cdots(T_k \circ_k T_{n_k}) \circ_{k-1} \cdots) \circ_1 T_{n_1})$, and let Σ_k act on $\mathcal{T}ree(k)$ by permuting the labelling of the leaves different from the root, we obviously obtain an operad structure on $\mathcal{T}ree$.

Consider the subset $\mathcal{F}ree(n) \subseteq \mathcal{T}ree(n)$, given by the n-trees where all knots are either bi-valent or tri-valent.

Note that $\mathcal{F}ree$ is still an operad. It is indeed *generated* by a 2-tree B with a single knot, plus the identity $e \in \mathcal{F}ree(1)$ having no knots – generated in the sense that any binary tree can be obtained through iterated operadic composition of B and e, followed by some action of Σ on the resulting k-tree.

In this sense, we say that $\mathcal{F}ree$ is the *free operad* on one bi-valent generator.

In order to add more generators to free operads, we expand $\mathcal{T}ree$, by allowing different markings of the knots. That is – if we let $\mathcal{T}ree_{S_*}(k)$ denote the operad, where we to each

[1] here we make the mild assumption that $I(n)$ is a set

knot v of $T \in \mathcal{T}ree_S(k)$, have specified a map $\sigma_v \colon \{v\} \to S_{|v|-1}$, here S_i is a set, for all $i \in \mathbb{N}$. We call elements of S_i the *i-markings*.

Assume about the cardinalities of S_*, that $|S_1| = c$ and $|S_2| = d$. We define the *free operad on c, d generators* – $\mathcal{F}ree_{c,d}$ – where the subspaces $\mathcal{F}ree_{c,d}(k)$ are the subspaces of $\mathcal{T}ree_{S_*}(k)$, generated by all 2-trees, as well as all 1-trees. Denote by $m_1, \ldots, m_c$ the elements of S_1, and by $w_1, \ldots, w_d$ the elements of S_2. We denote the generators of $\mathcal{F}ree_{c,d}$ by B_{m_i} for $i \in \{1, \ldots, c\}$ and B_{w_j} for $j \in \{1, \ldots, d\}$.

Consider elements $T_1, \ldots, T_n$ of $\mathcal{F}ree_{c,d}(k)$. By definition, we have that T_i are obtained as a series of operadic composition of the generators of $\mathcal{F}ree_{c,d}$ along with the identity e, followed by some action of Σ_k.

We form the operadic ideal $< T_1, \ldots, T_n >$. This is given as the composites of all trees of $\mathcal{F}ree_{c,d}$, obtained as some operadic composition involving some T_i – followed by some action of the corresponding symmetric group. This condition obviously makes $< T_1, \ldots, T_n >$ into an operadic ideal *generated by* $T_1, \ldots, T_n$.

For the quotient operad $\mathcal{F}ree_{c,d}/ < T_1, \ldots, T_n >$, we shall notate the fact that $< T_1, \ldots, T_n >$ is generated by $T_1, \ldots, T_n$, by $T_1 \sim T_2 \sim \cdots \sim T_n$ and write $\mathcal{F}ree_{c,d}/_\sim$ for the quotient operad.

As a first example of an application of free operads, we give

Definition 2.30 By the *associative operad*, we shall understand the quotient operad $\mathcal{A}ss := \mathcal{F}ree_{0,1}/_{\sim_a}$, generated by the 2-tree B, where $\sim_a$ is the relation

$$\omega(B; B, e) \sim_a \omega(B; e, B)$$

or perhaps more sensible – drawn among trees, the relation becomes

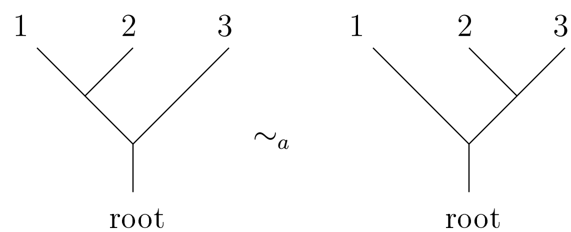

Figure 4: Trees encoding associativity

Construction 2.31 We have the free functor $F \colon (\mathrm{Set}, \times) \to (\mathrm{Mod}_R, \otimes)$, that to a set X associates $F(X) := R[X]$ – that is, the free module given by a basis indexed by X.

Indeed the free functor is a monoidal functor, we have $R[X \times Y] \cong R[X] \otimes R[Y]$, obtained since $\{v_i \otimes w_j\}$ forms a basis for $V \otimes W$, where $\{v_i\}$ is a basis for V and $\{w_j\}$ is a basis for W.

By 2.14, we therefore have that for an operad $\mathcal{O}$ in $(\mathrm{Set}, \times)$, $R[\mathcal{O}]$ is again an operad.

In particular for $R[\mathcal{F}ree_{c,d}]$ is an operad. Each tree T_1 is mapped to some basis vector e_{T_1}. Via this free functor, we can in $(\mathrm{Mod}, \otimes)$, expand the notion of an ideal generated by $T_1, \ldots, T_n$ and instead allow ideals generated by linear combinations of basis elements indexed by trees.

Notation 2.32 Let $c_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} \in \Sigma_n$ be the permutation that cycles all entries one up.

As sort of an illustration of 2.31, we define

Definition 2.33 By the *Lie operad*, we shall understand the quotient operad $\mathcal{L}ie := R[Free_{0,1})]/_{\sim_b}$, where $\sim_b$ is given as the relation

$$e_{\omega(B;B,e)} + e_{c_3.\omega(B;B,e)} \sim_b -e_{(c_3c_3).\omega(B;B,e)}$$

and

$$e_B \sim_b -e_{c_2.B}$$

or written up as a relation among linear combination of trees (in our drawings we omit the mentioning of basis elements):

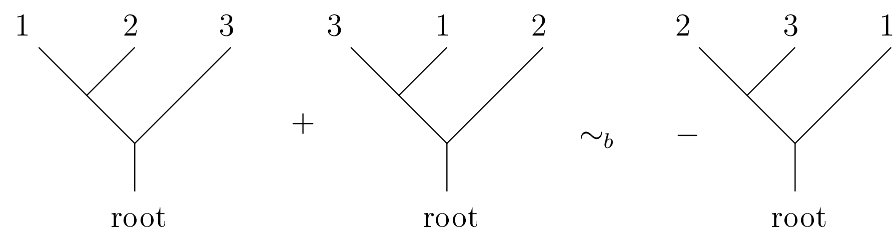

Figure 5: Trees encoding Jacobi-identity

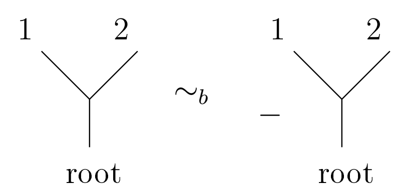

Figure 6: Trees encoding skew-symmetry

Proposition 2.34 Consider $\mathcal{A}ss$ as an operad in $(\mathrm{Mod}, \otimes)$, via 2.31. Giving an action of $\mathcal{A}ss$ on a module M is precisely the same as giving M the structure of an associative algebra.

The idea of the proof is the same as for 2.22. We however include the differences in the proofs as an illustration of the power of quotients of free operads.

Proof. Given an action α of $\mathcal{A}ss$ on M, we define a multiplication $\mu\colon M \otimes M \to M$ by $\mu = \alpha_2(B; -, -)$.

By 2.20, we still have a commutative diagram as (1) with every instance of $\mathcal{C}omm$ replaced by $\mathcal{A}ss$.

Using the diagram for $B \in \mathcal{Ass}(2)$ and $e \in \mathcal{Ass}(1)$, the two commutative squares involved in the diagram yields the two results

$$\alpha_3(\omega(B; B, e) \otimes a \otimes b \otimes c)$$

and

$$\alpha_3(\omega(B; e, B) \otimes a \otimes b \otimes c)$$

but the relation $\sim_a$ in the definition of $\mathcal{Ass}$, gives that in fact, these two results are the same. The second diagram of 2.20 simply defines a multiplication as invariant under permuting the factors, as it defines iterated multiplication for using all binary trees of $\mathcal{Ass}$ (i.e. the ones obtained from B plus a permutation of the leaves).

Assuming that we have an associative multiplication $\mu\colon M \otimes M \to M$, we similar to 2.22 define the action of $\mathcal{Ass}$ on M as $\alpha_n\colon \mathcal{Ass}(n) \otimes M^{\otimes n} \to M$ by

$$\alpha_n(\sigma.(B \circ_1 B) \circ_1 \cdots \circ_1 B \otimes v_1 \otimes \cdots v_n) = \mu(\cdots(\mu(v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)}) \cdots) v_{\sigma^{-1}(n)}).$$

which makes the diagrams of 2.20 commute; they respect the relations of $\mathcal{Ass}$, by associativity of μ. □

Remark 2.35 By the above proof, we see that for $\mathcal{Free}_{0,n}$ operads, the relations among them simply encode the conditions that we put upon our bilinear operators.

Indeed we will therefore redefine $\mathcal{Comm} := \mathcal{Free}_{0,1}/{\sim_c}$ where $\sim_c$ is the relation

$$e_B \sim_c e_{c_2.B}$$

equivalating all trees of $\mathcal{Free}_{0,1}(k)$.

In light of the above, we can mimic the proof of 2.34 or 2.22, to get

Proposition 2.36 Giving an action of $\mathcal{Lie}$ on a module M is precisely the same as giving V the structure of a Lie-algebra. [2]

So far, we have only talked about what happens for actions of quotients of free operads in the category of $(\mathrm{Mod}, \otimes)$. If we pass to the category of $(\mathrm{dgMod}, \otimes)$, we have to be careful with the grading, as noted in 2.6.

Definition 2.37 By the *Gerstenhaber operad*, we shall understand the quadratic operad in $(\mathrm{dgMod}, \otimes)$ – given by $\mathcal{Ger} := \mathcal{Free}_{0,2}/\sim_d$ – where we let the generators of $\mathcal{Free}_{0,2}$ be given by $e_{B_\star}$ in degree 0 and $e_{B_{[,]}}$ in degree 1.

We define the relation $\sim_d$ to be the one such that the basis element corresponding to $B_\star$ satisfies the relation $\sim_c$ of 2.35 and letting $B_{[,]}$ being subject to the relation(s) $\sim_b$ of 2.33.

Finally, we require $\sim_d$ to respect the *poisson-relation*

$$e_{\omega(B_{[,]}; e, B_\star)} \sim_d e_{\omega(B_\star; B_{[,]}, e)} + e_{c_3.\omega(B_\star; e, B_{[,]})}$$

[2]If the ring we are forming modules over is a field of characteristic 2, we would interfere with the usual definition of a Lie-Algebra – where shew-symmetry has a different interpretation, so we omit this case

Definition 2.38 In light of what we have seen in 2.35, we define a *Gerstenhaber algebra* to be an action of $\mathcal{Ger}$ on M, and can therefore rewrite it as M being equipped with two binary operators $- \star -\colon M \otimes M \to M$ and $[-,-]\colon M \otimes M \to M$, called the *product* and *Lie bracket*, satisfying the following identities for homogeneous elements $a, b, c \in M$:

- $|a \star b| = |a| + |b|, |[a,b]| = |a| + |b| + 1$ (the product is a degree 0 operator, and the Lie bracket is a degree 1 operator)
- $(a \star b) \star c = a \star (b \star c)$, $a \star b = (-1)^{|a||b|} b \star a (\star$ is graded commutative and associative)
- $[a,b] = -(-1)^{(|a|-1)(|b|-1)}[b,a]$ (graded skew-symmetry)
- $[[a,b],c] = [a[b,c]] - (-1)^{(|a|-1)(|b|-1)}[b,[a,c]]$ (graded Jacobi identity)
- $[a, b \star c] = [a,b] \star c - (-1)^{(|b|(|a|-1)} b \star [a,c]]$ (graded Poisson relation)

Remark 2.39 Note that in 2.38, there are a lot of signs, whereas the definition of the relations in 2.37 involves none. The signs all occur from the fact, noted in 2.6, that shuffle isomorphisms involve a sign.

For instance, as in the proof of 2.22, we see that since $\sim_d$ satisfies the relation $e_{B_*} \sim_d e_{c_2.B_*}$, we – defining $\star\colon V \otimes V \to V$ as $\alpha_2(e_{B_\star} \otimes - \otimes -)$ – have that the action of σ is trivial, so using the commutative diagram of (2), we have

$$\begin{tikzcd}
a \otimes b \arrow[r, "\star"] \arrow[d, "\text{shuffle}"] & a \star b = (-1)^{|a||b|} b \star a \\
(-1)^{|a||b|} b \otimes a \arrow[ur, "\star"] &
\end{tikzcd}$$

giving graded commutativity of $\star$.

Some of the signs are given with the degree of the homogeneous elements minus 1. As $e_{B_{[,]}}$ lives in degree 1, $\alpha_2(e_{B_{[,]}} \otimes - \otimes -)\colon V \otimes V \to V$ is not a graded map. In order to make it into a graded map, we suspend once, and consider define $[-,-] := \alpha_2(e_{b_{[,]}} \otimes - \otimes -)\colon \Sigma V \otimes \Sigma V \to \Sigma V$. It is then simply a matter of checking – using the diagrams of 2.20, and being careful with the shuffle isomorphisms as above – that indeed the signs match up.

Definition 2.40 By the *Batalin-Vilkovisky operad*, $\mathcal{BV}$, we shall understand the operad in (dgMod, $\otimes$), given as $\mathcal{Free}_{1,2}/\sim_e$ – with generators as for $\mathcal{Ger}$, plus a generator coming from a tree with a single 2-valent knot B_Δ, that lives as a basis vector in degree 1.

We let $\sim_e$ satisfy the relations of $\sim_d$ for $\mathcal{Ger}$, plus the relations

$$e_{\omega(B_\Delta;B_\Delta)} \sim_e 0$$

and

$$e_{\omega(B_{[,]})} \sim_e e_{\omega(B_\Delta;B_\star)} - e_{\omega(B_\star;B_\Delta,e)} - e_{\omega(B_\star;e,B_\Delta)}$$

Since we shall actually use it, we mention in analogy with the other propositions of this section

Definition 2.41 We can define what we mean by a *Batalin-Vilkovisky algebra* (or shorter *BV-algebra*, by adding the degree 1-operator $\Delta\colon M \to M$, and the relations

- $\Delta \circ \Delta = 0$
- $[a, b] = (-1)^{|a|}\Delta(a \star b) - (-1)^{|a|}\Delta(a) \star b - a \star \Delta(b)$

to the list of 2.38.

Proposition 2.42 Given an action of $\mathcal{BV}$ on a graded module M is precisely the same as given M the structure of a Batalin-Vilkovisky algebra.

Remark 2.43 As an alternative to this long list of relations in 2.41, it is shown in [Get94, Prop. 1.2], that having a Batalin-Vilkovisky algebra is the same as having two operators: One being $\Delta\colon M \to M$ of degree 1 and an associative graded commutative product $M \otimes M \to M$, such that $\Delta \circ \Delta = 0$ and such that the so-called *BV-master equation*:

$$\begin{aligned} \Delta(abc) &= \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) \\ &- \Delta(a)bc - (-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c) \end{aligned}$$

holds.

So equivalently the Batalin-Vilkovisky operad could be defined as a quotient operad of $\mathcal{F}\mathit{ree}_{1,1}$, with relations above incorporated instead.

2.4 Little Disk Operads

Notation 2.44 In the following, we let $D^m := \{x \in \mathbb{R}^m \mid \|x\| \leqslant 1\}$; that is D^m is the unit m-disk

Definition 2.45 An embedding $f\colon D^m \to D^m$ will be called a *little m-disk embedding*, if there is some $\sigma_f\colon \mathbb{R}^m \to \mathbb{R}^m$ given by $\sigma_f(x) = r(x + v)$, where $v \in \mathbb{R}^k$ and $r \in [1, \infty[$, such that $\sigma_f \circ f = \mathbb{1}_{D^m}$

That is, the left-inverse σ_f is given as a translation followed by a upscaling.

By the space of n little m-disks, $\mathcal{D}\mathit{isk}_m(n)$, we will understand, the space:

$$\left\{ f\colon \coprod_{i=1}^{n} D_i^m \to D^m \mid f \text{ is an embedding and } f|_{D_i^m} \text{ is a little } m\text{-disk embedding} \right\}$$

The topology on this space arises naturally from the metric given by

$$d_{\sup}(f, g) = \sup_{x \in \coprod_{i=1}^{n} D_i^m} d(f(x), g(x))$$

where $d(-,-)$ is given by a standard metric on D^m. We can note that this is the same as the open-closed topology since $\coprod_{i=1}^{n} D_i^m$ is compact.

Construction 2.46 We define an operad structure on $\{\mathcal{D}isk_m(n)\}_{n\in\mathbb{N}}$, simply by composition of the embeddings, that is

$$\omega\colon \mathcal{D}isk_m(k) \times \mathcal{D}isk_m(n_1) \times \cdots \times \mathcal{D}isk_m(n_k) \to \mathcal{D}isk_m(n_1 + \cdots + n_k)$$

is given by

$$\omega(f; f_1, \ldots, f_k) = f \circ (f_1 \amalg \cdots \amalg f_k)$$

– so that – each embedding of n_i little disks in $\mathcal{D}isk_m(n_i)$, is scaled down into the i'th embedding from an element of $\mathcal{D}isk_m(k)$. Note that composition of little m-disk embeddings again yield a little m-disk embedding; as composition of embeddings is associative, it easily follows that $\mathcal{D}isk_m$ is an operad, with the action of Σ_k on $\mathcal{D}isk_m(k)$ given by permuting the index of the domain of $f\colon \coprod_{i=1}^k D_i^m \to D^m$. The unit object is given by $\mathbb{1}_{D^m} \in \mathcal{D}isk(1)$

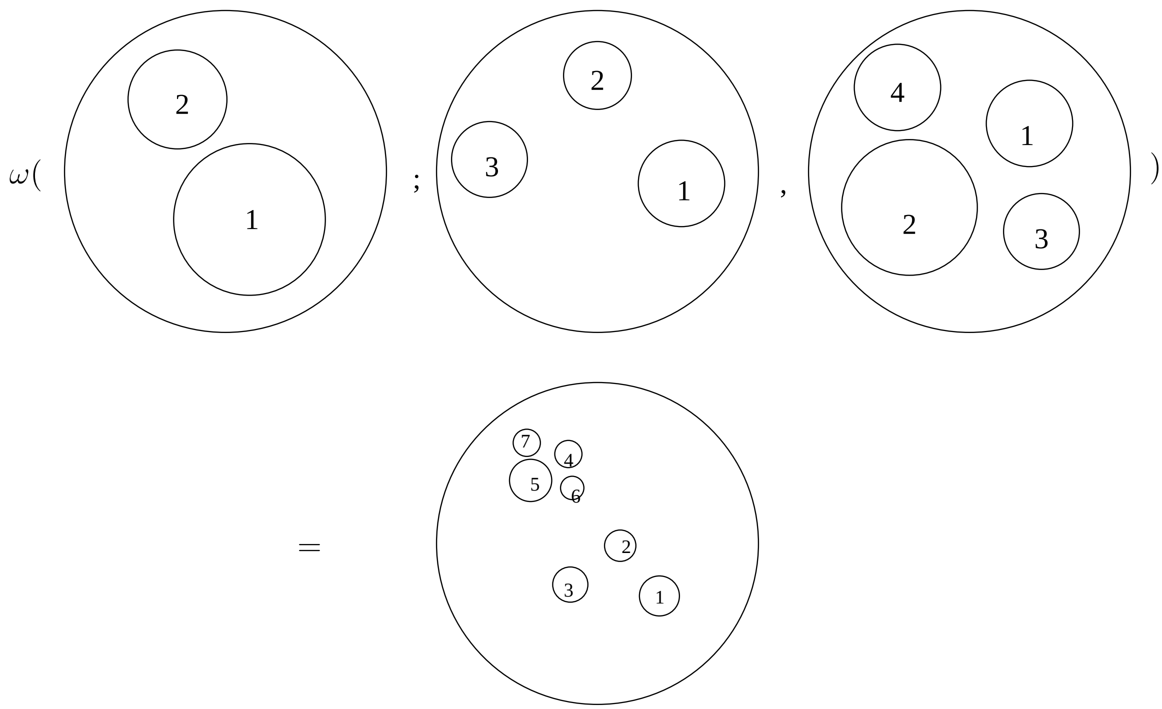

Figure 7: Example of an operadic composition $\omega\colon \mathcal{D}isk_2(2) \times \mathcal{D}isk_2(3) \times \mathcal{D}isk_2(4) \to \mathcal{D}isk_2(7)$. Note that we are simply inserting the little 2-disk embeddings forming the of the two rightmost objects into the leftmost object.

Definition 2.47 Given a topological space X, by the *ordered configuration space* over X with k points, we shall understand the space

$$\mathrm{Conf}_k(X) := \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j\}$$

Remark 2.48 Configuration spaces are classical objects of study. It is known – see for instance the graduate-project [Bar07, 2.4] – that $\mathrm{Conf}_k(\mathbb{R}^2) \cong \mathrm{Conf}_k(D^2)$ is an Eilenberg-Maclane space, such that $\pi_1(\mathrm{Conf}_k(\mathbb{R}^2)) \cong \mathrm{PB}_k$, where PB_k denotes the pure braid group.

Proving this is done by introducing a forgetful map $\rho\colon \mathrm{Conf}_k(X) \to \mathrm{Conf}_k(X)$, and then showing that for X a manifold, ρ is a fiber-bundle with fiber homeomorphic to $X\backslash\{x_1, \dots, x_{k-1}\}$, where $x_1, \dots, x_{k-1}$ are $k-1$ disjoint points in X.

For $X = \mathbb{R}^n$ this specializes to ρ having a fiber homotopy equivalent to $\bigvee_{k-1} S^{n-1}$.

Proposition 2.49 $\mathcal{D}isk_n(k)$ and $\mathrm{Conf}_k(\mathbb{R}^n)$ are homotopy equivalent.

Proof. As $\mathbb{R}^n \cong D^n$, we only need to show a homotopy equivalence of $\mathcal{D}isk_n(k)$ with $\mathrm{Conf}_k(D^n)$.

Denote by $c(f(D_i^n))$ the centre of the image of $f|_{D_i^n}$. We define a map $G\colon \mathcal{D}isk_n(k) \to \mathrm{Conf}_k(D^n)$ by letting $G(f) = (f|_{c(D_1^n)}, \dots, f|_{c(D_k^n)})$, that is, we restrict from each little disk embedding into the centre point.

In order to define a map $\hat{G}\colon \mathrm{Conf}_k(D^n) \to f\mathcal{D}isk_n(k)$, we map each of the points in the tuple $(x_1, \dots, x_k) \in \mathrm{Conf}_k(D^n)$ into the embedding given by letting $f|_{D_i^n}$ be the little disk embedding centered at x_i, and of a suitable radius, for instance $\frac{\min_{i,j}\{d(x_i,x_j)\}}{3}$ – such that no two of the k little disk embeddings in $\hat{G}(x_1, x_2, \dots, x_k)$ overlap.

$\hat{G}$ is a homotopy inverse to G, as $G \circ \hat{G}$ is the identity, and as $\hat{G} \circ G$ is homotopic to the identity, since the image of $\hat{G} \circ G$ is given by rescaling the radius of each little disk embedding $f|_{D_i^n}$. □

Definition 2.50 Similarly to 2.45, we call an embedding $f\colon D^m \to D^m$ a *framed little m-disk embedding*, if there is some left-inverse σ_f to f, given by $\sigma_f(x) = \rho.(r(x+v))$, where $\rho \in \mathrm{SO}(m)$, $v \in \mathbb{R}^m$ and $r \in [1, \infty[$. Here $\rho.-\colon \mathbb{R}^m \to \mathbb{R}^m$ denotes the action of $\mathrm{SO}(m)$, that rotates points in euclidean space.

That is – σ_f is given as a translation, a rescaling and a rotation. We define $f\mathcal{D}isk_m(k)$ to be the space

$$\left\{ f\colon \coprod_{i=1}^{k} D_i^m \to D^m \mid f \text{ is an embedding and} f|_{D_i^m} \text{ is a framed little } m\text{-disk embedding} \right\}$$

The structure of $f\mathcal{D}isk_m$ is similar to what we saw in 2.45.

Observation 2.51 There is a morphism of operads

$$\mathcal{D}isk_m \to f\mathcal{D}isk_m$$

given at the level of spaces by the embedding $\mathcal{D}isk_m(n) \to f\mathcal{D}isk_m(n)$. By 2.19, we automatically have that an algebra over $f\mathcal{D}isk_m$ is also an algebra over $\mathcal{D}isk_m$.

Remark 2.52 We have a homeomorphism $F\colon \mathcal{D}isk_m(k) \times (SO(m))^k \to f\mathcal{D}isk_m(k)$, where $F(f, \rho_1, \dots, \rho_k)$, is the given by $\tilde{f}$, determined by letting each of the framed little m-disk embeddings be given by $\tilde{f}|_{D_i^m} = \sigma^{-1}_{f|_{D_i^m}} \circ (\rho_i.(\sigma_{f|_{D_i^m}} \circ f|_{D_i^m}))$.

As in 2.48, we define a map $p_d\colon f\mathcal{D}isk_m(k) \to f\mathcal{D}isk_m(k-1)$, given by ignoring $f|_{D_i^m}$ from $f \in f\mathcal{D}isk_m(k)$. By the diagram

$$\begin{tikzcd}
\mathcal{D}isk_m(k) \times (\mathrm{SO}(m))^k \arrow[r, "p_d"] \arrow[d, "G \times \mathbb{1}"] & \mathcal{D}isk_m(k-1) \times (\mathrm{SO}(m))^{k-1} \arrow[d, "G \times \mathbb{1}"] \\
\mathrm{Conf}_k(D^m) \times (\mathrm{SO}(m))^k \arrow[r, "\rho \times \mathrm{pr}"] & \mathrm{Conf}_k(D^m) \times (\mathrm{SO}(m))^{k-1}
\end{tikzcd}$$

Where pr is the projection onto the first $k-1$ factors of $(\mathrm{SO}(m))^k$.

We are thus able to identify the homotopy fiber of p_d as

$$p_d^{-1}(\star) = \bigvee_{k-1} S^1 \times \mathrm{SO}(m).$$

In fact – for later use – the proof of [Bar07, 2.2] easily translates into this setting, and shows that p_d is in fact a fiber-bundle, so that the homotopy fiber indeed is an actual fiber.

In particular $\mathrm{SO}(2) \simeq S^1$, so for $m = 2$, we have $p_d^{-1}(\star) = (\bigvee_{k-1} S^1) \times S^1$. Therefore $f\mathcal{D}isk_2(k)$ is an Eilenberg-Maclane space with only non-trivial fundamental group.

Definition 2.53 We call $\mathrm{PRB}_k := \pi_1(f\mathcal{D}isk_2(k))$ the *pure ribbon braid group on k braids.*

Construction 2.54 Given a based space $(X, \star)$, the m-fold loop space is given as the mapping space

$$\Omega^m X := \{(D^m, \partial D^m) \to (X, \star)\}$$

or equivalently, the space of based maps $(S^m, \star) \to (X, \star)$. We construct an action α of $f\mathcal{D}isk_m$ on $\Omega^m X$ by giving a sequence of maps $\alpha_n\colon f\mathcal{D}isk_m(n) \times (\Omega^m X)^n \to \Omega^m X$ as follows.

Let $f \in f\mathcal{D}isk_m(n)$, and $g_1, \dots, g_n \in \Omega^m X$. We define

$$\alpha_n(f, g_1, \dots, g_n)(x) = \begin{cases} \star & x \notin \mathrm{Im}(f) \\ g_i\left(\sigma_f(x)\right) & x \in \mathrm{Im}\left(f|_{D_i^m}\right) \end{cases}$$

Note that this is truly a continuous map $(D^m, \partial D^m) \to (X, \star)$, as $g_i(\partial D^m) = \star$. And it easily follows by the fact that each $f|_{D_i^m}$ is scaled down via σ_f in operadic composition of $f\mathcal{D}isk_m$, that the diagrams of 2.20 indeed commutes.

In effect, we have an action of $f\mathcal{D}isk_m(n)$ as well as an action of $\mathcal{D}isk_m(n)$ on $\Omega^m X$.

In [CLM76] it shown that, using intricate computations of the cohomology ring of euclidean configuration spaces, that

Theorem 2.55 $\mathrm{H}_*(\mathcal{D}isk_2)$ is isomorphic to the operad $\mathcal{G}er$.

We refer however to the more accessible exposition [Sin06, Th. 6.3] as well.

In fact, 2.55 holds for $\mathcal{D}isk_m$ for all m, here $\mathrm{H}_*(\mathcal{D}isk_m)$ is isomorphic to an operad similar to $\mathcal{G}er$, the only difference being a change of the degree from 1 to $m-1$ of the generator $e_{B_{[,]}}$ in 2.37.

Furthermore, in [Get94], it is shown that

Theorem 2.56 Let $\mathrm{H}_*(-)$ denote homology with coefficients in a field.
$\mathrm{H}_*(f\mathcal{D}isk_2)$ is isomorphic to the operad $\mathcal{BV}$.

This is also shown in [SW03], by considering $f\mathcal{D}isk_m$ as – what is called – a semi-direct product of $\mathcal{D}isk_m$ with $\mathrm{SO}(m)$. Using this concept, they are able to – via the computations of 2.55 and computations of $\mathrm{H}_*(\mathrm{SO}(m))$ – provide higher dimensional computations for $\mathrm{H}_*(f\mathcal{D}isk_m)$ as well.

Remark 2.57 Note that in order to have an action of $\mathrm{H}_*(f\mathcal{D}isk_2)$ – with coefficient in a principal ideal domain – on a module M, induce the structure of a Batalin-Vilkovisky algebra, it is by 2.19, 2.14 and 2.42 enough to have a morphism of operads

$$\Phi\colon \mathcal{BV} \to \mathrm{H}_*(f\mathcal{D}isk_2).$$

Indeed, one can construct such a morphism, and we shall briefly say how to obtain Φ.

First of all by 2.43, we can consider $\mathcal{BV}$ as an operad generated by e_{B_*} of $\mathcal{BV}(2)$ in degree 0, and e_{B_Δ} of $\mathcal{BV}(1)$ in degree 1.

We therefore define

- $\Phi(e) = 1 \in \mathrm{H}_0(f\mathcal{D}isk_2(1))$
- $\Phi(e_{B_*}) = 1 \in \mathrm{H}_0(f\mathcal{D}isk_2(2))$
- $\Phi(e_{B_\Delta}) = 1 \in \mathrm{H}_1(f\mathcal{D}isk_2(1)) \cong \mathrm{H}_1(S^1)$

and extend by linearity.

$\mathcal{BV}$ is generated by the unit e together with the elements e_{B_Δ} and e_{B_*}, we can inductively – assuming $\Phi(k), \Phi(n_1), \ldots, \Phi(n_k)$ is defined, define the map

$$\Phi_{n_1+\cdots+n_k}\colon \mathcal{BV}(n_1+\cdots+n_k) \to h_*(f\mathcal{D}isk_2(n_1+\cdots+n_k)$$

to be the map that makes the diagram

$$\begin{tikzcd}
\mathcal{BV}(k) \otimes \mathcal{BV}(n_1) \otimes \cdots \otimes \mathcal{BV}(n_k) \arrow[r, "\omega"] \arrow[d, "\Phi_k \otimes \Phi_{n_1} \otimes \cdots \otimes \Phi_{n_k}"'] & \mathcal{BV}(n_1+\cdots+n_k) \arrow[d, dashed, "\Phi_{n_1+\cdots+n_k}"] \\
\mathrm{H}_* f\mathcal{D}isk_2(k) \otimes \mathrm{H}_* f\mathcal{D}isk_2(n_1) \otimes \cdots \otimes \mathrm{H}_* f\mathcal{D}isk_2(n_k) \arrow[r, "\omega"] & \mathrm{H}_* f\mathcal{D}isk_2(n_1+\cdots+n_k)
\end{tikzcd}$$

commute, and furthermore let $\Phi_{n_1+\dots+n_k}(\sigma.-) = \sigma.\Phi_{n_1+\dots+n_k}(-)$ where $\sigma \in \Sigma_{n_1+\dots+n_k}$.

Indeed, assuming that $\Phi(m)$ is well-defined for all m, this would define a morphism of operads.

However, we have three relations in $\mathcal{BV}$ that Φ needs to satisfy, in order for it to be well-defined.

First of all, since $\star$ is a graded commutative product, we need that $\Phi(e_{B_\star}) = \Phi(e_{c_2.B_\star}) = 1 \in h_0(f\mathcal{D}isk_2(2))$, but this obviously holds as $c_2.f \in f\mathcal{D}isk_2(2)$ is in the same component as $f \in f\mathcal{D}isk_2(2)$.Furthermore, we need $\Phi(e_{\omega(B_\Delta;B_\Delta)}) = 0$, but as Φ is a graded map, and $e_{\omega(B_\Delta;B_\Delta)}$ is of degree 2, we have that $\Phi(e_{\omega(B_\Delta;B_\Delta)}) \in \mathrm{H}_2(f\mathcal{D}isk_2(1)) = 0$.

Finally, for the relation corresponding to the BV-master-equation relating elements of $\mathcal{BV}(3)$ in degree 1, we refer to [SW03, 5.7], where it is noted that since $\mathrm{H}_1(f\mathcal{D}isk_2(3))$ is the abelianization of the mapping class group on a disk with three labelled holes: $\Gamma_{0,4} \cong \mathrm{PRB}_3 \cong \pi_1(f\mathcal{D}isk_2(3))$. The so-called lantern relation – a relation among certain Dehn-twists in the mapping class group $\Gamma_{0,4}$– gives a relation in $\mathrm{H}_1(f\mathcal{D}isk_2(3))$, that precisely says that Φ respects the relation of $\mathcal{BV}(3)$.

Combining 2.54 and 2.56, we note the first example of a BV-algebra; namely

Corollary 2.58 $\mathrm{H}_*(\Omega^2 X)$ is BV-algebra.

Of course, by the above, similar corollaries holds for higher iterations of based loop spaces, where we use the higher-dimensional analogue of a BV-algebra instead.

2.5 Alternative Definition of Operads

To keep the notational burden down, we introduce an equivalent notation for operads, similar to what we saw for the $\mathcal{T}ree$ operad in 2.29.

We – primarily for notational convenience – assume that operads $\mathcal{O}$ have $\mathcal{O}(n)$ given as sets (with some extra structure), as is the case for all operads we consider.

Proposition 2.59 Giving the structure of an operad on $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$, with an action of Σ_n on $\mathcal{O}(n)$, as in 2.9, is equivalent to giving a series of *composition maps* for every $k, n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$:

$$\circ_i \colon \mathcal{O}(n) \boxtimes \mathcal{O}(k) \to \mathcal{O}(n+k-1)$$

that satisfy the following:

(a) Given $a \in \mathcal{O}(n), b \in \mathcal{O}(k)$ and $c \in \mathcal{O}(h)$, and a fixed $i \in \{1, \dots, n\}$, we have *associativity* of the composition maps

$$(a \circ_i b) \circ_j c = \begin{cases} (a \circ_j c) \circ_{i+h-1} b & 1 \leqslant j \leqslant i \\ a \circ_i (b \circ_{j-(i-1)} c) & i \leqslant j \leqslant i+k-1 \\ (a \circ_{j-k-1} c) \circ_i b & i+k-1 \leqslant j \leqslant n+k-1 \end{cases}$$

(b) There is an identity element $e \in \mathcal{O}(1)$ satisfying for $g \in \mathcal{O}(n)$,

$$e \circ_1 g = g = g \circ_i e$$

for all $i \in \{1, \dots, n\}$

(c) For each $i \in \{1, \dots, n\}$, define for $\tau \in \Sigma_n$ and $\sigma \in \Sigma_k$, $\tau \circ_i \sigma \in \Sigma_{n+k-1}$ to be the block permutation $\tau.(e_1 \oplus \cdots e_{i-1} \oplus \sigma \oplus e_{i+1} \oplus \cdots \oplus e_n)$, where e_t is the only element of Σ_1. We want Σ_n-invariance, that is, we want for $f \in \mathcal{O}(n)$ and $g \in \mathcal{O}(k)$ the following identity:

$$(f.\tau \circ_i g.\sigma) = (f \circ_{\tau(i)} g).(\tau \circ_i \sigma)$$

There isn't really anything deep in the above proposition; it is simply a reformulation. From maps $\omega \colon \mathcal{O}(k) \boxtimes \mathcal{O}(n_1) \boxtimes \cdots \boxtimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$, we define $\circ_i \colon \mathcal{O}(n) \boxtimes \mathcal{O}(k) \to \mathcal{O}(n + k - 1)$ to be the composite

$$\mathcal{O}(n) \boxtimes \mathcal{O}(k) \to \mathcal{O}(n) \boxtimes \underbrace{\mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)}_{i-1} \boxtimes \mathcal{O}(k) \boxtimes \underbrace{\mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)}_{n-1} \to \mathcal{O}(n + k - 1)$$

given by first mapping $f \boxtimes g \mapsto f \boxtimes e \boxtimes \cdots \boxtimes e \boxtimes g \boxtimes e \boxtimes \cdots \boxtimes e$ and then applying ω, to get

$$f \circ_i g = \omega(f; e, \dots, e, g, e, \dots, e)$$

where $f \in \mathcal{O}(n)$, $g \in \mathcal{O}(k)$ and e is the unit in $\mathcal{O}(1)$.

From maps $\circ_i \colon \mathcal{O}(n) \boxtimes \mathcal{O}(k) \to \mathcal{O}(n+k-1)$, we obtain $\omega \colon \mathcal{O}(k) \boxtimes \mathcal{O}(n_1) \boxtimes \cdots \boxtimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$ by

$$\omega(f; g_1, \dots, g_k) = (\cdots((f \circ_k g_k) \circ_{k-1} g_{k-1}) \cdots) \circ_1 g_1$$

As we did for trees in 2.29.

The only thing that may not seem obviously equivalent in the list of requirements in the two different definition of operads is the associativity conditions. However, we refer to [Mar06, p.9-10], where it is noted that the three different possibilities in associativity of 2.59 is simply listing which 'slot' the $\circ_j$ composition falls into: before, into or after the 'slot' composed in by $\circ_i$.

2.6 $\mathcal{W}$-Resolutions of Topological Operads

We introduce $\mathcal{W}$-resolutions of topological operads, originally introduced in [Boa71]. For our purpose – as we shall see – the $\mathcal{W}$-resolution is an weakly Σ-equivariant equivalent replacement for the operad, that is easier to map out of, than the operad itself.

Construction 2.60 In order to do this, denote as usual by $\mathcal{T}ree(k)$ the set of k-trees.

To $T \in \mathcal{T}ree(k)$, let $V(T)$ denote the set of knots (i.e. vertices with valence greater than 1) of T, and let $E(T)$ denote the set of edges of T.

To $x \in V(T)$, we will by $|x|$ denote the valence of x. Note that there are $|x| - 1$ ingoing edges of x, and denote by $e(x)_i$ the i'th of these ingoing edges. Unless x is at level one of t, denote by $u(x)$ the outgoing edge. If x is at level one, and therefore the outgoing edge is the edge to the root, we let $u(x) = \varnothing$.

Assume that we are given a topological operad $\mathcal{O}$, and let

$$\mathcal{Tree}(\mathcal{O})(k) := \coprod_{T \in \mathcal{Tree}(k)} \left(\prod_{x \in V(T)} \left([0,1]^{|u(x)|} \times \mathcal{O}(|x|-1)\right) \right).$$

That is, $\mathcal{Tree}(\mathcal{O})(k)$ is a space indexed by all k-trees. We say that the knots of a k-tree has been labelled by elements of $\mathcal{O}(|x|-1)$ and that for each knot x of a k-tree, we have labelled the outgoing edge $u(x)$ by some real number $s_x \in [0,1]$.

Elements of $\mathcal{Tree}(\mathcal{O})(k)$ will be called *labelled (k-)trees.*

Since $\mathcal{Tree}$ is in an operad, $\mathcal{Tree}(\mathcal{O})(k)$ is an operad as well; we can define $\mathcal{Tree}(\mathcal{O})(k) \circ_i \mathcal{Tree}(\mathcal{O}(n))$ to be given by the element in $\mathcal{Tree}(\mathcal{O})(k+n-1)$, in the disjoint union indexed by $T_k \circ_i T_n$, where T_k and T_n are indexing the disjoint union of $\mathcal{Tree}(\mathcal{O})(k)$ and $\mathcal{Tree}(\mathcal{O})(n)$ respectively. The labellings of the composed trees are retained

All knots of $T_k \circ_i T_n$ occur from either T_k or T_n, and retain their valence. We let the labelling of these knots be retained. Note however that the vertex v at level 1 of T_n has $u(v) = \varnothing$ in T_n, but not in $T_k \circ_i T_n$. We assign $1 \in [0,1]^{\{u(v)\}}$ as label to v.

The identity $e \in \mathcal{Tree}(\mathcal{O})(1)$ is given by the one-point-set indexed by the 1-tree without any knots, and Σ_k acts by permuting the leaves of the index.

Construction 2.61 In 2.60, the role of $\mathcal{O}$ has been that of a spectator in $\mathcal{Tree}(\mathcal{O})$. To make it truly part of the play, we give the $\mathcal{W}$-resolution $\mathcal{W}(\mathcal{O})$ of $\mathcal{O}$ by defining $\mathcal{W}(\mathcal{O})(k)$ as a quotient operad of $\mathcal{Tree}(\mathcal{O})(k)$, under the following relations (i.e. we are forming an operadic ideal via the following relations):

(a) In a labelled n-tree T, assume that a vertex x is labelled by $o \in \mathcal{O}(k)$; x thus have the incoming edges $e(x)_1, \ldots e(x)_k$.

   To each $e(x)_i$, there is a branch $T|e(x)_i$ having a root at the position of x in T. Given $\sigma \in \Sigma_k$, we consider the tree $T^x.\sigma$, given by the tree where each branch $T|e(x)_i$ has been replaced by $T|e(x)_{\sigma(i)}$.

   We impose the relation that the elements indexed by T, and labelled by $o \in \mathcal{O}(k)$ at x is equivalent to the corresponding elements indexed by $T^x.\sigma$, and labelled by $o.\sigma^{-1}$ at x but otherwise equal.

(b) Let an internal edge $u(x)$ of a k-tree T connect the vertices x and y. Assume that $u(x)$ is labeled by $s_x = 0$. Assume that $u(x) = e_i(y)$, let x be labelled by $o_x \in \mathcal{O}(m)$ and y by $o_y \in \mathcal{O}(n)$.

   Consider the k-tree $T/u(x)$, where we perform an *edge-collapse* of $u(x)$, in the sense that we collapse the entire edge $u(x)$ to the vertices x and y (so in $T/u(x)$ $x = y$).

   Under these conditions, we impose the relation that elements indexed by T should be equivalent to the corresponding elements labelled by $T/u(x)$. In the sense that labellings of edges different from x and y for T should be retained for $T/u(x)$. All edges of $T/u(x)$ arise naturally from edges of T, and their labellings should be retained as well.

The only affected vertex in $T/u(x)$ is the one resulting from $x = y$, the element corresponding to T, but indexed by $T/u(x)$ is labelled by the operadic composition of the labels: $o_y \circ_i o_x$.

(c) Assume that we are given a tree T labelled at x by the identity $e \in \mathcal{O}(1)$, since x has valence 2, there is one ingoing edge $e(x)_1$, and one outgoing - $u(x)$. As long as $u(x)$ is not connecting to the root (i.e. T is the trivially labelled tree), we equivalate elements indexed by T with the elements indexed by the tree with $u(x)$ collapsed: $T/u(x)$. All labellings of unaffected vertices and edges remain the same. Assuming $u(x)$ connects to the vertex y, and $e(x)_1$ connects to z, the label of the edge from y to z in $T/u(x)$, attains the value $\max\{s_x, s_z\}$ where s_z is the label of $u(z)$ (if z is a leaf, we leave $s_z = 0$ in this computation) and s_x the label of $u(x)$. The labellings of the vertices y and z in $T/u(x)$ are the same as the labellings in T.

The relations in $\mathcal{W}(\mathcal{O})$ are invariant under grafting of trees, so it follows that we inherit an operadic structure from $\mathcal{T}ree(\mathcal{O})$:

Proposition 2.62 For any topological operad $\mathcal{O}$, $\mathcal{W}(\mathcal{O})$ is an operad with composition maps $\circ_i$, defined as grafting of trees in 2.61

Construction 2.63 We construct a morphism of operads $\varepsilon\colon \mathcal{W}(\mathcal{O}) \to \mathcal{O}$.

We specify $\varepsilon_k\colon \mathcal{W}(\mathcal{O})(k) \to \mathcal{O}(k)$, as the map that sets all labels of edges to zero.

That is, assume we are given a labelled k-tree T with n internal knots indexing an equivalence class $[o_1, (s_2, o_2), \ldots, (s_n, o_n)]$ of elements in $\prod_{i=1}^{n}[0,1]^{\{(u(x_i))\}} \times \mathcal{O}(|x_i| - 1)$, where we have chosen some ordering of the vertices in T, with x_1 being the vertex at level 1. We set

$$\varepsilon([o_1, (s_2, o_2), \ldots, (s_n, o_n)]) = ([o_1, (0, o_2), \ldots, (0, o_n)]).$$

Note by (b) in 2.61, that the image has a representative indexed by a k-corolla with its single knot labelled by the operadic composition specified via T. Operadic associativity of $\mathcal{O}$ ensures that this representative is unique. As obviously the subspace of $\mathcal{T}ree(\mathcal{O})(k)$ indexed by all k-corollas is isomorphic to $\mathcal{O}(k)$, we can indeed consider ε as mapping into $\mathcal{O}$.

It however still needs to by checked that ε is well-defined. Pick a tree T_a related to T by a relation as specified in 2.61(a). By Σ_k-equivariance of $\mathcal{O}$, it follows that the image of T and T_a under ε yield the same. We have already guaranteed by specifying the image of ε_k as k-corollas, that operadic associativity of $\mathcal{O}$ gives that ε is well-defined under relations given by 2.61(b). Finally the unit identities of $\mathcal{O}$ ensures that appending – or removing – bi-valent vertices indexed by e to T, as specified by 2.61(c), does not change the image of ε_k.

Having dealt with well-definedness, we turn to checking that ε is a morphism of operads. By 2.61(c), the identity tree with no internal knots is equivalent to the tree with one knot labelled by $e \in \mathcal{O}(1)$, checking that $\varepsilon(e_{\mathcal{W}(\mathcal{O})}) = e_{\mathcal{O}(1)}$.

In $\mathcal{Tree}(\mathcal{O})$, operadic composition is given by grafting of trees, so ε is constructed to commute with operadic compositions.

And 2.61(a) ensures that permutations via $\sigma \in \Sigma_k$ of the leaves in the k-corolla specified as the image of ε_k is the same as letting the knot be labelled by $o.\sigma^{-1}$, so as $(o.\sigma^{-1}).\sigma = o$, we have that acting by σ on o is the same as permuting the leaves of the k-corolla. As the action of Σ_k on $\mathcal{W}(\mathcal{O})(k)$ permutes the leafs of the indexing tree

Proposition 2.64 The morphism $\varepsilon\colon \mathcal{W}(\mathcal{O}) \to \mathcal{O}$ specified in 2.63 is a weak equivalence of operads.

Proof. For any n, we need to check that $\varepsilon(n)\colon \mathcal{W}(\mathcal{O})(n) \to \mathcal{O}(n)$ is a homotopy equivalence. Note that we have a map of spaces $\delta\colon \mathcal{O}(n) \to \mathcal{W}(\mathcal{O})(n)$ defined by to $o \in \mathcal{O}(n)$ assigning the k-corolla, labelled at the single knot by o. The composition

$$\mathcal{O}(n) \xrightarrow{\delta} \mathcal{W}(\mathcal{O})(n) \xrightarrow{\varepsilon} \mathcal{O}(n)$$

is $\mathbb{1}_{\mathcal{O}(n)}$ on the nose. The composition

$$\mathcal{W}(\mathcal{O})(n) \xrightarrow{\varepsilon} \mathcal{O}(n) \xrightarrow{\delta} \mathcal{W}(\mathcal{O})(n)$$

Takes a labelled k-tree T onto the labelled k-corolla, that can be obtained by setting all edge-lengths of t to zero.

We now notice that we can make a homotopy $\Phi_s\colon \mathcal{W}(\mathcal{O})(n) \to \mathcal{W}(\mathcal{O})(n)$, by letting Φ_s be given by multiplying the labels of the edges of T by the factor s. Or in the spirit of 2.63 as

$$\Phi_s([o_1, (s_2, o_2), \dots, (s_n, o_n)]) = ([o_1, (s \cdot s_2, o_2), \dots, (s \cdot s_n, o_n)])$$

Clearly, $\Phi_1 = \mathbb{1}_{\mathcal{W}(\mathcal{O})(n)}$ and $\Phi_0 = \delta\varepsilon$, checking that ε is a weak equivalence.

□

Construction 2.65 For later use, we shall specify a method for constructing a morphism of operads

$$B\colon \mathcal{W}(\mathcal{O}) \to \mathcal{P}.$$

We shall assume we are given maps $\Phi\colon \mathcal{Tree}(\mathcal{O})(k) \to \mathcal{P}(k)$. We want to obtain B as a morphism of operads, where B will be obtained as part of a factorization

$$\mathcal{Tree}(\mathcal{O}) \longrightarrow \mathcal{W}(\mathcal{O}) \xrightarrow{B} \mathcal{P}\ .$$

Assume therefore that we are given a labelled tree T, indexing an element of Φ.

We index the knot of T according to the following system:

- The knot nearest the root is called x_1.
- Having indexed one vertex x_i, we choose a vertex one level above x_i, and directly connected to it, and call it x_{i+1}
- If the above don't apply (i.e. if there are no internal unnamed vertices above x_i), we apply the process to the vertex connected to x_i one level below.

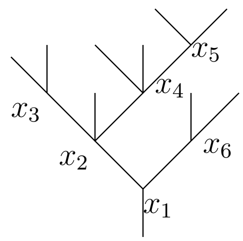

Figure 8: Example of the choice of indexing for T, indexing elements of $\mathcal{Tree}\ \mathcal{O}(k)$.

Note that at each vertex, a branch starting at x_i, with k internal edges, will be labelled by elements of $\{i, i+1, \ldots, i+k-1\}$.

We have assumed, for each labelled tree T with k internal edges, that we are given maps

$$\Phi\colon \prod_{1}^{k} \left([0,1]^{|\{u(x_i)\}|}\,\mathcal{O}(|x_i|-1)\right) \to \mathcal{P}(|x_1|+\cdots+|x_k|-2k+1)$$

Consider the internal edge x_i, and the branch $T|e(x_i)_s$ with m internal edges, where we assume that x_j is the first internal edge arising in $T|e(x_i)_s$. Φ should satisfy the following relations:

(1)

$$\begin{aligned} \Phi(f_1, t_2, f_2, \ldots, 1, f_i, \ldots, t_k, f_k) &= \\ \Phi(f_1, t_2, f_2, \ldots, t_{i-1}, f_{i-1}, t_{i+m+1}, f_{i+m+1}, \ldots, t_k, f_k) &\circ_s \Phi(f_i, t_{i+1}, f_{i+1}, \ldots, t_{i+m}, f_{i+m}) \end{aligned}$$

(2)

$$\begin{aligned} \Phi(f_1, t_2, f_2, \ldots, t_j, f_j, \ldots, 0, f_i, \ldots, t_k, f_k) &= \\ \Phi(f_1, t_2, f_2, \ldots, t_j, f_j \circ_s f_i, \ldots, t_{i-1}, f_{i-1}, t_{i+1}, f_{i+1}, \ldots, t_k, f_k) & \end{aligned}$$

(3) Assuming that x_j is a bi-valent vertex between x_i and x_{j+1}:

$$\begin{aligned} \Phi(f_1, t_2, f_2, \ldots, t_j, e, t_{j+1}, f_{j+1} \ldots, t_k, f_k) &= \\ \Phi(f_1, t_2, f_2, \ldots, \max\{t_j, t_{j+1}\}, f_{j+1}, \ldots, t_k, f_k) & \end{aligned}$$

(4) Assume that a permutation $\sigma_l \in \Sigma_k$ act on $f_l \in \mathcal{O}(k)$, then Φ should satisfy

$$\Phi(f_1, t_2, f_2, \ldots, t_l, f_l.\sigma, \underbrace{\ldots}_{\text{1'st branch}}, \ldots, \underbrace{\ldots}_{l\text{'th branch}}, \ldots, t_k, f_k) \quad =$$
$$\Phi(f_1, t_2, f_2, \ldots, t_l, f_l, \underbrace{\ldots}_{\sigma(1)\text{'st branch}}, \ldots, \underbrace{\ldots}_{\sigma(l)\text{'th branch}}, \ldots, t_k, f_k)$$

Note that the relations (2)-(4) makes B well-defined on $\mathcal{W}(\mathcal{O})$ – that is, via the universal property of quotient spaces, we obtain B as the uniqie factorization of Φ. The relation (1) ensures, as composition is given by grafting of trees and setting the new edge length 1 simply says that the diagram

$$\begin{tikzcd}
\mathcal{W}(\mathcal{O})(k) \times \mathcal{W}(\mathcal{O})(n) \arrow[r, "\circ_s"] \arrow[d, "B \times B"] & \mathcal{W}(\mathcal{O})(k+n-1) \arrow[d, "B"] \\
\mathcal{P}(k) \times \mathcal{P}(n) \arrow[r, "\circ_s"] & \mathcal{P}(k+n-1)
\end{tikzcd}$$

is commutative.

If furthermore Φ is Σ-equivariant Then by 2.11 B will actually be a morphism of operads.

3 From Framed Little Disks to Cacti

3.1 The Cacti Operad

In the following, we shall introduce the cacti operad, $\mathcal{C}acti_1$. This operad was originally announced in [Vor05, 2.7]. We give a slightly altered definition. Our motivation is to have better control over multiple intersection points of cacti.

Definition 3.1 As a ground space, we consider k disjoint oriented copies of $S^1 := \mathbb{R}/\mathbb{Z}$:

$$S(k) := \coprod_{i=1}^{k} S_i^1$$

where S_i^1 – an oriented copy of S^1 – is called the *i'th lobe*.

Consider the set

$$C'(k) := \left\{ \prod_{i=1}^{k-1} (S(k) \times_{\Sigma_2} S(k)) \right\}$$

That is, a point of $C'(k)$ consists of $(k-1)$ unordered pairs $[t_i, b_i]$ – called an *intersection tuple* of $S(k)$. The $2(k-1)$ point $t_1, \ldots, t_{k-1}, b_1, \ldots, b_{k-1}$ are called *intersection points*

Definition 3.2 From a point $c \in C'(k)$, we construct the *dual graph*, G_c:

G_c is given by having a vertex v_i for each S_i^1 of $S(k)$, and for each intersection tuple of c, with intersection points on S_i^1 and S_j^1, we join an edge between v_i and v_j.

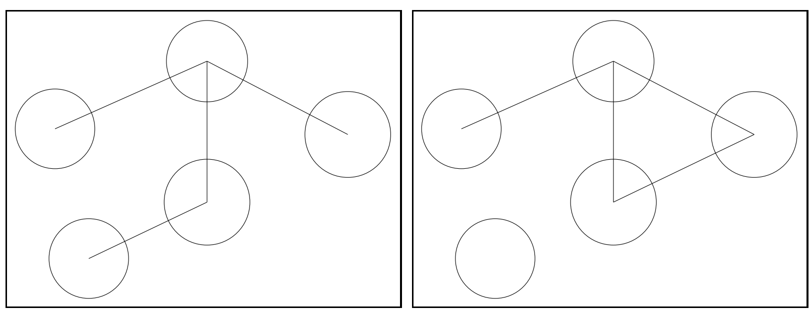

Figure 9: Two dual graphs. The leftmost is a tree (i.e. connected and no cycles)

Definition 3.3 Given a point in $c \in C'(k)$, we call a collection

$$L^n = \{[t_{i_1}, b_{i_1}], \ldots [t_{i_n}, b_{i_n}]\} \subseteq c$$

an (n)-*intersection cluster* if to $[t_{i_j}, b_{i_j}] \in L^n$, there exists a $[t_{i_l}, b_{i_l}] \in L^n$, with $l \neq j$, such that $\{t_{i_j}, b_{i_j}\} \cap \{t_{i_l}, b_{i_l}\} \neq \emptyset$.

We say that an intersection point p of an intersection cluster L^n is *free*, if there is only one intersection tuple of L^n that contains it.

A *cluster move* in L^n is given by picking $c' \in C'(k)$ with (almost) the same intersection tuples as c, except that we infer an alteration for one of the intersection tuples $[t_{i_j}, b_{i_j}]$ of L^n, where precisely one of the intersection points are free. Say b_{i_j} is the non-free intersection point of the intersection cluster L^n, we choose the intersection tuple $[t_{i_j}, p]$, where p is any other intersection point of L^n, different from t_{i_j}.

Given $c, c' \in C'(k)$, we say that they are *cluster-related*, if there is a sequence of cluster moves, that turns c into c'.

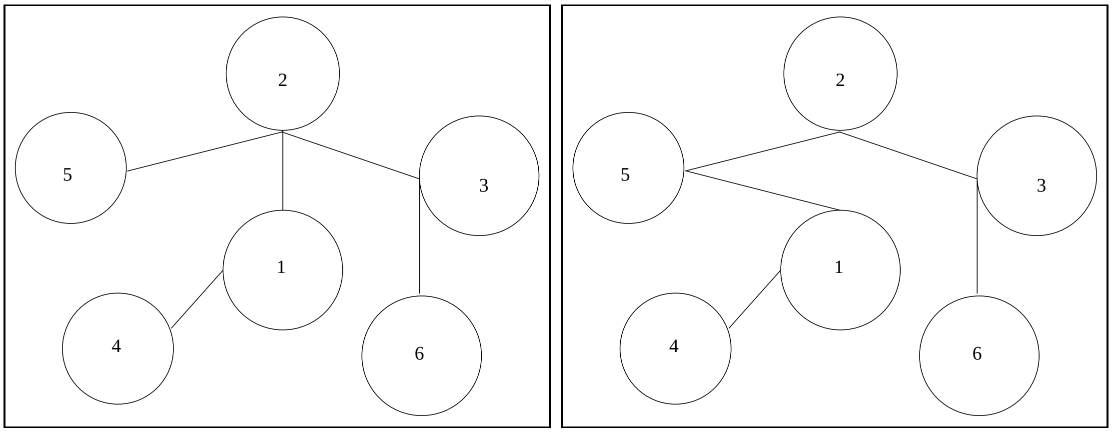

Figure 10: Example cluster-move in a cluster involving 5 intersection tuples. Intersection clusters are drawn as lines connecting the two involved intersection points

Obviously, the condition of being cluster-related defines an equivalence relation on $C'(k)$.

Definition 3.4 Let $c \in C'(k)$. Denote by $[c]$ the equivalence class of being cluster-related. Define

$$C(k) := \{[c] \mid c \in C'(k) \text{ and } G_c \text{ is a tree.}\}$$

A point of $C(k)$ will be called a *k-pre-cactus*. We shall take the freedom to – in notation – ignore the fact that points of $C(k)$ are equivalence classes, and simply let points of $C(k)$ be denoted by c.

It is standard in the literature to define – as above – cacti by requiring that G_c is a tree. With the definition we have given, we get the following alternative:

Proposition 3.5 c is a k-pre-cactus if and only if G_c is connected

Proof. G_c is by definition a graph with k vertices and $k-1$ edges. We get that the Euler characteristic $\chi(G_c) = k - (k-1) = 1$, so G_c is connected precisely if it has the homotopy type of a point; i.e. it is a tree. □

Remark 3.6 Note that connectivity of dual graphs obviously is invariant under cluster moves, so by 3.5, the definition of 3.4 is sound, in the sense that if for $c \in C'(k)$, G_c is a tree – then performing a cluster move on c giving $c' \in C'(k)$, $G_{c'}$ will be a tree again.

Definition 3.7 Let $c \in C'(k)$ be given, the intersection points b_i and t_i of an intersection tuple are points in $S(k)$. Consider the equivalence relation $b_i \sim_c t_i$ on $S(k)$ for all i, equivalating intersection tuples to a point. This gives rise to a connected graph $S(k)/\sim_c$.

Note that $S(k)/\sim_c$ is independent of the representative of c; all intersection points in an intersection cluster are in $S(k)/\sim_c$ identified to the same point.

In fact, the vertices $v_1, \ldots, v_m$ of $S/\sim_c$ corresponds to the intersection clusters $L^{l_1}, \ldots, L^{l_m}$ of c. The number of resulting edges on $S(k)/\sim_c$ is given by $k + m - 1$

We call the edges $e_1, \ldots, e_k, \ldots, e_{k+m-1}$ of $S(k)/\sim_c$ *internal edges*. Letting S^1_i have length 1, an internal edge inherit a length $\ell_{e_j} \in [0,1]$ given by the arc-length of S^1_i it spans. Likewise, e_j inherit an orientation from S^1_i.

Let $\pi\colon S^1 \to S(k)/\sim_c$ be an orientation preserving, surjective continuous map that is injective almost everywhere except at the vertices of $S(k)/\sim_c$. That is; $\pi(x) \neq \pi(y) \Rightarrow x \neq y$, except when $\pi(x) = v_i$ for some i. We allow $\pi(x) = \pi(y) = v_i$ for only finitely many different $x, y \in S^1$. π is called a *pinching map* of c.

We say that π is *equidistant* if to any arc A of S^1 of arc-length $a \in [0,1]$ we have that $\pi(A)$ is a path in $S(k)/\sim_c$ that in $S(k)$ corresponds to a union of arcs of $S(k)$, with accumulated arc-length $k \cdot a$.

As a set, we define

$$\mathcal{C}acti_1(k) := \{(c,\pi) \mid c \in C(k) \text{ and } \pi\colon S^1 \to S(k)/\sim_c \text{ is an equidistant pinching map}\}$$

A point $(c,\pi) \in \mathcal{C}acti_1(k)$ is called a (k-)*cactus*

Remark 3.8 Let the starting point $\pi(1) \in S(k)/\sim_c$ of an equidistant pinching map be given. Assume that $\pi(1)$ is at the j'th lobe – Following the orientation of S^1_i, completely determines π until it reaches a vertex $v \in S(k)/\sim_c$ – since we have specified that π is a bijection away from the vertices.

At v, there are some choices as to where π can travel further; say v has arisen from an l-intersection cluster L^l, then there are l different lobes coming together at v, and v has valence $2l$.

Note however that π definitely cannot traverse an edge that it has already traversed.

Taking the dual graph G_c into account, the vertex v of $S(k)/\sim_c$ determines l edges $w_1, \ldots, w_l$ of G_c, from which l branches $b_{w_1}, \ldots, b_{w_l}$ of G_c emanate from. As – again – we require that π is bijective away from the intersection points, it is impossible that π traverses further along the j'th lobe, before it has traversed all lobes corresponding to the vertices of b_{w_i} for all i.

As we furthermore require π to be oriented, once a lobe $S^1_{i_j}$ corresponding to w_{i_j} has been chosen, π is completely determined until the next intersection point lying on $S^1_{i_j}$.

Inducing on this argument, π is completely determined by choosing a starting point $\pi(1) \in S(k)/\sim_c$, called the *external starting point*, and for each vertex v in $S(k)/\sim_c$ arising from an l-intersection cluster L^l, choosing a *cyclical ordering* of the l intersection tuples in L^l, corresponding to order π traverses the corresponding lobes.

Besides the external starting point, there are also k *internal starting points* of a k-cactus, c, where the i'th internal starting point is given by the point of $S/\sim_c$ arising from the point $1 \in S^1_i$.

In figure 11, we give an illustration of a point in $\mathcal{C}acti_1(5)$, embedded inside of $\mathbb{R}^2$.

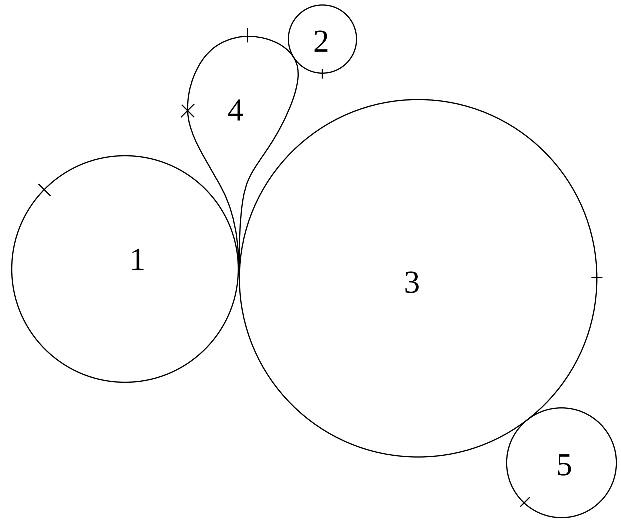

Figure 11: an example of a cactus embedded in $\mathbb{R}^2$. The external starting point $\pi(1)$ is marked by a cross. There are also markings of internal starting points, marked by a short line segment. Inside $\mathbb{R}^2$, the cyclic ordering at each intersection point is given by the natural counterclockwise cyclic ordering inside of $\mathbb{R}^2$ itself. Although we require that each lobe has length 1, it is helpful in visualizing cacti to allow embeddings, where the lobes have varying lengths.

We want to give $\mathcal{C}acti_1$ a topology. For this sake, we shall first consider cacti where the existence of internal and external starting points isn't automatic.

Definition 3.9 An *oriented metric graph* consists of a *vertex set* $V = \{v_0, \dots, v_n\}$, along with a *set of edges* with *lengths*, $E = \{(e_1, l_1), \dots, (e_k, l_k)\}$, where the *edges* are given as $e_i = [0, 1]$ (so they have an orientation), and the *lengths* $l_i \in]0, 1]$. To obtain a graph, we furthermore specify a gluing of the two points ∂e_i to some points of V.

An oriented metric graph may have multiple edges coming together at a point. In order to relate to cacti, we – at least – need to have some cyclical ordering at the vertices. We therefore define the set of *doubly oriented fat metric graphs* to be the set of metric graphs, Γ, such that each vertex has even valence. Plus the additional data that at each vertex $v \in \Gamma$, we group the $2n$ boundaries $b_1, \dots, b_{2n}$, of edges coming together at v in *gluing pairs* (b_i, b_j) – the first point of a gluing pair is called *outgoing* and the second *incoming*. For the sake of orientation, we require that for any edge e_i, the two boundaries $\partial_+ e_i$ and $\partial_- e_i$ is such that $\partial_+ e_i$ is always incoming in a gluing pair, and $\partial_- e_i$ is always outgoing.

That is – a point $x \in \Gamma$ will consist of a point in the metric graph, plus when x is a vertex v of valence $2n$, a number $x_v \in \{1, \dots, n\}$ indicating that x is at the x_v'th gluing pair. These numbers define a cyclical ordering of the gluing pairs, by calculating them mod n.

A neighborhood of a point x in a doubly oriented fat metric graph is thus given by a neighborhood of the edge x resides on, if x is not on a vertex. If x is at a vertex, and is at the gluing pair $(\partial_- e_i, \partial_+ e_j)$ we let to U_i and U_j – neighborhoods of $\partial_- e_i \in e_i$ and $\partial_+ e_l \in e_l$ respectively – $U_i \cup U_j$ be a neighborhood of x.

By the set of *spineless cacti*, $G(k)$ we shall understand the subset of fat metric graphs, such that Γ is connected and consists of in total k cycles, i.e. $\chi(\Gamma) = -k + 1$, where we have labelled each cycle by $1, \ldots, k$, and such that furthermore

- Each vertex has valence greater than two, except for the vertex v_0 – called the *external starting vertex* – which is allowed to have valence 2.
- The dual graph (defined similarly to 3.4, with one vertex pr. cycle, and one edge pr. vertex) is a tree.
- $l_{i_1} + \cdots + l_{i_s} = 1$ where $e_{i_1}, \ldots, e_{i_s}$ forms a single cycle of Γ.
- The gluing pairs of boundaries at the vertices is given such that for edges e_i and e_j, part of the same cycle of Γ (there are precisely two such edges) are only in a gluing pair together if they glue to vertex v_0, the two boundary components $\partial_+ e_i$ and $\partial_- e_j$ are grouped in gluing pairs, such that $(\partial_- e_j, \partial_+ e_h)$ comes immediately before $(\partial_- e_g, \partial_+ e_j)$ in the cyclical ordering at the vertex.

We are thus ready to specify a topology on $\mathcal{C}acti_1$, by first specifying a topology on the set of spineless cacti. Indeed, we shall form a cell-complex that is the basic building block for a topology on $\mathcal{C}acti_1(k)$.

Construction 3.10 We first of all note that the vertex v_0 of Γ is special in the sense that when $k > 1$, we can – assuming that v_0 has valence 2, i.e. it connects the edges e_j and e_i – remove the vertex v_0 from Γ, and obtain a new metric graph Γ_{v_0}, where we identify the edges e_j and e_i to one edge, and give it length $l_j + l_i$. If this is the case, and Γ has t edges, we say that Γ has $t - 1$ *actual edges*. Assume therefore for the time being that $k > 1$.

Let $E(l)$ denote the subset of $G(k)$ with $k + l$ actual edges $e_1, \ldots e_{k+l}$. We shall handle the case of v_0 later.

For $E(0)$, the lengths $l_1, \ldots, l_k$ are all fixed at 1. Therefore, $E(0)$ is a discrete set, consisting of $(k-1)!$ distinct points, each point coming from the different choices of a cyclic ordering at the (single) intersection point (but up to action of the cyclical group permuting the lobes). We call $E(0)$ the 0-skeleton.

$E(1)$ consists of graphs with $k + 1$ actual edges, $l_1, \ldots, l_{k+1}$. This leaves $k - 1$ actual edges fixed at length 1 and the remaining two edges e_i and e_j with $l_i + l_j = 1$. Choose one lobe that consists of e_i and e_j, and fixate a choice of ordering of the remaining $k - 1$ lobes.

$\{l_i, l_j \in [0,1]^2 \mid l_i + l_j = 1\}$ is a 1-simplex. In effect, $E(1)$ is naturally given the topology of a disjoint union of 1-simplices.

We have automatic gluing maps from elements of $E(1)$ to elements of $E(0)$, obtained by letting one of the lengths l_i converge to 0, deleting e_i, we obtain a cactus with k

actual edges. Before deleting e_i, Γ had two vertices v_1, v_2. After deletion there is only one vertex v. We pick the obvious ordering at the resulting vertex, namely if $(\partial_- e_i, \partial_+ e_j)$ and $(\partial_- e_h, \partial_+ e_i)$ are the gluing pairs at v_1 resp. v_2, the k gluing pairs at v is given by all the gluing pairs of v_1 and v_2 except the two above, where we instead specify the gluing pair $(\partial_- e_h, \partial_+ e_j)$. The cyclical ordering can be specified by requiring all neighborhoods at v to be the ones from v_1 and v_2. We omit neighborhoods $U_i^- \cup U_j^+$ and $U_h^- \cup U_i^+$, where $U_h^-, U_i^-, U_i^+, U_j^+$ are neighborhoods of $\partial_- e_h, \partial_- e_i, \partial_+ e_i, \partial_+ e_j$ respectively. Instead we add the neighborhoods $U_h^- \cup U_j^+$ at v.

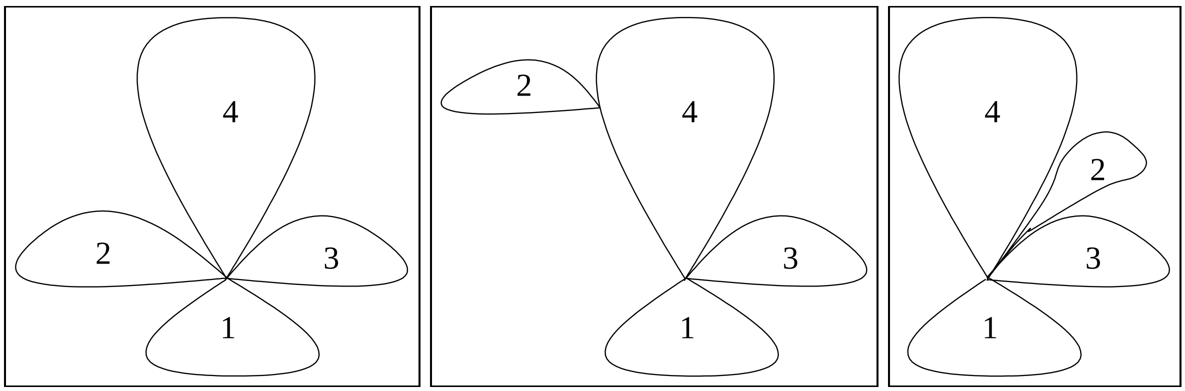

Figure 12: progressing between $E(0)$ and $E(1)$ by creating and collapsing internal edges

$E(2)$ consists of spineless cacti with $k+2$ actual edges. In the same fashion as for $E(1)$, we have 2-simplices given by a lobe partitioned up into 3 internal edges, e_i, e_j, e_h with $l_i + l_j + l_h = 1$, as shown in figure 13 and furthermore products of simplices from

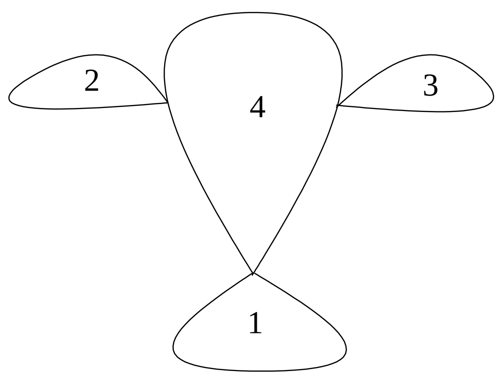

Figure 13: A point in a 2-simplex

$E(1)$ with simplices from $E(1)$, as shown in figure 14.

Letting an actual edge of $E(2)$ converge to 0 – as before – yields a simplex of $E(1)$.

In the general case, we get recursively that $E(l)$ consists of l-dimensional objects – l-simplices given by graphs with one cycle partitioned up into the actual edges $e_{i_1}, \ldots, e_{i_l}$,

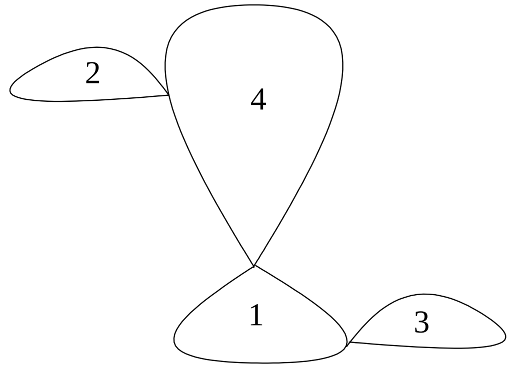

Figure 14: A point in a (1×1)-simplex

and the remaining cycles given as a single edge with length one. Call the set of these simplices $F(l)$.

$E(l)$ furthermore consists of products of lower-dimensional simplices precisely as the case was for $E(2)$, that is elements of $F(l_1) \times \cdots \times F(l_r)$, where $l_1 + \ldots + l_r = l$.

As we saw before, elements of $F(l)$ are naturally glued on to elements of $F(l-1)$, by letting an actual edge e_i of a spineless cacti parametrised by an element of $F(l)$ converge to zero, yielding a simplex in $F(l-1)$.

Hereby, we indeed glue each boundary of a simplex in $F(l)$ onto a simplex in $F(l-1)$.

As $E(k)$ is built up of $F(l)$, the gluings specify a cell-structure on $E(k)$, and there are no further cells for higher k. In order to describe a cell structure on $G(k)$, we need to account for the external starting vertex, v_0. When $k = 1$, we have seen that $E(1) = G(1)$ is simply a point.

When $k > 1$, we have that v_0 can be positioned anywhere on the k cycles. By the cyclical ordering at the vertices, these choices can naturally be parametrized via S^1 as the (unique – by the last bullet of 3.9) orientation- and length-preserving path traversing Γ starting and ending at the same point – v_0 – in the spineless cacti.

As we have specified the topology on $E(k)$ such that this path doesn't change the ordering with which it traverses the cycles of Γ – even upon edge-deletion, we can naturally specify $G(k) = S^1 \times E(k)$ for $k > 1$.

Likewise, the k internal starting points can be chosen anywhere on one of the k cycles, so we have that choosing k internal starting points specifies a space $(S^1)^k \times G(k)$.

Finally we want to translate this information back to $\mathcal{C}acti_1(k)$. That is, we specify a bijection $T\colon (S^1)^k \times G(k) \to \mathcal{C}acti_1(k)$

That is, we want to define the k-cactus $T(p_1, \ldots, p_k, \Gamma) = (c_\Gamma, \pi_\Gamma)$.

We start by identifying the i'th cycle of Γ with $S^1_i \subseteq S(k)$, where we identify p_i with $1 \in S^1_i$, in the unique way s.t. connected $f_k \subseteq e_k = [0, 1]$ of edges on the i'th cycle gets mapped to the arc $A_{f_k} \subseteq S^1_i$ where A_{f_k} has arc-length $|f_k| \cdot l_k$.

Indeed, this identifies every point of Γ with a point of $S(k)$.

Let $\pi_\Gamma(1)$ be the point corresponding to $v_0 \in \Gamma$. If $|v_0| > 2$, then v_0 lies at some gluing point $(\partial_- e_i, \partial_+ e_j)$. We choose $\pi_\Gamma(1)$ to be the point in $S(k)$ corresponding to $\partial_+ e_j$.

Starting at v_0, we travel Γ in positive orientation. To the first gluing point $(\partial_- e_i, \partial_+ e_j)$ we encounter, $\partial_- e_i$ corresponds to some point $t_1 \in S(k)$ and $\partial_+ e_j$ to some $b_l \in S(k)$. The intersection points lies on different lobes, and we define an intersection tuple $[t_1, b_1]$.

Continuing to traverse Γ in positive orientation, for (almost) every gluing pair encountered, we likewise form an intersection tuple. However, if intersection tuples for all but one gluing pair have been formed, we do not form an intersection tuple for the last gluing pair.

If Γ has m vertices, it has $k+m-1$ edges and hence $k+m-1$ gluing pairs. As we are omitting an intersection tuple for each vertex, we have in total $k+m-1-m = k-1$ intersection tuples $\{[t_1, b_1], \ldots, [t_{k-1}, b_{k-1}]\} =: c_\Gamma$. By travelling Γ, we have automatically defined $\pi_\Gamma \colon S^1 \to S(k)/\sim_c$ as the unique equidistant pinching map.

Obviously, we have defined c_Γ to have one intersection point at each lobe, and as we are defining the cyclical ordering – by the last bullet in 3.9 – such that we indeed all lobes are traversed – one by one – in the cyclical ordering, π_Γ is bijective away from the vertices $S(k)/\sim_{c_\Gamma}$, and (c_Γ, π_Γ) is a k-cactus.

Similarly we can traverse $S(k)/\sim_{c_\Gamma}$ via the pinching map π_Γ, and obtain the spineless cacti Γ decomposed as what we called *internal edges* in 3.7, so it follows by construction – keeping the notion of cluster relation in mind – that T is a bijection

To conclude our endeavours, we state

Definition 3.11 The topology on $\mathcal{C}acti_1(k)$ is the one induced by the bijection

$$T \colon G(k) \times (S^1)^k \to \mathcal{C}acti_1(k)$$

of 3.10

Using this identification, we shall often allow ourselves to switch between these two different interpretations of the topological space of k-cacti.

For later use, we give a continuous procedure for appending more and more cycles to spineless cacti.

Definition 3.12 Assume that we are given an element $\Gamma \in E(l)$. Let $x \in \Gamma$. By

$$\Gamma \vee x \in E(l+1)$$

we shall understand the doubly fat oriented metric graph, obtained by adding an edge e_x of length $l_x = 1$, at x. In the sense that as x is a point in a doubly oriented fat metric graph, whenever x is at a vertex$v$, it also contains information on what gluing pair $(\partial_- e_i, \partial_+ e_j)$ it is at. So in this case, we remove $(\partial_- e_i, \partial_+ e_j)$ and add the two gluing pairs $(\partial_- e_x, \partial_+ e_j)$ immediately before $(\partial_- e_i, \partial_+ e_x)$ at the position of the cyclical ordering $(\partial_- e_i, \partial_+ e_j)$ had at v.

Definition 3.13 By the *flagged spineless k-cacti*, we shall understand the space

$$fG(k) := \{(\Gamma, x) \mid \Gamma \in G(k) \text{ and } x \in \Gamma\}$$

That is – a point of $fG(k)$ consists of a spineless cacti – and a point in the cacti.

Similarly, we define flagged version of the related spaces $E(k)$ and $\mathcal{C}acti_1(k)$, written as $fE(k)$ and $f\,\mathcal{C}acti_1(k)$ respectively.

Lemma 3.14 Let X be a topological space.

Assume that we are given a continuous map $f\colon X \to fE(k)$. In coordinates, we write f as $x \mapsto (\Gamma_x, f_{\Gamma_x}(x))$

Let $\Phi_f\colon X \to E(k+1)$ be given by

$$\Phi_f(x) = \Gamma_x \vee f_{\Gamma_x}(x).$$

Φ is a continuous map.

It is easy to see that similar results holds with $fG(k)$ and $f\,\mathcal{C}acti_1(k)$ replaced by $fE(k)$ as well. We shall however only use the result for $fE(k)$.

Proof. Let $f_{\Gamma_x}(x)$ be a point of some edge e_{i_j}. e_{i_j} is part of some cycle with k edges, $e_{i_1}, \ldots, e_{i_k}$. The cycle given by these edges, represents a simplex $\Delta_{k-1} := \{(l_{i_1}, \ldots, l_{i_k} \in [0,1]^k \mid l_{i_1} + \ldots + l_{i_k} = 1\}$ in $F(k-1)$.

Let $d_{(f_{\Gamma_x}(x))}$ denote the distance from the incoming boundary of the edge of e_i to the point $f_{\Gamma_x}(x)$. G maps a point of the simplex $(l_{i_1}, \ldots, l_{i_k}) \in \Delta_k$ to the point $(l_{i_1}, \ldots, l_{i_j} - d_{f_{\Gamma_x}(x)}, d_{f_{\Gamma_x}(x)}, \ldots, l_{i_k})$ in a k-simplex of $F(k)$. As long as $f_{\Gamma_x}(x)$ stays on the same cycle, on all other simplices arising from other cycles than the one described by the vertices $e_{i_1}, \ldots, e_{i_k}$, simplices are mapped identically to each other.

The fact that $f_{\Gamma_x}(x)$ is continuous implies that $d_{(f_{\Gamma_x}(x))}$ depends continuously on x, so it follows that indeed – as the gluing along lower-dimensional simplices is automatic in the sense of 3.10 – that G is a continuous map. $\square$

Remark 3.15 We have an extra cycle on $\Gamma \vee x$, compared to Γ. We have not specified a labelling of this extra cycle. We can let the cycle obtained from e_x be labelled by any number $i \in \{1, \ldots, k\}$, and thereby relabel the cycles in Γ labelled by i through k to $i+1$ through $k+1$.

Indeed, these choices all define different maps. We shall however, not use different notation for them. In applications, the actual labelling of the appended cycle will be obvious.

We are hereby ready to define the operadic structure on $\mathcal{C}acti_1$.

Definition 3.16 The action of Σ_k on $\mathcal{C}acti_1(k)$ comes from permuting the ordering of the k lobes of the cactus.

To define $\mathcal{C}acti_1(n) \circ_i \mathcal{C}acti_1(k)\colon\ \mathcal{C}acti_1(n) \times \mathcal{C}acti_1(k) \to \mathcal{C}acti_1(n+k-1)$, assume that we are given a tuple $((c_k, \pi_k), (c_n, \pi_n)) \in \mathcal{C}acti_1(k) \times \mathcal{C}acti_1(n)$

The base spaces are $S(n) := \coprod_{j=1}^n S_j^1$ for c_n and $S(k) := \coprod_{j=1}^k S_j^1$ for c_k.

The base space in the image of $\mathcal{C}acti_1(n+k-1)$ will be given by $S(n+k-1) := \coprod_{j \in \{1,\ldots,i-1,i+1,\ldots,n+k-1\}} S_j^1$, i.e. we ignore the i'th circle of $S(n)$, and append $S(k)$. Where the i'th lobe of $S(n)$ occurred. As usual, this specifies an ordering of the lobes of $S(n+k-1)$.

In order to give a $(n+k-1)$-pre-cactus, c_{n+k-1} we need to give $n+k-2$ intersection tuples. $k-1$ of the intersection tuples will be given by those of c_k. There is likewise $n-1$ intersection tuples of c_n. We will use these intersection tuples as well. However, some of these tuples may have an intersection point at S_i^1. Via the pinching map $\pi_k\colon S^1 \to S(k)/\sim_c$, where we replace the domain to S_i^1 – and thereby map $1 \in S_i^1$ to the external starting point of (c_k, π_k) – we get a method to – continuously – identify these intersection points at S_i^1 with points on on $S(k)$.

In order to get a pinching map π_{n+k-1}, we start at the external starting point of (c_n, π_n), and retain the cyclical ordering at the intersection point attained from π_n and π_k. Understood in the sense that we travel along c_n, using the cyclical ordering of π_n, until we reach an intersection tuple with a point originally on S_i^1. We then travel from the corresponding point at $S(k)$ using the cyclical ordering of π_k, until we reach a point corresponding to an intersection point originally at S_i^1, where we go back to the ordering of π_n – and henceforth.

In effect, what we do is simply identifying internal edges of S_i^1 with the corresponding arcs of $S(k)$ under the pinching map π_k, and the operadic structure follows.

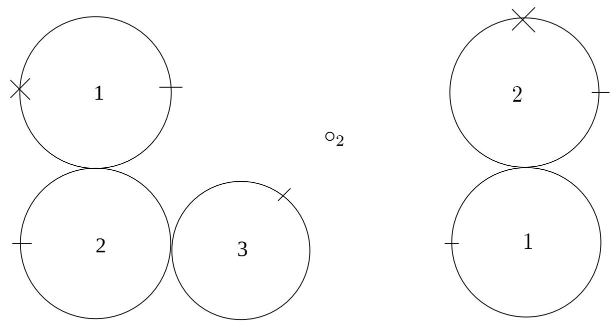

Figure 15: Example composition of Cacti

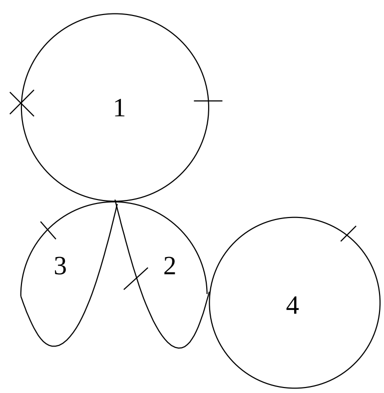

Figure 16: The composed cactus

3.2 Throwing Lobes Away

Construction 3.17 We construct a map

$$p_l\colon \mathcal{Cacti}_1(k) \to \mathcal{Cacti}_1(k-1).$$

On the level of pre-cacti – $p_l\colon C(k) \to C(k-1)$ – $p_l(c)$ is given by a $(k-1)$-pre cactus, where we forget the k'th lobe from $S(k)$.

Pick a specific representative of the pre-cactus c. Denote by $[t_{i_1}, b_{i_1}], \dots [t_{i_n}, b_{i_n}]$ the intersection tuples with either one of the points of the tuple residing on the k'th lobe, or being in an intersection cluster with one that has.

Assume for notational simplicity that if $[t_{i_j}, b_{i_j}]$ contains an intersection point on the k'th lobe, it is the intersection point denoted b_{i_j}. If neither lies on the k'th lobe, the intersection point is part of an intersection cluster. Precisely one of the intersection points are free – if there were none, the dual graph would contain a cycle – in this case we assume that it is the intersection point denoted b_{i_j} that is the non-free.

We specify the intersection tuples of $p_l(c)$ to be given by $[t_{i_1}, t_{i_n}], \dots, [t_{i_{n-1}}, t_{i_n}]$ and $[t_r, b_r]$ for $r \neq i_j$. In effect, let t_{i_n} reside one the m'th lobe. The dual tree $G_{p_l(c)}$ is given by collapsing the edge arising from $[t_{i_n}, b_{i_n}]$ (as well as the vertex arising from the k'th lobe) down to the vertex arising from the m'th lobe.

All $[t_{i_1}, t_{i_n}], \dots, [t_{i_{n-1}}, t_{i_n}]$ are part of the same intersection cluster. Therefore $p_l(c)$ is both independent of the choice of representative of c, as well as the choice of t_{i_n} as the second choice of intersection point.

In order to specify cyclic orderings of the intersection points, we note that we might as well let $p_l(c)$ be given by letting all but one edge-lengths of the edges $e_{i_1}, \dots, e_{i_q}$ forming the k'th lobe converge to 0, then removing the final length 1 edge e_{i_q} of the k'th lobe. By the topology of letting edge-lengths converge to zero given in 3.10, this uniquely defines a cyclic ordering of the intersection points, and furthermore via the edge-collapses specifies the external starting point – if the external starting points happens to be on the edge e_{i_q}, we let the external starting point be given at the vertex v where ∂e_{i_q} comes together, and position the external starting point in the cyclically ordered at v where the gluing pair consisting of ∂e_{i_q} was.

The remaining $k-1$ internal starting points remains the same.

We describe the image of p_l on the simplices described in 3.10, as elements of the simplex set $F(t-1)$. If the $(t-1)$-simplex $\Delta = \{(l_0, \dots, l_t) \in \mathbb{R}^t \mid \sum l_i = 1\}$ is given as a cactus with the k'th lobe having only one intersection point intersecting at a vertex v having no other lobes intersecting at v, we have that removing v makes two edges, say e_i and e_j become one edge. Therefore – to such simplices – p_l is a face map

$$p_l(\Delta) = \{(l_1, \dots, l_i + l_j, \dots, l_t) \in \mathbb{R}^{t-1} \mid \sum l_i = 1\}$$

This makes p_l continuous; we collapse the simplex arising from the k'th cycle to a single point, and on all other simplices, p_l maps as the identity. As $E(k)$ is made out of $F(t)$, it follows – as we also have specified the image of the external and internal starting points as continuous – that $p_l\colon \mathcal{Cacti}_1(k) \to \mathcal{Cacti}_1(k-1)$ indeed is a continuous map.

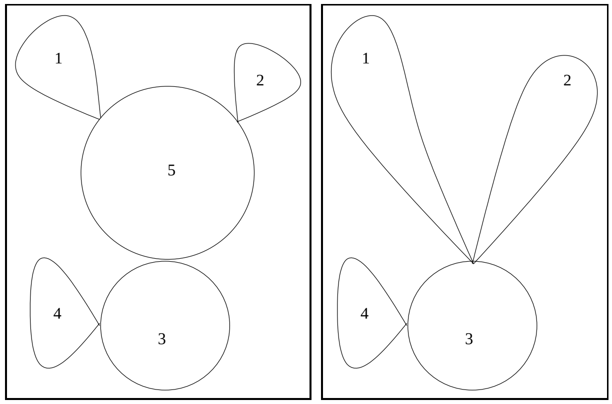

Figure 17: a cactus before and after application of p_l

Remark 3.18 It is stated in [CV06, p. 37], that p_l is a fibration. As noted in [Kau05], this is too much to hope for, if it should be, we would in particular have a lift as illustrated in the commutative diagram

$$\begin{tikzcd}
* \arrow[r, hook, "(*\times 0)"] \arrow[d] & * \times I \arrow[d, "\gamma"] \arrow[dl, dashed] \\
\mathcal{C}acti_1(k) \arrow[r] & \mathcal{C}acti_1(k-1)
\end{tikzcd}$$

For any path γ in $\mathcal{C}acti_1(k-1)$, meaning that we should be able to lift the path γ in $\mathcal{C}acti_1(k-1)$, with $\gamma(0) = p_l(c,\pi)$.

However, consider a 3-cactus (c,π) given by having the lobes labelled by 1 and 2 intersecting the lobe labelled 3 at respectively the point 0 and $\frac{1}{2}$ of the lobe S^1_3. $p_l(c,\pi)$ consists of a cactus with two lobes labelled by 1 and 2. See figure 18

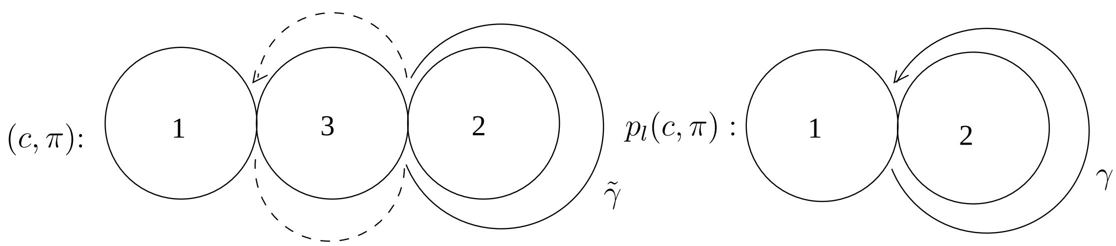

Figure 18: As indicated, to define $\tilde{\gamma}$, it would have to 'jump across' the lobe S^1_3

We can specify a path $\gamma\colon I \to \mathcal{C}acti_1(2)$ with $\gamma(0) = p_l(c,\pi)$, for instance by rotating the lobe labelled 1 once around the lobe labelled 2. γ don't lift to a map $\tilde{\gamma}\colon I \to \mathcal{C}acti_1(3)$, since at $\tilde{\gamma}(0)$, the lobe labelled by 1 has to be at the point $0 \in S^1_3$, but given $\varepsilon > 0$, $\tilde{\gamma}(\varepsilon)$ has to intersect somewhere at the lobe labelled 2. In the metric $d(-,-)$ induced from $\mathbb{R}^2$ on the simplex of $F(2)$ that (c,π) is part of, $d(\tilde{\gamma}(0), \tilde{\gamma}(\varepsilon)) > \frac{1}{2}$, so $\tilde{\gamma}$ cannot be continuous, and hence p_l cannot be a fibration.

Luckily, the less restrictive notion of a quasifibration will show more applicable:

Definition 3.19 A surjective map $p\colon E \to B$ between topological spaces is called a *quasifibration*, if the induced map on pairs $\hat{p}\colon (E, p^{-1}(b)) \to (B, b)$ is a weak equivalence (i.e. induces an isomorphism on all higher homotopy groups) for all $b \in B$

Note that if we consider the long exact sequence of homotopy groups of the pair $(E, p^{-1}(b))$, we can replace $\pi_n(E, p^{-1}(b))$ by – as usual is done for fibrations – observing that $p_*\colon \pi_n(E, x_0) \to \pi_n(B, b)$ is factorized as the composition

$$\pi_n(E, x_0) \xrightarrow{i_*} \pi_n(E, p^{-1}(b)) \xrightarrow{\hat{p}_*} \pi_n(B, b)$$

and then using the fact that p_* is an isomorphism to obtain the following from the long exact sequence of homotopy groups:

Proposition 3.20 A quasifibration $p\colon E \to B$ induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(F, x_0) \longrightarrow \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b) \longrightarrow \cdots \longrightarrow \pi_0(E, x_0) \longrightarrow \pi_0(B, b)$$

Where F denotes the fiber $p^{-1}(b)$.

Furthermore, we note the following useful

Proposition 3.21 Given a quasifibration $p\colon E \to B$, the fibers $p^{-1}(b)$ are all homotopy-equivalent to the homotopy fiber of p.

Proof. Consider the commutative diagram

$$\begin{tikzcd} E_p \arrow[r, "\pi_p"] & B \\ E \arrow[u, "\beta"] \arrow[r, "p"] & B \arrow[u, equal] \end{tikzcd}$$

where π_p is the pathspace fibration associated to p, and β is the homotopy equivalence $a \mapsto (a, p(a))$, where $p(a)$ is the constant curve in B with value $p(a)$.

We get an induced map along the fibers, $\beta\colon p^{-1}(b) \to \pi_p^{-1}(b)$

So applying the long exact sequences of homotopy groups to π_p and p, the 5-lemma tells us that $\beta_*\colon \pi_n(p^{-1}(b)) \to \pi_n(\pi_p^{-1}(b))$ is an isomorphism, and hence β is a weak equivalence along the fibers as well. $\square$

For our purpose – having a quasi-fibration is by the above good enough. Therefore, 3.18 will not be a concern, as the following theorem can be found in [Kau05, 3.3.18]

Theorem 3.22

$$p_l\colon \mathcal{C}acti(k) \to \mathcal{C}acti(k-1)$$

is a quasifibration

Therefore, identifying the homotopy fiber amounts to nothing more than identifying the fiber of p_l for one selected basepoint of $\mathcal{C}acti(k-1)$.

Let $(\frac{1}{2})_i$ denote the point $\frac{1}{2} \in S^1_i$. As basepoint we choose the cactus $(c_*, \pi_*) \in \mathcal{C}acti(k-1)$ with precactus c_* given by having intersection tuples

$$[(\frac{1}{2})_1, (\frac{1}{2})_{k-1}], \ldots, [(\frac{1}{2})_{k-2}, (\frac{1}{2})_{k-1}] \tag{3}$$

that is, all intersection tuples are part of the same intersection cluster, and we choose a representative with non-free intersection points at $\left(\frac{1}{2}\right)_{k-1}$.

In order to progress to an actual $k-1$-cactus, let the external starting point be given by the internal starting point – 1 – of S^1_1. Let the cyclical ordering at the single vertex of $S(k-1)/\sim_{c_*}$, arising from the $k-2$ intersection tuples listed above, be given by letting the pair of edges arising from the $(j \mod (k-1))$'th lobe be the immediate predecessor of the ones arising from the $(j+1 \mod (k-1))$'th lobe for all j.

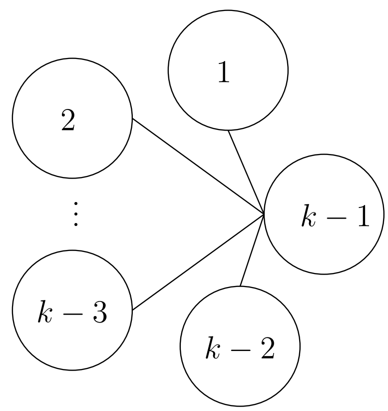

Figure 19: The basepoint c_*.

Remark 3.23 We aim at describing the homotopy type of $p_l^{-1}(c_*, \pi_*)$. In order to this, we give a cellular description on the image of $p_l^{-1}(c_*, \pi_*)$ inside of $E(k)$ from 3.10, and then handle choice of starting-points afterwards. We list the cells of different dimension, and how they are attached to each other:

0-cells: The only way to obtain a 0-cell is to add an intersection tuple (note that we are not taking internal starting points into account, so there really is only one choice) to the list (3). This gives $k-1$ different choices of cyclic ordering. We therefore have in total $(k-1)$ 0-cells, $v_1, \ldots, v_{k-1}$. Here v_i denotes the 0-cell given by letting the pair of edges arising from the k'th lobe be the immediate successor to the pair arising from the i'th lobe.

1-cells: Choosing the k'th lobe to have a single intersection tuple lying somewhere at the i'th lobe gives a 1-cell, s_i. We therefore obtain the 1-cells $s_1, \ldots, s_{k-1}$, where s_1 is attached to v_{k-1} and v_1, and for higher i, s_i is attached to v_{i-1} and v_i.

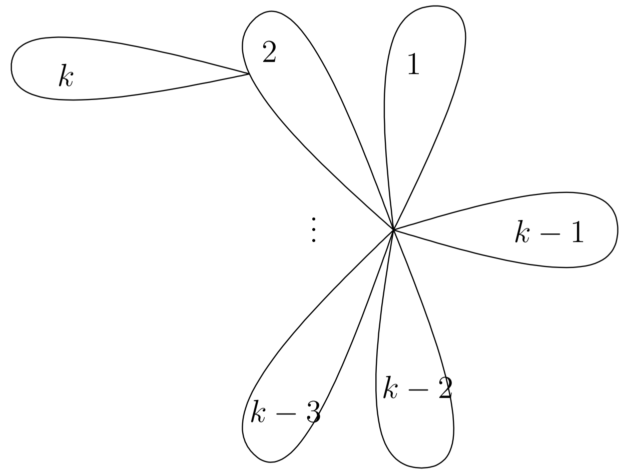

Figure 20: moving k around the lobe labelled 2 yields a 1 cell attached to v_1 and v_2

(k-1)-cell: By S, we denote the $(k-1)$-simplex given by a cactus with the lobes labelled by $1, \ldots, k-1$ having an intersection tuple somewhere on the lobe labelled by k. This gives $k-1$ edges $e_1, \ldots, e_{k-1}$ of $S(k)/\sim_{c_\star}$ at the k'th lobe. Where we let e_i denote the edge between the lobe labelled by $i \mod (k-1)$ and $i+1 \mod (k-1)$, and hence a single $(k-1)$-simplex.

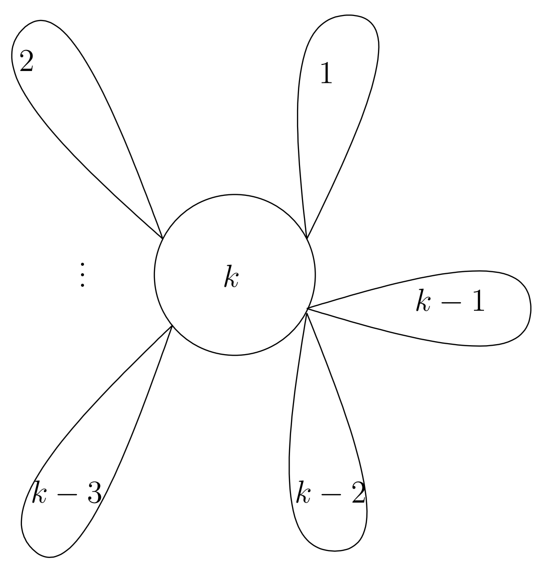

Figure 21: A point in the $k-1$-simplex S, given by the vertices at the k'the lobe.

S is attached only to the 0-cells by letting the length of one of e_i converge to 1 (and hence the rest to 0). In this case, this yields the vertex v_i.

Call the above cell-complex $B_\star$. Note that $B_\star$ describes all points of $p_l^{-1}(c_*, \pi_*)$ projected down to $E(k)$.

We therefore turn to the effect of choosing external and internal starting points.

Unless the k'th lobe has an intersection tuple with an intersection point at the external starting point of (c_*, π_*), $1 \in S^1_1$ (this is only the case for an internal point of the 1-cell s_1), the external starting point of points in $p_l^{-1}(c_*, \pi_*)$ can be nothing but $1 \in S^1_1$.

In the case where the k'th lobe has an intersection tuple point of intersection at $1 \in S^1_1$, the point can be chosen to be anywhere along the k'th lobe. That is, anywhere along the internal points of s_1: s_1°. Therefore, parametrising s_1° via $]0,1[$ choosing an external starting point gives a wedge-product of $B_\star$ with s_1°.

All other but the k'th internal starting point are predetermined, and the choice of internal starting point for the k'th lobe yields a cartesian product with S^1

All in all, we have that as a subspace of $\mathcal{C}acti_1(k)$:

$$p_l^{-1}(c_*, \pi_*) \cong (B_\star \vee]0,1[) \times S^1$$

As $]0,1[$ is contractible, and the simplex S of $B_\star$ is attached only to the points $v_1, \dots, v_{k-1}$, it can be contracted to a single point, yielding a homotopy equivalence $B_\star \simeq \bigvee_{k-1} S^1$, in effect, we have shown:

Proposition 3.24 The homotopy fiber of $p_l \colon \mathcal{C}acti(k) \to \mathcal{C}acti(k-1)$ is given by

$$\left(\bigvee_{k-1} S^1 \right) \times S^1$$

3.3 The Gravity Map on the Level of Spaces

Notation 3.25 We shall use quite a lot of geometric constructions of objects inside of D^2. We allow ourselves to use the following notation:

- To many geometric objects G we consider, there is a natural notion of the center point. If applicable, we shall denote $c(G)$ for this centre point
- For two points $a, b \in D^2$, we can talk about the line-segment in D^2, between a and b. This will be denoted $\ell(a,b)$. For $A, B \subseteq D^2$, we set $\ell(A,B) := \bigcup_{a \in A, b \in B} \ell(a,b)$
- For a point $p \in D^2 \backslash c(D^2)$, we can measure the angle of $\ell(c(D^2), p)$ from the line $\ell(c(D^2), (1,0))$. Denote the angle of $\ell(c(D^2), p)$ by $\operatorname{ang}(p) \in [0, 2\pi[$.
- By $|\cdot|$, we shall understand the standard euclidean distance on D^2, induced from $\mathbb{R}^2$.

Weighted averages will come in handy, for the constructions of this section:

Definition 3.26 Consider a line-segment $L \subseteq \mathbb{R}$. To a finite set $S := \{l_k, w_k\}_{k \in \{1,\dots,n\}}$, where $l_k \in L$ and $w_k \in [0,1]$ is called a *weight*. The *weighted average* of S is given by the function $\alpha \colon L^n \times [0,1]^n \to L$ defined by

$$\alpha(S) = \frac{\sum_{i=1}^n w_i l_i}{\sum_{i=1}^n w_i}$$

provided $\omega_i \neq 0$ for some i; otherwise, we leave it undefined.

α is made out of continuous functions, so it is continuous. As

$$\frac{\sum_{i=1}^n w_i}{\sum_{i=1}^n w_i} \min\{l_1, \ldots, l_n\} \leqslant \frac{\sum_{i=1}^n w_i l_i}{\sum_{i=1}^n w_i} \leqslant \frac{\sum_{i=1}^n w_i}{\sum_{i=1}^n w_i} \max\{l_1, \ldots, l_n\},$$

α indeed takes values in L.

The following trivial observation holds as well, from the reasoning above

Observation 3.27 Let $L \subseteq \mathbb{R}$ be connected. Consider a set $S := \{x_k, \omega_k\}_{k \in \{1,\ldots,n\}}$ with all $x_k \in L'$ as well as $x_k \in L$.

Let L' be the least connected subset of $\mathbb{R}$, containing all x_k, letting $\alpha_{L'}$ and α_L denote the weighted average functions with respect to these two different line-segments. Then

$$\alpha_L(S) = \alpha_{L'}(S) \in L'$$

Another trivial observation we will need is simply saying that weight zero points, i.e. point that contribute with a zero on the numerator and denominator of α are irrelevant

Observation 3.28 Let

$$S := \{l_k, \omega_k\}_{k \in \{1,\ldots,n\}}.$$

If $\omega_{k_0} = 0$ (and $n > 1$), letting

$$S' := \{l_k, \omega_k\}_{k \in \{1,\ldots,k_0-1,k_0+1,\ldots,n\}}$$

we have that

$$\alpha(S) = \alpha(S')$$

It is our goal during the remainder of this section to construct an explicit morphism of operads $\Gamma\colon \mathcal{W}(f\mathcal{D}isk_2) \to \mathcal{C}acti_1$, that is a Σ-equivariant local equivalence.

The general structure will be to define a Σ-equivariant homotopy equivalence mapping $\gamma_n\colon f\mathcal{D}isk_2(n) \to \mathcal{C}acti_1(n)$, called the *gravity map*. This will not be a morphism of operads though.

However, after constructing γ_n, it is shown in the following subsection how to turn γ_n into the weak equivalence of operad; what we call the *gravity morphism of operads*, $\Gamma\colon f\mathcal{D}isk_2 \to \mathcal{C}acti_1$.

Remark 3.29 Before our construction of the local equivalence between $f\mathcal{D}isk_2$ and $\mathcal{C}acti_1$, we first mention that in [CV06, p.37], it is suggested that in order to prove this equivalence, one applies the so-called framed Fiedorowich recognition principle. Developed in [Wah01, 1.5.17].

In order to state this principle, observe first of all that the action of Σ_k on $f\mathcal{D}isk_2(k)$ is a covering space action (also called proper action). Define the *ribbon braid group* to be $\mathrm{RB}_k := \pi_1(f\mathcal{D}isk_2(k)/\Sigma_k)$. By what we saw in 2.52 it follows that $f\mathcal{D}isk_2(k)/\Sigma_k$ is a $K(\mathrm{RB}_k, 1)$.

With this in mind, we state the principle:

Assume $\mathcal{O}$ is a topological operad. If the following two condition holds

- $\mathcal{O}(k)/\Sigma_k$ is a $K(\mathrm{RB}_k, 1)$ for all k
- There is a morphism of operads $\mathcal{D}isk_1 \to \mathcal{O}$

then $\mathcal{O}$ is Σ-equivariant weakly equivalent to $f\mathcal{D}isk_2$.

We can see the recognition principle as a method for detecting enough homotopy theoretic properties of $\mathcal{O}$ similar to that of $f\mathcal{D}isk_2$, that a Σ-equivariant weak equivalence between them is forced.

$f\mathcal{D}isk_n(k)$ has all sorts of higher homotopy groups for $n > 2$, so there seems to be no hope of generalizing the Framed Fiedorowich recognition principle to higher dimensions – the homotopy theoretic properties of $f\mathcal{D}isk_n(k)$ are simply too complicated.

Our main reason to differ from applying the Framed recognition principle, and instead take a more direct approach in constructing a weak equivalence between $\mathcal{C}acti_1$ and $f\mathcal{D}isk_2$ is – as mentioned in the introduction – that we aim at generalizing this weak equivalence to higher dimensions.

Of course, such a – potential – generalization would require a (suitable) definition of higher dimensional analogues of the cacti-operad.

Construction 3.30 We give the general method for constructing a k-cactus from a little disk embedding f. In fact, we shall use a bit more, that is, assume that we are given:

- A point $p \in D^2$, called the *center of mass*
- An element $f \in f\mathcal{D}isk(k)$.

We first show how to construct a k-pre-cactus c_f^p from this data. First of all note that the boundary of the domain of f is given as the ground space $S(k) = \coprod_{i=1}^k S_i^1$. We choose each S_i^1 to be given an orientation induced by requiring that $f(S_i^1)$ is positively oriented, as a subspace of $\mathbb{R}^2$. f is an embedding and, using f and p, we shall compute $k-1$ intersection tuples in $S(k)$.

We first of all compute either k or $k-1$ intersection points called t_i. That is, we compute a t_i lying on each S_i^1 of $S(k)$. However, if $p \in f(D_i^2)$, we do not compute a t_i.

Consider the line segment $\ell(p, c(f(D_i^2)))$. Assuming that $p \notin f(D_i^2)$, there is precisely one point in the intersection with $f(S_i^1)$, and we set

$$t_i := f^{-1}\left(f(S_i^1) \cap \ell(p, c(f(D_i^2)))\right)$$

In order to have a pre-cactus, we however need to choose $k-1$ intersection tuples. The computed t_i will participate in each of these intersection tuples.

For each $f(S_i^1)$, let h_i denote the lower hemisphere of $f(S_i^1)$, defined by setting the south pole equal $f(t_i)$. Let B_i be the beam $\ell(p, h_i)$. The method of computing intersection tuples will be split up into two parts, (a) and (b):

(a) Assume that $i_1, \ldots, i_l$ are precisely the indices with $B_{i_m} \cap f(D_j^2) = \varnothing$ for all $j \neq i_m$, we give $l-1$ intersection tuples as the intersection cluster defined by $[t_{i_2}, t_{i_1}], \ldots, [t_{i_l}, t_{i_1}]$. If $l = 1$, we choose no intersection tuples in this way.

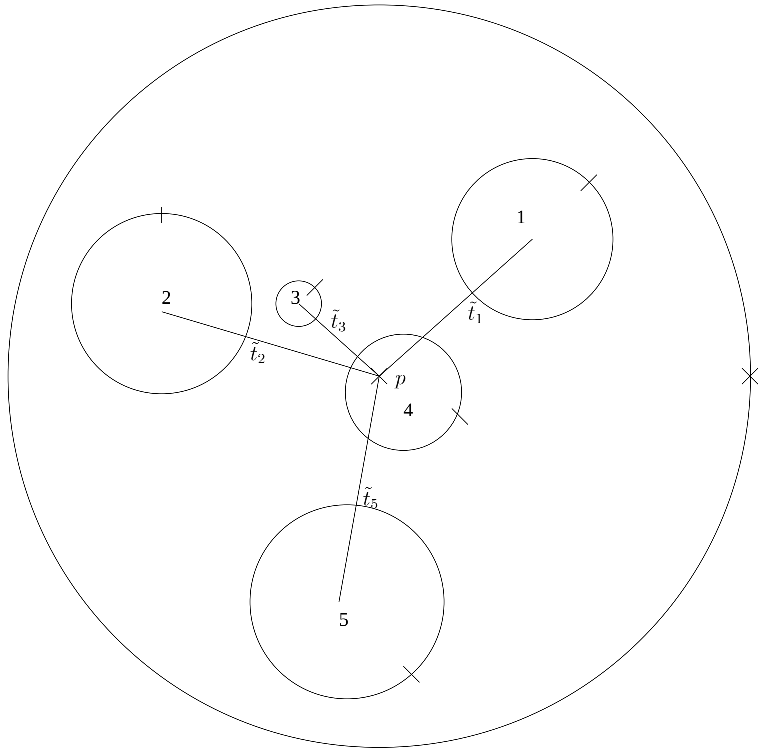

Figure 22: choosing the intersection points t_i. In the image we have indicated points $\tilde{t}_i$ on $\partial f(D_i^2)$. Note that we omit computing intersection point for the framed little disk embedding labelled 4 as it is over the point marked p. We set $t_i = f^{-1}(\tilde{t}_i)$

Denote by $c_f^p(l)$, the pre-cactus given by restricting $S(k)$ to $S(l) := S_{i_1}^1 \coprod \cdots \coprod S_{i_l}^1$, with the chosen intersection tuples. Note that there is a natural cyclical ordering of the lobes $S_{i_1}^1, \ldots, S_{i_l}^1$, coming together at the single vertex of $S(l)/\sim_{c_f^p(l)}$, namely the one given by the ordering of the angles $\operatorname{ang}(f(t_{i_1})), \ldots, \operatorname{ang}(f(t_{i_l}))$ – where we, of course, consider the angles as centered around p instead of $c(D^2)$.

(b) For each t_j, where $j \neq i_1, \ldots, i_l$ from (a), we shall choose a point b_j, such that $[t_j, b_j]$ is an intersection tuple. Note that this will define in total $k-1$ intersection tuples – in (a) we chose $l-1$ intersection tuples, and choosing b_j for $j \neq i_1, \ldots, i_l$ will choose in total $k-l+l-1 = k-1$ intersection tuples.

The construction of b_j will be a recursive one. We shall use (a) as the start of the recursion. Assume that an intersection tuple $[t_j, b_j]$ has been chosen for all j with

$$|c(f(D_j^2)) - p| \leqslant |c(f(D_i^2)) - p| \tag{4}$$

Assume that there are $l-1$ b_j satisfying (4).

Assume furthermore that if we restrict to the pre-cactus $c_f^p(l)$ given by the lobes $S(l)$ satisfying (4), we have chosen a cyclical ordering of the vertices of $S(l)/\sim_{c_f^p(l)}$.

We shall apply a weighted average, to determine b_j. In order to make the weighted average, we shall first find a line-segment L_j to take the weighted average over. L_j will be pieced together by arcs of $\partial \operatorname{Im}(f)$.

Note that the beam B_j contains only $f(S_i^1)$ with b_i chosen for each i. Using the orientation of the hemisphere h_j, we parametrize part of h_j as the distance- and orientation-preserving homeomorphism $\gamma\colon [0,1] \to h_j \cap B_j$.

The first thing we want to do is find a starting point q^0 for L_j, first of all, if $l = 1$ in (a), we pick the point $t_{i_1} =: q^0$. If $\ell(p, \gamma(0)) \neq \varnothing$, we set q^0 to be the point of the finite set $\ell(p, \gamma(0)) \cap f(D_i^2)$ nearest $\gamma(0)$.

If neither of these applies, we shall pick $q^0 = t_i$, where t_i is part of the intersection cluster computed in (a). We pick the t_i with least angle in clockwise direction, compared to the line $\ell(p, \gamma(0))$.

In either case, q^0 is a point at some $f(S_{i^0}^1)$. From q^0, we travel $f(S_{i^0}^1)$ in the positive direction of the orientation to the first occurring intersection point q^1 of $f(S_{i^0}^1)$. This yields a line-segment L_j^1 with length equal to the arc-length from q^0 to q^1.

q^1 might be part of some intersection cluster, but by the cyclical ordering at the corresponding vertex of $S(l)/\sim_{c_f^p(l)}$, there is some q^2 forming $[q^1, q^2]$, such that q^2 connects to the next lobe of $S(l)/\sim_{c_f^p(l)}$.

From q^1 we jump to the intersection point q^2. We traverse from q^2 to the next intersection tuple, to obtain a line-segment L_j^2 as before, and progressing on, the process terminates – as the dual graph of c_f^p is a tree – when there are no further points in B_j to travel. In effect, we obtain line-segments $L_j^1 \coprod \ldots \coprod L_j^h$. These line segments glue together to L_j, by identifying the right-most point of L_j^g with the left-most of L_j^{g+1}. Call the point of L_j obtained by gluing line-segments together *gluing points.*

By construction, each gluing point has arisen from some intersection tuple. For future reference, we mark all gluing points in L_j by p, if they have arisen from an intersection tuple of the intersection cluster described in (a) (for the case with no intersection cluster computed in (a), we mark them by p if they have arisen from the point t_{i_1} as well).

The line segment L_j contains all point of $B_j \cap \partial \operatorname{Im}(f)$; Having obtained a line-segment, we just need to produce a set S_j of points with weights for our weighted average.

For each $t \in [0,1]$, we consider the (finite) set of points $\ell(p, \gamma(t)) \cap f(S_i^1)$ for $i \neq j$, from this set, we choose the point nearest $\gamma(t)$.

These points all lie on $f(S_i^1)$ for some i, and therefore constitute some connected components of $L_j : a_1, \ldots, a_s$. Should one of the a_l intersect with a gluing point of L_j, we split it up in two components a_l and a_{l+1}. Each point of a_l, correspond to some $t \in [0,1]$; call the subset of $[0,1]$ under this correspondence c_l.

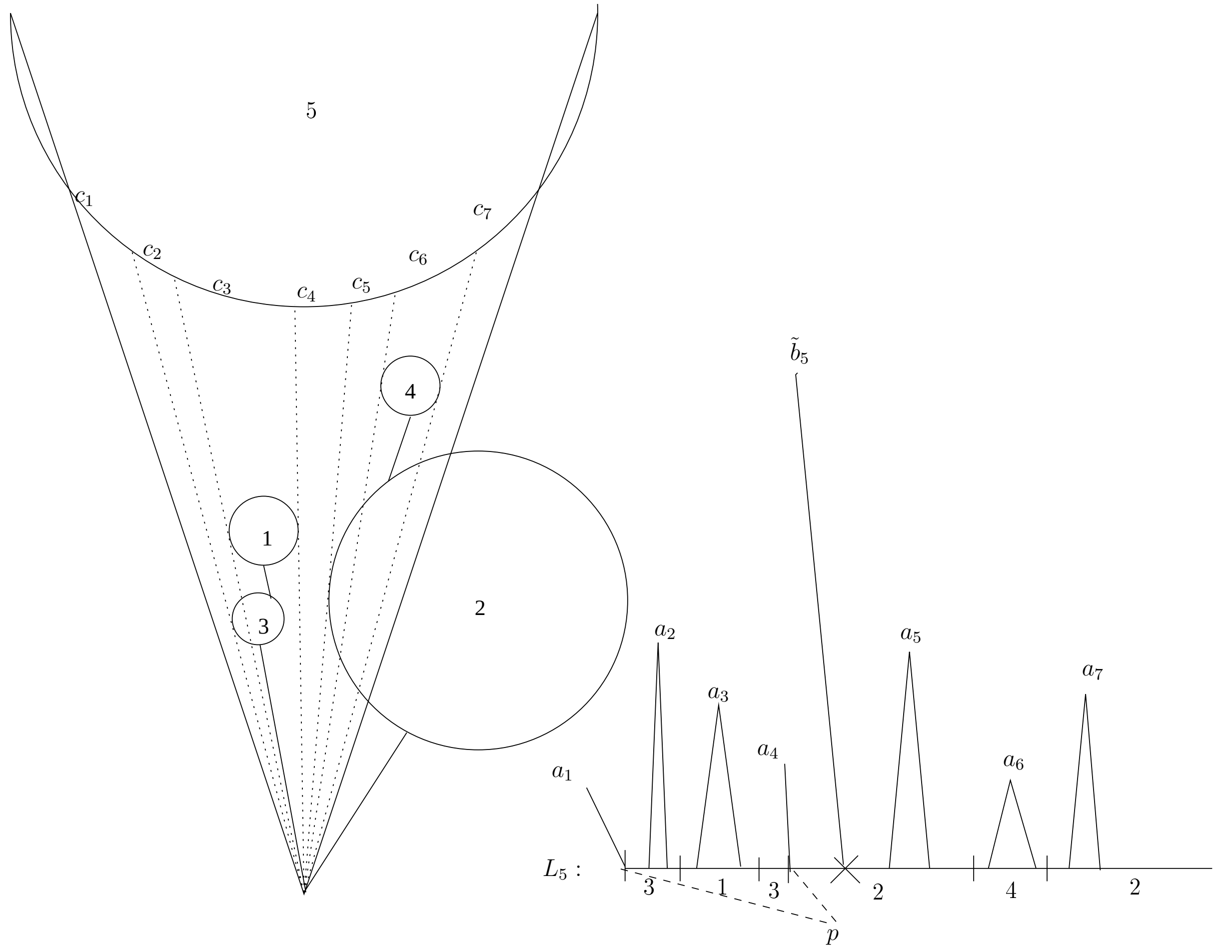

Figure 23: Illustration of a computation of the point $\tilde{b}_5$, to the left, the beam B_5, with the disks intersecting B_5. To the right, the line L_5 spanning the portion of the cactus along the boundary of the little disks. The segments $a_1, \ldots, a_7$ corresponding to $c_1, \ldots, c_7$ are given along L_5 have been marked. – X marks the result of the computation.

Note that considering the beam $C_i := \ell(p, \gamma(c_i))$ we get a one-to-one correspondence between c_i and a_i, the latter considered as a set of $\mathrm{Im}(f)$.

For some t, $\ell(p, \gamma(t)) \cap \mathrm{Im}(f) = \varnothing$, so these constitute some connected components $c_s, \ldots, c_{s+t}$ of $[0, 1]$.

To these points, the whole beam $C_{s+i} = \ell(p, \gamma(c_{s+i}))$ $(i > 0)$, does not intersect with $\mathrm{Im}(f)$, and it follows by the ordering at the intersection clusters that we get a unique intersection tuple $[q_1, q_2]$ – part of the intersection cluster described in (a) – such that $\ell(f(q_1), f(q_2)) \cap C_{s+i} \neq \varnothing$. $[q_1, q_2]$ determines some gluing point marked p of L_j. By a_{s+i}, we shall understanding this gluing. In effect we end up with an additional t points $a_{s+1}, \ldots, a_{s+t}$.

The points for the weighted average will be given as the centre points of each component – or point – $c(a_i) \in L_i$.

We set the weights corresponding to $c(a_i)$ to be $\omega_i := |c_i|$; that is – the length of the interval c_i.

We therefore have the set $S_j := \{(c(a_i), \omega_i) \in L_j \times [0,1]\}_{i \in \{1,\dots,s+t\}}$, to this data, we can apply the weighted average, α of 3.26 and set

$$\tilde{b}_j := \alpha(S_j) \in L_j \tag{5}$$

$\tilde{b}_j$ is a point of some L_j^g, and therefore by construction we can also regard it as a point of $f(S_i^1)$ for some i – where we for the moment assume that $\tilde{b}_j$ is not at a gluing point of L_j. We define $b_j = f^{-1}(\tilde{b}_j)$. Should $\tilde{b}_j$ happen to be at a gluing point, we have that $[b_j, t_j]$ becomes part of an intersection cluster, and therefore it doesn't matter which of the two possible lobes we choose b_j to lie on – they are related by a single cluster move. For the sake of having the recursive procedure sound, we choose the one that is not a b_i itself.

Once the recursive process is done, and we have obtained $k-1$ intersection tuples, we actually have the cyclic ordering at each vertex of $S(k)/\sim_{c_f^p}$ necessary to define a k-cactus. Therefore, the only thing missing, to attain an actual cactus, is the choice of an external starting point.

We can however parametrize the semi-hemisphere[3] of $\partial D^2 = S^1$ containing $(1,0) \in S^1$ as a pole – call it h_{k+1} – by a homeomorphism $\gamma\colon [0,1] \to h_{k+1}$. And as we already have chosen a cyclical ordering at the vertices of $S(k)/\sim_{c_f^p}$, we can apply (b) above to this parametrization as well, and obtain a point $v_0 \in S(l)/\sim_{c_f^p}$ as some weighted average. We let v_0 be the external starting point of c_f^p.

All in all we therefore have describe a procedure from $f \in f\mathcal{D}isk_2(k)$, and $p \in D^2$, to produce a cactus $(c_f^p, \pi) \in \mathcal{C}acti_1(k)$.

Having a procedure as prescribed above is not enough, if one wants to do topology. In order to get anything working, we need to ensure that the procedure gives us a continuous map. This will be our concern in the course of the next couple of pages.

Definition 3.31 We define $(f\mathcal{D}isk_2(k) \times D^2)^{1 \leqslant \cdots \leqslant k} \subset f\mathcal{D}isk_2(k)$ to be the subspace given by the (f,p) such that $|c(f(D_i^2)) - p| \leqslant |c(f(D_{i+1}^2) - p|$ for all i.

Remark 3.32 We shall first consider continuity of the map defined in 3.30 restricted to the subspace $(f\mathcal{D}isk_2(k) \times D^2)^{1 \leqslant \cdots \leqslant k}$. Even more so, instead of considering the map to have image in $\mathcal{C}acti_1(k)$, we first consider it as having image in the base-space $E(l)$ of 3.10, where we ignore the internal and external starting points.

Note that when restricting to the mentioned subspace, we can order the recursive procedure, such that we first apply (a) of 3.30 to the disk labelled 1, then apply (a)

[3] i.e. one quarter of a sphere

or (b) (whichever applies) to the disk labelled 2, and henceforth until finally the disk labelled k.

We can consider the j'th step in the recursive procedure as being given by identifying the element $\Gamma^j \in E(j)$ obtained so far, and via (a) resp. (b) picking a point of Γ^j, given at the intersection cluster described in (a) resp. the intersection point obtained by the formula (5) in (b). Selecting this point thus provides a map $\varphi^j_{\Gamma^j}\colon (f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots\leqslant k} \to \Gamma^j$.

Therefore, the j'th step in fact gives a map

$$\varphi^j\colon (f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots\leqslant k} \to fE(j)$$

given by letting $\varphi^j(f,p) = (\Gamma^j, \varphi^j_{\Gamma^j}(f,p))$. Note that by construction, in order to progress from Γ^j to Γ^{j+1}, we apply the map Φ_{φ^j}, defined in 3.14; thus we let $\Gamma^{j+1} := \Gamma^j \vee \varphi^j_{\Gamma^j}(f)$.

We can therefore write the diagram defining the map of 3.30 up in a diagram – that becomes smaller by letting $X := (f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots\leqslant k}$:

$$\begin{tikzcd} X \arrow[r] & E(1)\times X \arrow[r, "\varphi^1\times\mathbb{1}"] & fE(1)\times X \arrow[r, "\Phi_{\varphi^1}\times\mathbb{1}"] & E(2)\times X \arrow[r, "\varphi^2\times\mathbb{1}"] & \cdots \arrow[r, "\varphi^{k-1}\times\mathbb{1}"] & fE(k-1)\times X \arrow[d, dashed, "\Phi_{\varphi^{k-1}}\times\mathbb{1}"] \\ & & & & E(k) & E(k)\times X \arrow[l, "\mathrm{pr}_1"'] \end{tikzcd} \tag{6}$$

Here the first arrow is the first application of (a) in 3.30 on the first coordinate (note that $E(1)$ is just a singleton), and the identity on the second coordinate, and pr_1 is the projection onto the first coordinate.

Lemma 3.33 $\tilde{\gamma}_k\colon (f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots\leqslant k} \to E(k)$ defined via (6) is a continous.

Proof. 3.14 tells us that Φ_{φ^j} are automatically continuous, once each φ^j are continuous.

We show that φ^j are continuous by complete induction. That is, assume that $\varphi^{j-i}(i > 0)$ are continuous. In particular, this means that the image $\Gamma^j \in E(j)$ depends continuously on (f,p) as arising as the image of a composition of continuous functions.

Therefore, we need only show that $\varphi^j_{\Gamma^j}\colon (f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots\leqslant k} \to \Gamma^j$ is a continuous map.

That is, we need to show that the points computed in (a) and (b) depends continuously on $(f,p) \in (f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots\leqslant k}$.

When (a) applies, $\varphi^j_{\Gamma^j}$ is given by being constant at a single vertex $v \in \Gamma^j$; even the position in the cyclical ordering at v constant, so here $\varphi^j_{\Gamma^j}$ is continuous.

Assume now that (b) applies. First of all, as the line by construction L_j is given by gluing the boundaries of the edges $e_{i_1}, \ldots, e_{i_k}$ of Γ^j together in the gluing pairs that they are given by. By the topology of Γ^j, it is therefore enough to show that the map into L_j yields a continuous map. The map into L_j is given by a weighted average, and by 3.26, taking a weighted average is continuous. Taking a weighted average involves points on the line segment L_j, as well as weights for those. In 3.30 we noted that by considering the beam $C_i := \ell(p, \gamma(c_i))$ the intervals and point $a_1, \ldots, a_{s+t}$ are completely determined by the partition $c_1, \ldots, c_{s+t}$ of the interval $[0,1]$. By construction, the set S_i is determined by $c_1, \ldots, c_{s+t}$.

In effect, we have reduced the problem to showing that the map that to $(f,p) \in (f\mathcal{D}isk_2(k) \times D^2)^{1 \leqslant \cdots \leqslant k}$ associates a partition $c_1, \ldots, c_{s+t} \subset [0,1]$ is continuous.

Denote by $\mathrm{part}_{s+t}([0,1])$ the space of partitions of $[0,1]$ by $s+t$ intervals

Keep for the moment the cardinality of $\{c_1, \ldots, c_{s+t}\} \subset [0,1]$ fixed. We show that the map $\rho\colon (f\mathcal{D}isk_2(k) \times D^2)^{1 \leqslant \cdots \leqslant k} \to \mathrm{part}_{s+t}([0,1])$ defined in (b) is continuous.

For continuity of ρ in the first variable, both $(f\mathcal{D}isk_2(k) \times D^2)^{1 \leqslant \cdots \leqslant k}$ and $\mathrm{part}_{s+t}([0,1])$ are metric spaces – endowed with the supremum metric. In this case, it suffices to find a δ that respects a given ε:

Pick first f such that no $f(D_i^2)$ overlaps with p. This means that there is some least distance $b > 0$ from p to f, and it follows – by considering the maximal effect on a unit-cube around D^2 of taking an ε-ball around $\mathrm{Im}(f)$ on the beams $C_1, \ldots, C_{s+t}$ – that given $b > \varepsilon > 0$, we can pick fx. $\delta := \frac{\varepsilon}{4(b-\varepsilon)}$.

If $p \in f(D_i^2)$, then for each $t \in [0,1]$, $f(D_i^2) \cap \ell(p, \gamma(t))$, and therefore as long as $p \in f(D_i^2)$, the partition $\{c_1, \ldots, c_{s+i}\} \subset [0,1]$ is independent of $f(D_i^2)$, reducing the case to the above.

We therefore turn to the effect of adding or deleting intervals c_i to or from the list $c_1, \ldots, c_{s+t}$. The effect of doing this, is to add or subtract another point and a weight to the weighted average. By 3.27, we need to check that in the transition of doing this, we always add a weight with value tending to 0. But as the weight of each point $c(a_r)$ is given as $\omega_r := |c_r|$; each a_r lies on precisely one $f(S_i^1)$, so adding or deleting c_r can only occur by introducing or removing a_r on some $f(S_i^1)$, or by removing a point of L_j marked p. By construction, this can only occur when the beam C_r – corresponding to the point marked p or the interval a_r on L_j – tends to have zero volume, making c_r tend to have zero length.

In the transition between (a) and (b) of 3.30, the C_r corresponding to the point marked p of L_j tends to have length 1, so $\tilde{b}_r$ tends to be at the point marked p.

In the transition between $p \in f(D_i^2)$ and $p \notin f(D_i^2)$, the line L_j has fixed length, and $\{c_1, \ldots, c_{s+t}\}$ remain constant as well. The only difference being that points marked p transition to points on L_j correspond to points of $f(\partial D^2)$

Finally, note that in a sufficiently small ball B_p around $p \in D^2$, translating p via v inside of this ball, yields precisely the same cactus as translating f via $-v$, so continuity in the second variable of ρ follows from continuity of the first variable. □

Remark 3.34 The notation $(f\mathcal{D}isk_2(k) \times D^2)^{1 \leqslant \cdots \leqslant k}$ suggests that for $\sigma \in \Sigma_k$, we should define $(f\mathcal{D}isk_2(k) \times D^2)^{\sigma(1) \leqslant \cdots \leqslant \sigma(k)} \subset f\mathcal{D}isk_2(k) \times D^2$ as the subspace given by requiring that

$$|c(f(D_{\sigma(i)})) - p| \leqslant |c(f(D^2_{\sigma(i+1)})) - p|,$$

and indeed we do.

There are – of course – no problems in transferring the statement of 3.33 with all instances of $(f\mathcal{D}isk_2(k) \times D^2)^{1 \leqslant \cdots \leqslant k}$ replaced by $(f\mathcal{D}isk_2(k) \times D^2)^{\sigma(1) \leqslant \cdots \leqslant \sigma(k)}$.

As furthermore the choice of external starting point in $\mathrm{Im}(\tilde{\gamma}_k)$ is – by 3.30 – given precisely via the same procedure as (b) in 3.30, we have that from the proof of 3.33, that $\tilde{\gamma}_k$ can be considered as a continuous map to $G(k)$, and as the choice of internal starting points are automatically the points $1 \in S^1_i$ for all i, the map can even be considered as a continuous map into $\mathcal{C}acti_1(k)$.

Note that $(f\mathcal{D}isk_2(k) \times D^2)^{\sigma(1)\leqslant\cdots\leqslant\sigma(k)}$ are all closed subspaces of $f\mathcal{D}isk_2(k) \times D^2$, and we have

$$\bigcup_{\sigma\in\Sigma_k}(f\mathcal{D}isk_2(k) \times D^2)^{\sigma(1)\leqslant\cdots\leqslant\sigma(k)} = f\mathcal{D}isk_2(k) \times D^2$$

using this, we are finally able so state

Proposition 3.35

$$\tilde{\gamma}_n\colon f\mathcal{D}isk_2(k) \times D^2 \to \mathcal{C}acti_1(k)$$

constructed in 3.30 is continuous

Proof. We suggestively write

$$(f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots\leqslant i=i+1\leqslant\cdots\leqslant k} :=$$

$$(f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots i\leqslant i+1\leqslant\cdots k} \cap (f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots i+1\leqslant i\cdots\leqslant k}. \tag{7}$$

Any permutation can be split up into simple permutations switching two consecutive letters. Therefore, by the Pasting lemma, we by 3.33 and 3.34 only need to check that the two definitions coming from the factors of intersection of (7) of $\tilde{\gamma}_k$ agree at the intersection $(f\mathcal{D}isk_2(k) \times D^2)^{1\leqslant\cdots\leqslant i=i+1\leqslant\cdots\leqslant k}$.

Note that this is the subspace of the two factors in the intersection (7) given by further requiring that $|c(f(D^2_i)) - p| = |c(f(D^2_{i+1}) - p|$.

When this is the case, the beams B_i and B_{i+1} of 3.30 satisfies $B_i \cap B_{i+1} = \varnothing$ as f being an embedding in particular implies that the hemispheres h_i and h_{i+1} are disjoint.

Therefore, the cactus obtained by first calculating b_i and then b_{i+1} is the same as the one with calculation of b_i and b_{i+1} reversed; the line segments L_i and L_{i+1} of 3.30 satisfy $L_i \cap L_{i+1} = \varnothing$, considered as points on $\partial\,\mathrm{Im}(f)$. □

In the forthcoming, we shall indeed use that the map $\tilde{\gamma}_n\colon f\mathcal{D}isk_2(k)\times D^2 \to \mathcal{C}acti_1(k)$ is continuous in both variables.

However the second variable is only needed for technical purposes. Therefore, we define

Definition 3.36

$$\gamma_k\colon f\mathcal{D}isk_2(k) \to \mathcal{C}acti_1(k)$$

As the map $\gamma_k(f) := \tilde{\gamma}_k(f, c(D^2))$, where as usual $c(D^2)$ denotes the centre of D^2. That is, γ_k is given as $\tilde{\gamma}_k$ with centre of mass fixed at $c(D^2)$.

Remark 3.37 Note that γ_k indeed is Σ_k-equivariant, as Σ_k permutes the labellings of components in the domain of $f\colon \coprod_{i=1}^k D_i^2 \to D^2$ in $f\mathcal{D}is\mathcal{k}_2(k)$, and the action on $\mathcal{C}acti_1(k)$ permutes the labelling of lobes given by the boundary of this domain.

As $\gamma_k(f)$, maps each component in the domain labelled i of f to the lobe labelled i, Σ_k-equivariance follows

Proposition 3.38

$$\gamma_k\colon f\mathcal{D}is\mathcal{k}_2(k) \to \mathcal{C}acti_1(k)$$

is a Σ_k-equivariant homotopy equivalence

Proof. Consider the diagram

$$\begin{tikzcd} F_d \arrow[r] & f\mathcal{D}is\mathcal{k}_2(k) \arrow[r, "p_d"] \arrow[d, dashed, "\gamma_k"] & f\mathcal{D}is\mathcal{k}_2(k-1) \arrow[d, dashed, "\gamma_{k-1}"] \\ F_l \arrow[r] & \mathcal{C}acti_1(k) \arrow[r, "p_l"] & \mathcal{C}acti_1(k-1) \end{tikzcd} \tag{8}$$

Where the map denoted p_l is the quasifibration defined in 3.17, and p_d is the fiber bundle forgetting the k'th. little disk, defined in 2.52.

It is easy to see that the square in (8) is not a commutative one.

F_l and F_d are the fibers of p_l respectively p_d. Given the fiber of any two base-points, there is a homotopy equivalence connecting the fibers. For our purpose, it shall therefore be enough to consider fibers of specific basepoints.

In 3.22, we defined $(c_*, \pi_*) \in \mathcal{C}acti_1(k-1)$.

In order to define an explicit basepoint – $*_d \in f\mathcal{D}is\mathcal{k}_2(k-1)$, we consider $D^2 \subset \mathbb{C}$, and let ξ_i^{k-1} denote the i'th of the $k-1$'th unit roots. Let $*_d$ be given by the $k-1$ framed little disk embeddings f_i^*, specified by being centered around $\frac{1}{2}\xi_i^k$, and with radius $\frac{|\xi_1^{k-1} - x_2^{k-1}|}{6}$ (so none of the $k-1$ little disks intersect). This specifies f_i^* as a little disk embedding. We specify the framing on $f_i^\star$ by a rotation of $\xi_i^{k-1} \in S^1$. See figure 24

We therefore (re-)define $F_l := p_l^{-1}(c_*, \pi_*)$ and $F_d := p_d^{-1}(*_d)$. If we co-restrict the map p_d to $*_d$, and pick $f \in F_d$, we have that indeed the restricted square of (8) commute: $\gamma_{k-1} \circ p_d(f) = \gamma_{k-1}(*_d)$, and $p_l \circ \gamma(f)$ are both cacti with all $k-2$ intersection tuples being part of the same intersection cluster of (a) in 3.30

Therefore, we get induced maps $\iota_k\colon F_d \to F_l$

We now claim that the square in (8) is in fact homotopy-commutative.

To see this, we shall define homotopies deforming both $\gamma_{k-1} \circ p_d$ and $p_l \circ \gamma_k$.

First of all, given $f \in f\mathcal{D}is\mathcal{k}_2(k)$, let $\rho_f\colon [0,1] \to D^2$ be given as a convex combination from the center of D^2 to the center of the k'th framed little disk embedding:

$$\rho_f(t) = tc(D^2) + (1-t)c(f(D_k^2)).$$

We define a homotopy $\Phi_t\colon [0,1] \times f\mathcal{D}is\mathcal{k}_2(k) \to \mathcal{C}acti_1(k-1)$ by setting

$$\Phi_t := p_l \circ \tilde{\gamma}_k(f, \rho_f(t)).$$

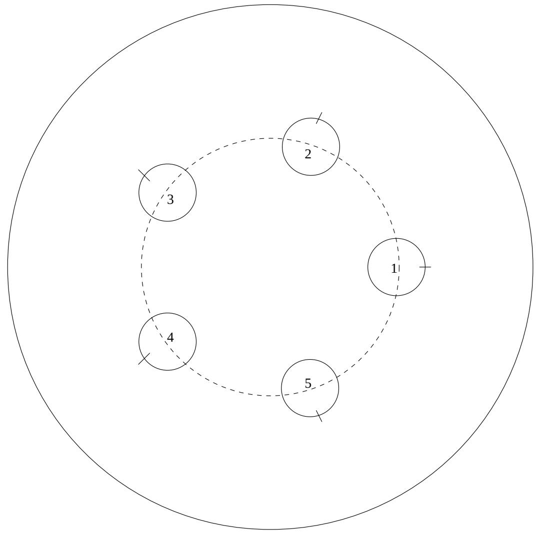

Figure 24: The basepoint $*_d \in f\mathcal{D}isk_2(5)$. Note the similarities with figure 19.

Note that $\Phi_0 = p_l \circ \gamma_k$, and $\Phi_1 = p_l \circ \tilde{\gamma}_k(f, c(f(D_k^2)))$.

For the other composite $\gamma_{k-1} \circ p_d$ in the square, we also define a homotopy $\Theta\colon [0,1] \times f\mathcal{D}isk_2(k) \to \mathcal{C}acti_1(k-1)$. We define $\Theta_t(f) = \tilde{\gamma}_{k-1}(p_d(f), \rho_f(2t))$ for $t \in [0, \frac{1}{2}]$. Again, we have $\Theta_0 = \gamma_{k-1} \circ p_d$.

The only difference between $\Theta_{\frac{1}{2}}(f)$ and $\Phi_1(f)$ is that given a framed little disk embedding $f|_{D_i^2}$, the b_i calculated in 3.30 is in $\Theta_{\frac{1}{2}}$ computed as a weighted average along a line-segment L_i where there are some points marked p. In $\Phi_1(f)$, we take precisely the same weighted average, only instead of the n points marked p, there are line-segments arising from $f(S_k^1)$, of length c_i^j, where $j \in \{1, \ldots, n\}$.

Define therefore $\Theta_t(f)$ to be the map that at time $1 \geqslant t \geqslant \frac{1}{2}$ for each step in the recursive procedure described in 3.30, computes b_i as for $\Theta_{\frac{1}{2}}$ – only the j'th point marked p at L_i, we insert an *artificial line-segment* of length $(2t-1)\left|c_i^j\right|$. We position the point a_r at the $j'th$ point marked p, used to compute the weighted average – centered at the artificial line segment. If $\tilde{b}_i$ is computed to be at an artificial line-segment, we interpret it as being at the corresponding point marked p of L_i. By considerations similar to what we saw in the proof of 3.33 and 3.35, Θ is continuous.

That $\Phi_1 = \Theta_1$ tells us that the square in (3.38) homotopy commutative.

Therefore, we have the following commutative diagram of homotopy groups, with exact rows:

$$\begin{tikzcd}
\pi_{n+1}\left(f\mathcal{D}isk_2(k-1)\right) \arrow[r] \arrow[d, "(\gamma_{k-1})_*"] & \pi_n\left(F_d\right) \arrow[r] \arrow[d, "(\iota_k)_*"] & \pi_n\left(f\mathcal{D}isk_2(k)\right) \arrow[r] \arrow[d, "(\gamma_k)_*"] & \pi_n\left(f\mathcal{D}isk_2(k-1)\right) \arrow[r] \arrow[d, "(\gamma_{k-1})_*"] & \pi_{n-1}\left(F_d\right) \arrow[d, "(\iota_k)_*"] \\
\pi_{n+1}\left(\mathcal{C}acti_1(k-1)\right) \arrow[r] & \pi_n\left(F_l\right) \arrow[r] & \pi_n\left(\mathcal{C}acti_1(k)\right) \arrow[r] & \pi_n\left(\mathcal{C}acti_1(k-1)\right) \arrow[r] & \pi_{n-1}\left(F_l\right)
\end{tikzcd}$$

γ_1 is obviously a homotopy equivalence between two copies of S^1. Therefore we – by induction along k, the 5-lemma, and the Whitehead theorem (for homotopy groups) – only need to show that $(\iota_k)_*\colon \pi_n(F_d) \to \pi_n(F_l)$ is an isomorphism for all n and k. [4]

To do this, note that by 3.24 and 2.52, the homotopy type of F_l and F_d are both $\bigvee_{k-1} S^1 \times S^1$. Obviously, the ι_k is a homeomorphism along the S^1-factor, and independent of the first factor, as it is given by the rotating the k'th little disk embedding.

Therefore, it only remains to be checked that along the $\bigvee_{k-1} S^1$-factor, we have that the i'th summand of the domain is mapped as a degree one map to the i'th summands in the image, and mapped as a degree 0 map for $i \neq j$.This should however be clear from figure 25

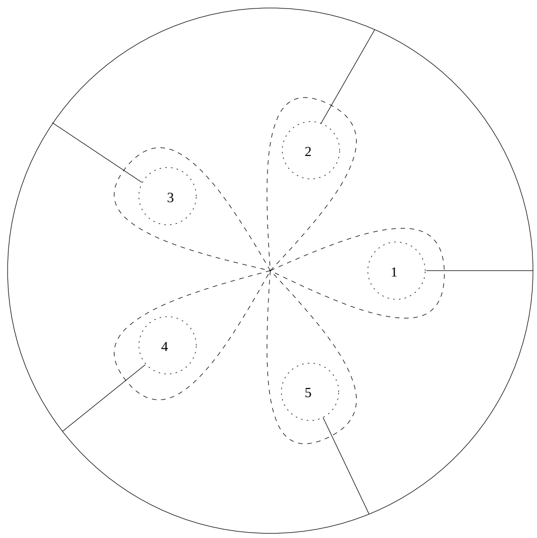

Figure 25: $\bigvee_{k-1} S^1$ as a deformation retract of F_d shown around the dotted circles – by 3.23, we can also consider this as the base-point for $p_l^{-1}(c_*, \pi_*)$. As illustrated – under γ_k – only framed disks centered at the lines going out from the j'th hole gets mapped to the point corresponding to the j'th lobe. Therefore, if we consider ι_k as a map from $\bigvee_{k-1} S^1$ to $\bigvee_{k-1} S^1$ it has local degree 1 at the point corresponding to this line.

□

3.4 The Gravity Morphism of Operads

We shall alter the construction of 3.30, so that we get a morphism of operads $\Gamma\colon \mathcal{W}(f\mathcal{D}isk_2) \to \mathcal{C}acti_1$. We shall in fact specify a morphism from $\Phi\colon \mathcal{T}ree(f\mathcal{D}isk_2) \to \mathcal{C}acti_1$, and then use 2.65 to obtain Γ.

[4] we have noted that since the upper and lower rows are given from maps between Eilenberg-Maclane spaces, we in fact only need to check this for $n = 1$, however this will not make any difference in the following, so we ignore it

Notation 3.39 Elements of $\mathcal{Tree}(f\mathcal{D}isk_2)(k)$, will be denoted by T_f, T is the tree indexing the element f given as some cartesian product dependent on the shape of T. Generally, we shall refer to T_f as a labelled k-tree. To each vertex v of T, we have associated a labelling $f_v \in f\mathcal{D}isk_2(|v|-1)$.

Generally, we use the convention of 2.65 for indexing the vertices.

By $T_f(n)$, we shall understand the $(k-i)$-tree, given by the subtree of T_f only having knots at level n and lower (the labellings are retained for the lower knots as well). All outgoing edges of level n vertices of $T_f(n)$ are retained. However, we consider these edges as connecting to leaves, given some – for our purposes rather irrelevant – labelling.

Observation 3.40 A geometric observation that will play a role in the construction of the gravity morphism is that we via what we noted in the end of 3.30, have that under the image of $\tilde{\gamma}(f, c(D^2)) = (c_f^p, \pi_f)$, we can consider $\partial D^2 = S^1$ the domain of the pinching map π_f – as noted – by using the construction in (b) for the semi-hemisphere centered around $1 \in \partial D^2$.

This gives us a way of relating the different labelling of vertices of T_f to each other. Given a vertex v at level q of T_f, we have a labelling $f_v \in f\mathcal{D}isk_2(|v|-1)$ of v. Let w be the vertex of T_f at level $q-1$ that w via the edge $u(v)$ connects to. Assuming that $q \geqslant 2$, there is some labelling $f_w \in f\mathcal{D}isk_2(|w|-1)$ of w. For the tree $T_f(q-1)$, we have that the $|w|-1$ leaves coming from the vertex w are labelled in increasing order $i_1 < \cdots < i_{|w|-1}$. f_w is composed as a disjoint union of framed little disk embeddings $f_w^1, \ldots, f_w^{|w|-1}$.

v is in $T_f(q)$ inserted at some leaf of $T_f(q-1)$ labelled i_j. Composing with the embedding $f_w^j \colon D^2 \to D^2$, we can equally well consider $f_v \colon \coprod_{i=1}^k D_i^2 \to D^2$ as embedded into $f_w^j(D^2)$, and the construction of 3.30 applies to this situation as well, and we get that the pinching map π_w associated to $\tilde{\gamma}(f_v, p)$, has domain $f_w^j(\partial D^2)$. For $q = 1$, we – as usual – use ∂D^2 as domain for the pinching map.

We shall use this interpretation of T_f, and thereby a typical element $T_f \in \mathcal{Tree}(f\mathcal{D}isk_2)(k)$ will give a tree-like arrangement of $f\mathcal{D}isk_2$ elements inside of each other, as in figure 26

Construction 3.41 From $T_f \in \mathcal{W}(f\mathcal{D}isk_2)(k)$, we want to associate a cactus $\Phi(T_f) \in \mathcal{C}acti_1(k)$. The description of Γ relies heavily on the construction given in 3.30.

That is, start by considering $T_f(1)$, i.e. we restrict T_f to have only one internal vertex, v_1 with i leaves on outgoing edges from v_1. We let $\Phi T_f(1)) = \gamma_i(f_{v_1})$. We shall take this as the start of a recursion, where we add more and more vertices of T_f to a tree starting with $T_f(1)$.

That is, assume that we have computed $\Phi(T_f(q))$ where $q \in \{1, \ldots, n-1\}$. Let v_j be a vertex of T_f at level $n-1$. We want to show how to extend $\Gamma(T_f(n-1))$ to $T_f(n-1) \cup v_{j+1}$, given by taking the subtree of T_f obtained from $\Gamma(T_f(n-1))$ by adding the vertex v_{j+1} at level n, connected by an edge to the vertex v_j.

Let there be m leafs of $T_f(n-1)$. If v_j has valence l, this gives $m + l - 1$ leafs of $T_f(n-1) \cup v_{j+1}$.

Each leaf – labelled i – sits on some vertex v_j, and therefore each leaf corresponds to a framed little disk embedding $f_i \colon D_i^2 \to D^2$, which in turn combines together to a framed

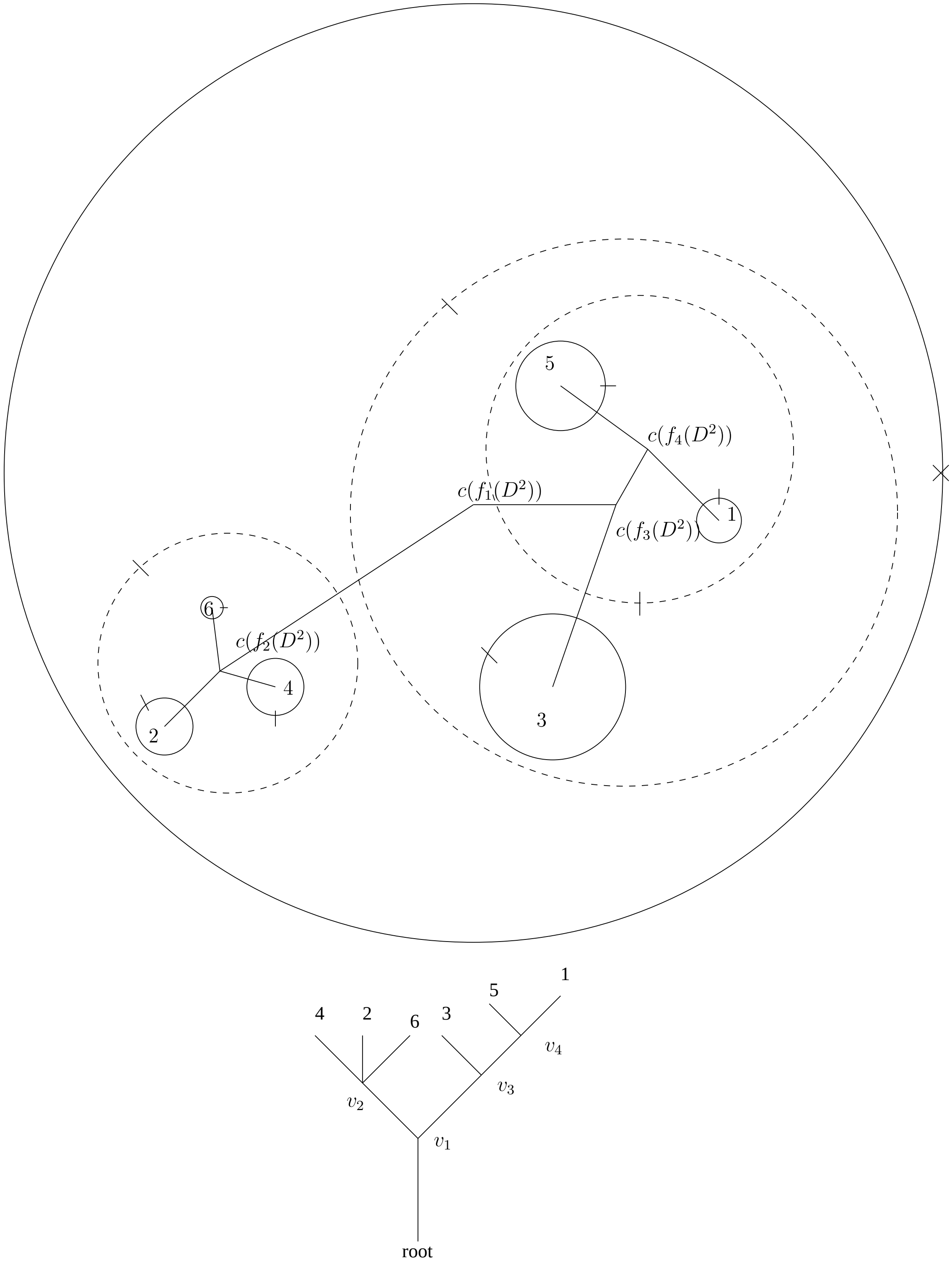

Figure 26: an element of $\mathcal{W}(f\mathcal{D}isk_2)(6)$ labelled by a tree with 4 internal knots

little disk embedding $f_q \in f\mathcal{D}isk_2(m-1)$ – we assume that for f_q, we have computed $m-1$ intersection tuples

$$[t_1, b_1], \dots, [t_{m-1}, b_{m-1}] \tag{9}$$

that lie on the domain of $\partial f_q \colon \partial_{i=1}^m D_i^2 \to D^2$.

To each vertex v_j, we further assume that we have assigned a *v_j-center of mass*, c_{v_j}. For the vertex v_1 of $T_f(1)$, this is given by $c_{v_1} = c(D^2)$.

having introduced the vertex v_{j+1} to get $T_f(n-1) \cup v_{j+1}$, let the label of v_{j+1} be denoted by $f_{v_{j+1}} \in f\mathcal{D}isk_2(l-1)$.

Let v_{j+1} be replacing a leaf of $T_f(n-1)$ labelled by s, so f_s is the framed little disk embedding that we are replacing by introducing v_{j+1}. We rename f_s to $f_{v_j}^s$. Likewise the intersection tuple $[t_s, b_s]$ that (possibly) is associated to f_s, in the sense that $t_s \in \partial \operatorname{Im}(f_s)$ we rename to $[t_{v_j}^s, b_{v_j}^s]$.

This allows us to rename the $l-1$ newly framed little disk embeddings introduced from $f_{v_{j+1}}$ by $f_s, \dots, f_{s+l-1}$, without introducing two alike names to different things.

Generally, we shall let the computation of the intersection point t_{s+i} corresponding to one of these $l-1$ framed little disk embeddings – f_{s+i} – be given – as in 3.30 – by if the v_j-center of mass $c_{v_j} \notin f_{s+i}(D^2)$:

$$t_{s+i} = f_{s+i}^{-1}\left(f_{s+i}(\partial D^2) \cap \ell(c_{v_j}, c(f_{s+i}(D^2)))\right)$$

If $c_{v_j} \in f_{s+i}(D^2)$, we – as in 3.30 – do not compute t_{s+i}, and leave it out.

By specifying the computation for $T_f(n-1) \cup v_{j+1}$ the further assumptions we make in the computation of $\Phi(T_f(n-1))$ should become evident as we progress on.

If this is not the case, we shall need to split up into two cases, corresponding to labelling s_{v_j} of $u(v_j)$. Namely if the label $s_{v_{j+1}} \in [\frac{1}{2}, 1]$ respectively $s_{v_{j+1}} \in [0, \frac{1}{2}]$ we call these two parts the internal respectively the external deformation:

<u>Internal deformation:</u> First of all, if $c_{v_{j-e}} \in f_{v_j}^s(D^2)$, where v_{j-e} is the vertex that v_j connects to via $u(v_j)$, we set $c_{v_j} = (2s_{v_j}-1)c(f_{v_j}(D^2)) + (1-(2s_{v_j}-1))c_{v_{j-e}}$ and compute $l-1$ intersection tuples via $\tilde{\gamma}(f_{v_{j+1}}, c_{v_j})$.

If $c_{v_{j-e}} \notin f_{v_j}^s(D^2)$, we have some $\tilde{t}_{v_j} \in \partial \operatorname{Im}(f_{v_j}^s)$.

We set

$$\nu := \inf_{x \in \operatorname{Im}(f_{v_{j+1}}), y \in \partial \operatorname{Im}(f_{v_j}^s)} |x-y|.$$

That is, ν is the shortest distance from any point in $\operatorname{Im}(f_{v_{j+1}})$ to the boundary of $f_{v_j}^s$.

We set $\mu := |\tilde{t}_{v_j} - c(f_{v_j}^s(D^2))|$. Set $\hat{t}_{v_j} := \frac{\nu}{2\mu}(c(f_{v_j}^s(D^2)) - \tilde{t}_{v_j}) + \tilde{t}_{v_j}$ – that is – we have choose $\hat{t}_{v_j}$ such that it is a point between $\tilde{t}_{v_j}$ and $c(f_{v_j}^s(D^2))$, with $\operatorname{Im}(f_{v_{j+1}}) \cap \ell(\tilde{t}_{v_j}, \hat{t}_{v_j}) = \varnothing$.

We define the v_{j+1}-center of mass via a convex combination, w.r.t $s_{v_j} \in [\frac{1}{2}, 1]$:

$$c_{v_j} = (2s_{v_j}-1)c(f_{v_j}^s(D^2)) + (1-(2s_{v_j}-1))\hat{t}_{v_j} \tag{10}$$

We compute $\tilde{\gamma}(f_{v_{j+1}}, c_{v_j})$ and thereby obtain $l-2$ intersection tuples for these little disk embeddings.

By assumption, there is some intersection tuple $[t_{v_j}, b_{v_j}]$ with b_{v_j} lying somewhere on $\Gamma(T_f(n-1))$.

By 3.40 we have a pinching map $\pi_{v_j}\colon \partial f^s_{v_j}(D^2) \to S(l-1)/\sim_{\tilde{\gamma}(f_{v_{j+1}}, c_{v_j})}$. In particular – up to choice of intersection cluster – we have that $\pi_{v_j}(\tilde{t}_{v_j})$ is a point of some arc of $\partial f_{v_{j+1}}(D^2)$, such that adding $[\pi_{v_j}(t_{v_j}), b_{v_j}]$ to the list of $l-2$ intersection tuples for $\tilde{\gamma}(f_{v_{j+1}}, c_{v_j})$, we get in total $l-1$ extra intersection tuples. Adding these to 9, we have in total $m+l-2$ intersection tuples, such that the dual graph given by having these tuples on $S(k+l-1)$ is connected.

Note that when $s_{v_j} = \frac{1}{2}$, we have that $c_{v_j} = \hat{t}_{v_j}$, so in this case $[\pi_{v_j}(t_{v_j}), b_{v_j}]$ is part of the intersection cluster arising in (a) (or if there is only one f_{s+i} to which (a) applies, $t_{s+i} = t_{v_j}$) of 3.30. We therefore can choose all intersection points of (a) in the computation of $\tilde{\gamma}(f_{v_{j+1}}, c_{v_j})$ to be part of an intersection cluster by picking the second non-free point of each tuple to be given by b_{v_j} instead of equal to one of the t_{s+j}. This gives us in total $l+m$ intersection tuples, so – for $s_{v_j} = \frac{1}{2}$ – we delete $[\pi_{v_j}(t_{v_j}), b_{v_j}]$, and thereby have $l-1$ extra intersection tuples.

This accounts for the intersection tuples – and the cyclical ordering at the intersection clusters – of the additional lobes introduces by $f_{v_{j+1}}$.

Let a framed little disk embeddings, $f^r_v\colon D^2 \to D^2$ be part of the label $f_v\colon \mathit{fDisk}_2(|v|-1)$ of v in $T_f(n-1)$. We make the recursive assumption that we have a beam $B^r_v = \ell(c_v, h^r_v)$, where h^r_v is the hemisphere centered around the intersection point $f^r_v(t^r_v)$ that as in 3.30 computes the intersection point b^r_v for $\Phi(T_f(q))$. B^r_v may overlap with $f^s_{v_j}(D^2)$. As noted in 3.40, we have via the pinching map $\pi_{s_{v_j}}$, identified $f^s_{v_j}(\partial D^2)$ with the arcs of the $f_{v_{j+1}}(\partial \coprod_{i=1}^{l-1} D^2)$ corresponding to edges of $\tilde{\gamma}(f_{v_{j+1}}, c_{v_j})$.

Computing an intersection point b^r_v corresponding to f^r_v, can therefore be done by either considering the intersection with lobes arising from $f^s_{v_j}(\partial D^2)$ and obtain a weighted average $\alpha(S^s_{v_j}(r))$, or we can choose to ignore $f^s_{v_j}(\partial D^2)$ from the computation of the weighted average, and only consider the lobes of $f_{v_{j+1}}$ – that lies inside of $f^s_{v_j}(D^2)$ and thereby obtain a weighted average $\alpha(S_{v_{j+1}}(r))$.

In either case, the line segment L^r_v – given by gluing the edges of lobes lying inside of B^r_v together – that the two different weighted averages are taken over, are equal by the identification of the arcs of $f^s_{v_j}(\partial D^2)$ with the edges of $\tilde{\gamma}(f_{v_{j+1}}, c_{v_j})$. Therefore, we compute the intersection point b^r_v by

$$\tilde{b}_r = (1-(2s_{v_j}-1))\alpha(S_{v_{j+1}}(r)) + (2s_{v_j}-1)\alpha(S^s_{v_j}(r)) \tag{11}$$

and set – as usual $b^r_v = (f^s_{v_j} i)^{-1}(\tilde{b}_r)$ (compare (5) of 3.30). What this formula says is simply that in the start of the internal deformation, when $s_{v_j} = 1$, we use the b^r_v arising by using $f^s_{v_j}$, and in the end when $s_{v_{j+1}} = \frac{1}{2}$, we use the one arising from $f_{v_{j+1}}$, so that in the end of the internal deformation, $f^s_{v_j}$ has no influence on computations of intersection tuples.

<u>External deformation:</u> The point of the external deformation will be that when $s_{v_j} = 0$, we have that $f_{v_{j+1}}$ is computed as if it was part of the embedding in $\mathit{fDisk}_2(|v_j|-1)$ labelling the vertex v_j.

In the internal deformation, we have specified that instead of computing intersection points as in (a) in 3.30, we should choose the intersection cluster arising in (a) as having its second point of the intersection tuple being b_{v_j}. b_{v_j} is obtained by considering the beam $B^s_{v_j}$ as in (b) of 3.30, given from the embedding $f^s_{v_j}$, and computing $b^s_{v_j}$ as some weighted average $\alpha(S^s_{v_j})$ dependent on the little disk embeddings intersecting with B_{v_j}. To any f_{s+i} of $f_{v_{j+1}}$, where (a) of 3.30 applies w.r.t. the v_j-center of mass, we instead use the computation as $\alpha(S^s_{v_j})$ to obtain an intersection tuple $[t_{s+i}, b_{v_j}]$, when $s_{v_j} = \frac{1}{2}$.

v_j connects via $u(v_j)$ to some vertex v_{j-e}. In order to have the intersection points t_{s+i} defined for f_{s+i}, we – during the external deformation, i.e. when $s_{v_j} \in [\frac{1}{2}, 1]$, set the v_j-center of mass equal to

$$c_{v_j} := (2(s_{v_j}))f^s_{v_j}(t^s_{v_j}) + (1 - 2s_{v_j})c_{v_{j-e}} \tag{12}$$

As for pinching maps, we let the pinching map $\pi_{v_{j-e}}$ be computed via (11). At $s_{v_j} = \frac{1}{2}$ – as we have that $f^s_{v_j}$ has no effect on the computation of $\pi_{v_{j-e}}$ – in effect, we compute $\pi_{v_{j-e}}$ for all $|v_{j-e}| - 1 + l - 1$ little disks embeddings forming either $f_{v_{j+1}}$ or $f_{v_{j-e}}$ (but not $f^s_{v_j}$), $\pi_{v_{j-e}}$ identifies arcs on the boundary of the surrounding disk with arcs on these little disk embeddings. We shall for $s_{v_j} < \frac{1}{2}$ compute pinching maps in the same way.

Computing b_{s+i} will be a slight moderation of what is done in 3.30.

We let h_{s+i} denote the hemisphere of $f_{s+i}(\partial D^2)$ centered around the intersection point $f_{s+i}(\tilde{t}^0_{s+i})$. Here $\tilde{t}^0_{s+i}$ denotes the intersection point of $f_{s+i}(\partial D^2)$ obtained when $s_{v_j} = 0$.

Similar to (b) of 3.30, we set $B_{s+i} = \ell(c_{v_{j-e}}, h_{s+i})$, and The calculation of $\tilde{b}_{s+i}$ will depend on the framed little disk embeddings intersecting B_{s+i}. The worried reader will note that the calculation of $\tilde{b}_{s+i}$ will thus not depend on s_{v_j}; we shall return to this dependency in (15) and (16).

Corresponding to B_{s+i}, we shall find a set S_{s+i}, to take a weighted average over.

First of all, we however need to find a line segment L_{s+i} to take a weighted average over. B_{s+i} passes through arcs of framed little disk embeddings f^r_v, arising from a labelling $f_v \in f\mathcal{D}isk_2(|v|-1)$ at a vertex v, at level – say – q. In $T_f(q)$, the label of f^r_v corresponds to some leaf labelled r of $T_f(q)$, and in $T_f(q+1)$, this leaf becomes some vertex w with a label $f_w \in f\mathcal{D}isk_2(|w| - 1)$.

If the label s_w of the edge $u(w)$ has $s_w > \frac{1}{2}$, we are in the case where f_w undergoes internal deformation, and we have already specified how we use 3.40 to – via pinching maps we identify the arcs of $f^r_v(\partial D^2)$ with arcs of $f_w(\coprod_{i=1}^{|w|-1} \partial D^2_i)$.

When $s_w < \frac{1}{2}$, i.e. when f_w undergoes external deformation, we have – by definition of pinching maps for f_w undergoing external deformation, that the pinching map π_x, where x is a vertex at a level lower than v, that π_x is independent of f^r_v, so we ignore the arcs arising from $f^r_v(\partial D^2)$.

In effect, we have an identification that tells us that any arc B_{s+i} intersects with – no matter what level the corresponding vertex is at, using the pinching maps iteratively – we can identify it with an arc on some framed little disk embedding $f_j(\partial D^2)$, corresponding to a leaf of $T_f(n-1) \cup v_{j+1}$.

As in (b) in 3.30, we therefore obtain for each level q a set of line-segments – and points – $a_1^q, \dots, a_{i_q}^q$ lying on L_{s+i}. To each of these $a_{j_q}^q$, we precisely as in 3.30 obtain a weight $\omega_{j_q}^q$.

In effect, we have a list $(a_1^1, \dots, a_{i_1}^1, \dots, a_1^{n-1}, \dots, a_{i_{n-1}}^{n-1})$ of points on the line-segment L_{s+i}, with corresponding weights. We are however need to modify the list, before we can compute the weighted average.

Each $a_{j_q}^q$ is obtained from some little disk embedding – call it $f_{j_q}^q$ – arising as one of the little disk embeddings of a label $f_v \in f\mathcal{D}isk_2(|v|-1)$ of the vertex v – at level q in $T_f(n-1) \cup v_{j+1}$. To have some notation, we can consider $f_{j_q}^q : D^2 \to D^2$ as being the framed little disk embedding coming from the vertex v^{j_q} – possibly a leaf of $T_f(n-1) \cup v_{j+1}$ – that connects to v via $u(v^{j_q})$.

Let $E(w,v)$ denote the set of edges along the (shortest) path in the tree indexing $T_f(n-1) \cup v_{j+1}$ from the vertex w to the vertex v (v and w can be leafs in which case we shall denote them by their labelling in $\{1, \dots, k+l-1\}$). Given $e \in E(w,v)$, let $s_e \in [0,1]$ denote the label of e. Let the *accumulated label to v w.r.t. w* be given by

$$s(w,v) := \max_{e \in (E(w,v) \setminus \{u(w), u(v)\})} \{s_e\} \tag{13}$$

We need to distinguish between whether v^{j_q} is a leaf or not, therefore let to a number $x \in \mathbb{R}$, $\lfloor x \rfloor := \max\{x, 0\}$, and set

$$h(s_{v^{j_q}}) := \begin{cases} \lfloor 2s_{v^{j_q}} - 1 \rfloor & \text{if } v^{j_q} \text{ is an internal vertex of } T_f(n-1) \cup v_{j+1} \\ 1 & \text{if } v^{j_q} \text{ is a leaf of } T_f(n-1) \cup v_{j+1} \end{cases}$$

Here, as usual $s_v \in [0,1]$ denotes the labelling of the edge outgoing edge $u(v)$ from v.

Let now w denote the vertex at the highest level, such that both v^{j_q} and the leaf labelled $i+s$ are on branches of w, we set

$$\omega_{j_q}^{\prime q} := \lfloor (h(s_{v^{j_q}}) - \lfloor 2s(v^{j_q}, w) - 1 \rfloor \rfloor \cdot \lfloor 1 - \lfloor 2s(i+s, w) - 1 \rfloor \rfloor \omega_{j_q}^q. \tag{14}$$

We let $\omega_{j_q}^{\prime q}$ replace the weight $\omega_{j_q}^q$ as the weight corresponding to $a_{j_q}^q$.

The important notes to make here, is that

- If $s_{v^{j_q}} < \frac{1}{2}$, then $\omega_{j_q}^{\prime q} = 0$. That is, if $f_{j_q}^q$ participates in an external deformation, we let the weight of $a_{j_q}^q$ have weight zero.
- If $s(i+s, w) = 1$ or $s(v^{i_q}, w) = 1$, then $\omega_{j_q}^{\prime q} = 0$. I.e. if any edge on the shortest path from i to v is 1, the weight of $a_{j_q}^q$ has weight zero.

We let $S_{s+i} := \{c(a_u^q), \omega_u^{\prime q}\}$, and thereby can give meaning to taking the weighted average $\alpha(S_{s+i})$ corresponding to B_{s+i} over the line segment L_{s+i}.

Note that we can extend the line segment L_{s+i} to include all the framed little disk embeddings that $B_{v_j}^s$ contains – by considering $B_{v_j}^s \cup B_{s+i}$ instead – the added line-segment has no influence on the computation of the weighted average. For lobes that at $s_{v_j} = \frac{1}{2}$ has b_{i+s} computed via $B_{v_j}^s$, we let

$$\tilde{b}_{i+s} := 2s_{v_j}\alpha(S^s_{v_j}) + \left(1 - 2s_{v_j}\right)\alpha(S_{s+i}) \tag{15}$$

Similarly in other cases, i.e. where b_{s+i} is computed for $s_{v_j} = \frac{1}{2}$, via some $B^{\frac{1}{2}}_{s+i} := \ell(h_{s+i}, c_{v_{j+1}})$ as some $\alpha(S^{\frac{1}{2}}_{s+i})$, such that $B^{\frac{1}{2}}_{s+i}$ intersects non-trivially with framed disk embeddings of f_{v_j}. We can again expand L_{s+i} from above to include all the lobes that $B^{\frac{1}{2}}_{s+i}$ intersect, and set

$$\tilde{b}_{i+s} := 2s_{v_j}\alpha(S^{\frac{1}{2}}_{s+i}) + \left(1 - 2s_{v_j}\right)\alpha(S_{s+i}) \tag{16}$$

Remark 3.42 The idea for proving continuity for Γ is the same as for the map $\tilde{\gamma}_k$ that we saw in 3.35. We just have to make a similar ordering for each set of labellings of edges at level q for all q. As the assignment of each b_i is continuous in the labels s_e of the edges of T_f as well, and by construction – when $s_{v_j} = \frac{1}{2}$, transitioning between internal and external deformation of f_{s+i} gives the same computation of b_i in either case. Therefore one can obtain a result for Γ – similar to 3.33 as well.

Proposition 3.43 $\Gamma\colon \mathcal{W}\left(f\mathcal{D}isk_2\right) \to \mathcal{C}acti_1$, obtained as the last part of a factorization

$$\mathcal{T}ree(f\mathcal{D}isk_2) \to \mathcal{W}\left(f\mathcal{D}isk_2\right) \to \mathcal{C}acti_1$$

of Φ in 3.41 is a morphism of operads.

Proof. We simply need to check that Φ given in 3.41 satisfies the relations of 2.65.

Assume we are given $T_f \in \mathcal{W}\left(f\mathcal{D}isk_2\right)$.

We start by verifying (1) of 2.65, so assume T_f has an edge $u(v)$ labelled by $s_v = 1$. In particular, this means that for any two vertices x, w with $u(v) \in E(x,w)$, we have $s(v,w) = 1$.

Denote by $T_f|u(v)$ the branch of T_f emanating from $u(v)$ – that is – with $u(v)$ the edge going down to the root (with all labellings, except s_v of T_f retained). Similarly, denote by $T_f^{u(v)}$ the subtree of T_f given by replacing the entire branch $T_f|u(v)$ by a edge going to a leaf labelled s. (with all labellings, except s_v retained).

Computing $\Phi(T_f)$ and $\Phi(T_f^{u(v)})$, we see by the formula (14) (or the second bullet below that) that all intersection tuples $[t_1, b_1], \ldots, [t_{k-m}, b_{k-m}]$ of $\Phi(T_f^{u(v)})$ agree with the corresponding list of intersection tuples of $\Phi(T_f)$, since the only difference between the weighted averages used to compute b_j is some weight zero points, which by 3.28 has no effect, as the c_w-center of masses in $T_f^{u(v)}$ agree with the ones in T_f, we have that the t_j agree as well.

Since the pinching map of $\Phi(T_f^{u(v)})$ is computed via the formula (11), it follows that the labellings of $T_f|u(v)$ has no effect on the pinching map of $\Phi(T_f^{u(v)})$, so by definition of operadic composition in the $\mathcal{C}acti_1$ operad, it follows that

$$\Phi(T_f) = \Phi(T_f^{u(v)}) \circ_s \Phi(T_f|u(v))$$

To check formula (2) of 2.65, assume that for a vertex v_i, that the label s_{v_i} of $u(v_i)$ is given by $s_{v_i} = 0$, let $u(v_i)$ be connecting v_i to v_j. As above, denote by $T_f|u(v_i)$ the branch emanating from $u(v_i)$. Again as above, we let $u(v_i)$ replace the edge to the leaf labelled s of the tree $T_f^{u(v_i)}$.

$\Phi(f_1, t_2, f_2, \dots, t_j, f_j, \dots, 0, f_i, \dots, t_k, f_k)$ has v_j-center of mass equal to the v_i-center of mass, by (12). Therefore, all intersection points called t_i agree with $\Phi(f_1, t_2, f_2, \dots, t_j, f_j \circ_s f_i, \dots, t_{i-1}, f_{i-1}, t_{i+1}, f_{i+1}, \dots, t_k, f_k)$

Using formula (15) and (16), we have that the b_v for v a vertex of $T_f|u(v_i)$ gives the same as for $\Phi(f_1, t_2, f_2, \dots, t_j, f_j \circ_s f_i, \dots, t_{i-1}, f_{i-1}, t_{i+1}, f_{i+1}, \dots, t_k, f_k)$ as well.

For vertices of $T_f^{u(v_i)}$, we use (14) – or the first bullet below that. And by definition of the pinching map for vertices undergoing external deformation, we have that indeed formula (2) of 2.65 holds.

Asumme that v_j is labelled by the identity morphism $\mathbb{1}\colon D^2 \to D^2 \in \mathcal{fDisk}_2(1)$, i.e. the identity of $\mathcal{fDisk}_2$, we have that $u(v_{j+1})$ is connecting to v_j.

Obviously, the only difference between $\Phi(f_1, t_2, f_2, \dots, t_j, \mathbb{1}, t_{j+1}, f_{j+1}, \dots, t_k, f_k)$ and $\Phi(f_1, t_2, f_2, \dots, t_{j+1}, f_{j+1}, \dots, t_k, f_k)$, lies in the computation (14), since for a vertices v in $T_f|u(v_{j+1})$ and w a vertex in $T_f^{u(v_j)}$, we have that a path from v to w automatically passes through both $u(v_j)$ and $u(v_{j+1})$, so by (13), formula (3) of 2.65 holds.

(4) of 2.65 follows since if we let $\sigma \in \Sigma_k$ act on a label $f_v \in \mathcal{fDisk}_2(|v| - 1)$ of the vertex of v, by permuting the labellings of the little disk embeddings. We have that if v is at level q, that this permutes the leafs according to σ. In effect we have that (4) of 2.65.

Similar to checking (4) of 2.65 above, we have that acting by Σ_k on $\mathcal{W}(\mathcal{fDisk}_2)(k)$ is given by permuting the labellings of the leafs of the given tree. Under Φ, the labellings of the leafs determines the labellings of the corresponding lobes of the cactus, and as the action of Σ_k on $\mathcal{Cacti}_1(k)$ permutes the lobes, we have Σ-equivariance, commutes, checking that indeed Γ is a morphism of operads.

$\square$

Theorem 3.44 The operads $\mathcal{fDisk}_2$ and $\mathcal{Cacti}_1$ are related via the following diagram consisting of local equivalences of operads:

$$\begin{tikzcd} & \mathcal{W}(\mathcal{fDisk}_2) \arrow[dl, "\varepsilon"'] \arrow[dr, "\Gamma"] & \\ \mathcal{fDisk}_2 & & \mathcal{Cacti}_1 \end{tikzcd}$$

Proof. By 2.64, we just need to check that Γ indeed is a local equivalence of operads.

At each n, $\Gamma(n)\colon \mathcal{W}(f\mathcal{D}isk_2)(n) \to \mathcal{C}acti_1$ can be related via a homotopy

$$\Psi_t\colon \mathcal{W}(f\mathcal{D}isk_2)(n) \times I \to \mathcal{W}(f\mathcal{D}isk_2)(n)$$

that is given by scaling the labels of the edges of elements in $\mathcal{W}(f\mathcal{D}isk_2)(n)$ with $t \in [0,1]$. Thereby, we have $\Psi_1 = \Gamma(n)$ and Φ_0 given as γ_n mapping out of the n-corolla, which is the result of the series of operadic compositions having all edges labelled by zero. From 3.38 we have that $\Gamma(n)$ is homotopic to a homotopy equivalence, and hence itself a homotopy equivalence.

$\square$

Corollary 3.45 There is a morphism of operads

$$\Psi\colon \mathcal{BV} \to \mathrm{H}_*(\mathcal{C}acti_1)$$

such that if we let $\mathrm{H}_*(-)$ have coefficients in a field, Ψ becomes an isomorphism of operads.

Proof. Combine 3.44, 2.57 and 2.17 For the statement with coefficients in a field, consider 2.56 as well. $\square$

4 Cacti 'Acting' on Free Loop Spaces

As we have seen, operads are interesting because actions of them control algebraic structure of the object they act upon. In the previous chapter, we worked with the cacti-operad. In this chapter, we show why the cacti-operad is interesting; we show how to construct an action of $\mathrm{H}_*(\mathcal{C}\boldsymbol{acti}_1)$ on $\mathbb{H}_*(LM)$, where LM is the free loop space over a smooth orientable manifold, M. Indeed this action is gives rize to the celebrated Chas-Sullivan loop product on $\mathbb{H}_*(LM)$.

4.1 Pullbacks

In this section, we give some – rather basic – connections between pullbacks and fiber-bundles, needed for later use.

We shall need explicit computations in pullbacks, so we start by recalling their construction in the category Top:

Definition 4.1 Suppose we are given a diagram of topological spaces

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

We define the *pullback space* of this diagram to be the space

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

We define the *pullback* of the diagram to be the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f_g"] \arrow[d, "g_f"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

where f_g and g_f are the restrictions to $X \times_Z Y$ of the projections $X \times Y \to Y$ and $X \times Y \to X$.

By definition of the pullback space, the diagram commutes

Proposition 4.2 Assume that we have a fiber bundle $p\colon E \to B$ with fiber F, and a map $f\colon X \to B$. Then in the pullback diagram

$$\begin{tikzcd} X \times_B E \arrow[r, "f_p"] \arrow[d, "p_f"] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

p_f is a fiber bundle with fiber F.

In this case the pullback space $X \times_B E$ is often denoted $f^*(E)$, and p_f is called the *pullback bundle* $p_f\colon f^*(E) \to X$.

In the proof, we shall give an explicit formula for the local trivializations; which basically says that local trivializations of p_f are inherited from local trivializations of p. There will be applications using the explicit description of the local trivializations given in the proof below.

Proof. Note first of all that since p is surjective, $f^*(E)$ consists of – to any $f(x)$ – tuples (x,e) such that $f(x) = p(e)$.

We want to find a local trivialization of p_f. We have by assumption a local trivialization of p, i.e. a covering $\{U_i\}_{i\in I}$ of B, and local trivializations $\tau_i\colon U_i \times F \to p^{-1}(U_i)$.

The family $\{f^{-1}(U_i)\}_{i\in I}$ is an open covering of X, and we can define local trivializations

$$\tilde{\tau}_i\colon f^{-1}(U_i) \times F \to p_f^{-1}\left(f^{-1}(U_i)\right) \subseteq f^*(E)$$

by setting

$$\tilde{\tau}_i(x,y) = (x, \tau_i(f(x), y)). \tag{17}$$

The image of $\tilde{\tau}_i$ is in $f^*(E)$, since by local triviality $p(\tau(f(x),y)) = f(x)$. Furthermore, the image of $\tilde{\tau}_i$ is in $p_f^{-1}\left(f^{-1}(U_i)\right)$, as τ_i maps into $p^{-1}(U_i)$. Local triviality follows as we to $(x,e) \in f^*(E)$ with $f(x) = p(e) \in O_i$ can define an inverse

$$\tilde{\tau}_i^{-1}\colon p_f^{-1}\left(f^{-1}(U_i)\right) \to f^{-1}(U_i) \times F$$

by setting

$$\tilde{\tau}_i^{-1}(x,e) = (x, (\tau_i^{-1}(e))_2) \tag{18}$$

where we use the notation $(-)_i$ for the projection onto the i'th coordinate in a cartesian product.

For $(x,e) \in p_f^{-1}(f^{-1}(U_i)$, we have $p(e) = f(x)$, so

$$\tilde{\tau}_i \circ \tilde{\tau}_i^{-1}(x,e) = \tilde{\tau}_i(x, (\tau_i^{-1}(e))_2) = (x, \tau_i(f(x), (\tau_i^{-1}(e))_2)) = (x, \tau_i(p(e), (\tau_i^{-1}(e))_2) = (x,e)$$

where the last equality follows by local triviality of τ_i. As furthermore

$$\tilde{\tau}_i^{-1}\tilde{\tau}_i(x,y) = (x, (\tau_i^{-1} \circ \tau_i(f(x),y))_2) = (x, (f(x),y)_2) = (x,y)$$

we have indeed defined inverse homeomorphisms.

As the first coordinate remains fixed, $\tilde{\tau}_i$ is, by definition of p_f, locally trivial. ☐

Definition 4.3 Let $p\colon E \to B$ be a rank k vector bundle. An *orientation* of p is given as a function that to each $b \in B$ assigns an orientation of the vector spaces $p^{-1}(\{b\}) \cong \mathbb{R}^k$, subject to the local compatibility condition saying that B has a covering of locally trivial neighborhoods $\{U\}$ such that we – using locally trivial coordinates $\tau\colon U \times \mathbb{R}^k \to p^{-1}(U)$ – have that the linear map A_b from $\mathbb{R}^k$ (with fixed orientation) to $p^{-1}(\{b\})$ given by $r \mapsto \tau(b,r)$ is *orientation preserving*, in the sense that $\det(A_b)$ has the same sign for all b.

We say that p is *orientable* if there exists an orientation for p.

Proposition 4.4 Let $f\colon X \to B$, assume that $p\colon E \to B$ is a fiber-bundle, with fiber F. For the pullback bundle $f_p\colon f^*(E) \to E$, we have the following:

(a) if f is a topological embedding, then $f_p\colon f^*(E) \to E$ is a topological embedding.

(b) if f is a topological embedding, and p is an orientable vector bundle of rank k, then p_f is an orientable vector bundle of rank k.

Proof. To prove (a), we need to show that f_p is an injective open mapping.

Injectivity follows as to $(x,e),(x',e') \in f^*(E)$, assuming $e = e'$, we get $f(x) = p(e) = p(e') = f(x')$, so it follows from injectivity of f that $(x,e) = (x',e')$.

To see that f_p is open, pick an open set $U \subset B$, that has a local trivialization τ w.r.t. p. The pre-image $f^{-1}(U)$ is an open neighborhood of X that trivializes p_f via $\tilde{\tau}$ given in the proof of 4.2. By local triviality, and the formula (17) of 4.2, using the fact that f_p is the projection to the second coordinate, we get that

$$f_p(\tilde{\tau}(x,y)) = (x, \tau(f(x),y))_2 = \tau(f(x),y)$$

so we have the following commutative square:

$$\begin{tikzcd} f^{-1}(U) \times F \arrow[r, "\tilde{\tau}"] \arrow[d, "f \times \mathbb{1}"'] & p_f^{-1}(f^{-1}(U)) \arrow[d, "f_p"] \\ U \times F \arrow[r, "\tau"] & p^{-1}(U) \end{tikzcd}$$

showing that f_p is open, as f is.

To see (b), fix an orientation of p. Pick $b \in f(X) \cap B$, and choose the same orientations for $p_f^{-1}(f^{-1}(b)) \cong p^{-1}(b) \cong \mathbb{R}^k$. For a locally trivial neighborhood U of p, with local trivialization $\tau\colon U \times \mathbb{R}^k \to p^{-1}(U)$, the neighborhood $f^{-1}(U)$ is locally trivial for p_f. We see that the map $r \mapsto \tilde{\tau}(f^{-1}(b), r) = (f^{-1}(b), \tau(b,r)) \in p_f^{-1}(f^{-1}(b))$, is orientation preserving as $r \mapsto \tau(b,r)$ is. □

4.2 Free Loop Spaces

Throughout this section, we let M denote a smooth orientable manifold of dimension d.

As mentioned in the introduction, by the *free loop space* on M, we will understand the space of unpointed continuous maps to M, $LM := \mathrm{Map}(S^1, M)$.

Definition 4.5 Given a representative c of a k-pre-cactus (i.e. we fix potential intersection clusters). To c, there are by definition, upon the k lobes of $S(k)$ in total $k-1$ intersection tuples $[t_1, b_1], \dots, [t_{k-1}, b_{k-1}]$ and in total $2(k-1)$ intersection points. We define a *cactus-evaluation* map

$$\mathrm{ev}_c\colon (LM)^k \to M^{2(k-1)}$$

ev_c is given by identifying the i'th lobe S^1_i of $S(k)$ with the domain of the i'th loop $l_i\colon S^1 \to M$ of $(l_1,\ldots,l_k) \in (LM)^k$, and evaluate at all intersection points $p^i_1,\ldots,p^i_d$ lying on S^1_i. Doing so for all i, we obtain a continuous map $\mathrm{ev}_c(l_1,\ldots,l_k) \in M^{2(k-1)}$ given by the $2(k-1)$ points at each loop l_i by $l_i(p^i_1),\ldots,l_i(p^i_d)$.

For later purposes, we order the $2(k-1)$ intersection points such that under the diagonal map $\Delta\colon M^{k-1} \to M^{2(k-1)}$ a x_i of $(x_1,\ldots,x_{k-1}) \in M^{k-1}$ hit points of $\mathrm{Im}(ev_c)$ arising from the same intersection tuple.

The fiber $\mathrm{ev}_c^{-1} i(\{(x_1,\ldots,x_{2(k-1)})\})$ will be denoted $\Omega^c_{x_1,\ldots,x_{2(k-1)}}X$ and it is given by maps $f\colon S(k) \to M$, such that the image under f of the i'th intersection point is mapped to x_i.

Before proving anything about ev_c, we need a technical construction – a map that pushes points around in the unit disc, but is constant at the boundary. There are many ways to define such maps, the following will do for us.

Let D^n denote the closed unit-disk, $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}$.

Lemma 4.6 Let $\mathrm{Homeo}(D^n)$ denote the space of self-homeomorphisms of the disk. Letting $\Phi\colon D^n \backslash \partial D^n \to \mathrm{Homeo}(D^n)$ be given by

$$v \mapsto \Phi_v,\ \Phi_v(s) = s + (1 - \|s\|)v$$

indeed defines a continuous map.

Proof. Φ_v goes into D^n, since

$$\|\Phi_v(s)\| = \|s + (1-\|s\|)v\| \leqslant \|s\| + (1-\|s\|)\|v\| \leqslant \|s\| + 1 - \|s\| = 1,$$

and Φ_v is obviously continuous.

We now check that Φ_v is a bijection. Let $S_k := \{x \in \mathbb{R}^n \mid \|x\| = k\}$. Consider $\Phi_v(S_k)$; that is, the image under Φ_v of some circle centered around 0. For x with $\|x\| = k$, $\Phi_v(x) = x + (1-k)v$; that is, $\Phi(S_k)$ is a circle of radius k centered around $(1-k)v$. Restricted to each S^k, Φ_v is by this justification a bijection.

Let $k > k'$, and $s \in S_k$, and $s' \in S_{k'}$.

We get by the inverse triangle inequality that

$$\begin{aligned}\|\Phi_v(s) - \Phi_v(s')\| = \|s - s' - (k-k')v\| \geqslant \big|\|s-s'\| - (k-k')\|v\|\big| &\geqslant \\ \big|(k-k') - (k-k')\|v\|\big| &> 0\end{aligned}$$

where the last inequality follows as $\|v\| < 1$. Therefore $\Phi_v(S_k) \cap \Phi_v(S_{k'}) = \varnothing$, showing that Φ_v is injective.

Furthermore, we get from the above that for D_k a disk of radius k, centered at 0, $\Phi_v(D_k)$ is the disk of radius k centered around $(1-k)v$, so as $\Phi_v(D_1)$ is the unit disk centered at 0, it follows that Φ_v is surjective.

So as Φ_v is a continuous and bijective map between compact hausdorff spaces, it is a homeomorphism, and by construction, Φ_v depends continuously on v. □

Theorem 4.7 Suppose that M is an n-manifold, then

$$\mathrm{ev}_c\colon (LM)^k \to M^{2(k-1)}$$

is a fiber-bundle.

In order to handle the local information, we first need to introduce some notation; let M_U^A denote the space of maps $\gamma\colon A \to M$ where A is an open subset of $[0,1]$ such that $\frac{1}{2} \in A$ with $\gamma(\frac{1}{2}) \in U$, and furthermore extending γ to $\overline{\gamma}\colon \overline{A} \to M$, we require $\overline{\gamma}(\partial\overline{A}\backslash\{0,1\}) \in \partial\overline{U}$.

Proof. We use the notation of 4.5. Along the domain of a loop l_j, there is a number – between 1 and $k-1$ – of points p_j^i, such that $l_j(p_j^i) = x_i$. Fix neighborhoods $U_i^j \subset S_j^1$ centered around p_j^i, such that $U_i^j \cap U_k^j = \varnothing$ if $p_j^i \neq p_j^k$.

Let $(x_1, \ldots, x_{2(k-1)}) \in M^{2(k-1)}$ be given. Choose $V_1, \ldots, V_{2(k-1)}$ such that V_i is a neighborhood of x_i, with $\varphi_i\colon V_i \to D^n$ a homeomorphism such that $\varphi_i(x_i) = 0$. Set $V := V_1 \times \cdots \times V_{2(k-1)}$. We desire a local trivialization ξ; i.e. satisfying that the following diagram commutes

$$\begin{tikzcd}
V \times \Omega^c_{(x_1,\ldots,x_{2(k-1)})}M \arrow[rr, "\xi"] \arrow[dr, "\mathbb{1}\times 0"'] & & ev_c^{-1}(V) \arrow[dl, "\mathrm{ev}_c"] \\
& V &
\end{tikzcd}$$

Note that $\mathrm{ev}_c^{-1}(V)$ is given by curves $f\colon S(k) \to M$ satisfying that the i'th intersection point of c is mapped to V_i.

Each $l_j \in \Omega^c_{(x_1,\ldots,x_{2(k-1)})}M$ has $l_j(p_j^i) = x_i$, and if we let l_j^i denote the curve where we first restrict l_j to U_i and then co-restrict the image to V_i, we get $l_j^i\colon A_j^i \to V_i$. We identify U_j^i with $[0,1]$ – such that p_j^i becomes identified with $\frac{1}{2}$, and thus A_j^i with an open subset of $[0,1]$, so we have that $l_j^i \in M_{\{x_i\}}^{A_j^i}$.

In order to define ξ, we define a 'local' homeomorphism $\chi_j^i\colon V_i \times M_{\{x_i\}}^{A_i^j} \to M_{V_i}^{A_i^j}$, doing this, we easily extend to all of ξ.

We define χ_j^i by working in local coordinates – i.e. where the curves run under the image of φ^i – in the coordinate ball D^n – and we describe the image of $(l_j^i, v) \in V_i \times M_{\{x_i\}}^{A_j^i}$ by

$$(\chi_j^i)_{(l_j^i)}(v, s) = l_j^i(s) + (1 - \|l_j^i(s)\|)(2s-1)^2 v \tag{19}$$

Fixate s and v, it follows as $(2s-1)^2 v \in \varphi_i(V_i)$, from 4.6, that $(\chi_j^i)_{(-)}(v,s)$ has an inverse, $((\psi_j^i)_{(-)}(s))$.

We have that $((\psi_j^i)_{l_j^i}(\frac{1}{2})) = x_i$. Therefore we can define an inverse of χ_j^i pointwise, by setting $(\chi_j^i)^{-1}_{l_j^i}(s) = \Big(((\psi_j^i)_{l_j^i}(s)), l_j^i(\frac{1}{2})\Big)$.

Having defined the local χ_j^i's, we simply let ξ be given by χ_j^i at each neighborhood U_j^i and constant outside of these neighborhoods.

First of all, as $U^i_j \cap U^k_j = \emptyset$, for different valued intersection points p^i_j and p^k_j, the different χ^i_j are completely independent of each other. From (19) it follows that if $0, 1 \in \overline{A^j_i} \subseteq [0, 1]$ respectively, then $(\chi^i_j)_{l^i_j}(1) = l^i_j(1)$ resp. $(\chi^i_j)_{l^i_j}(0) = l^i_j(0)$, and as furthermore the map $\Phi_v(s)$ of 4.6 becomes the identity as s approaches the boundary, we have that leaving the curve constant outside of U^j_i indeed gives us a continuous curve.

□

Construction 4.8 $\mathcal{C}acti_1(k)$ is decomposed up in cells by 3.10. Pick a cell $\tilde{\Psi} \in F(n_1) \times \cdots \times F(n_m)$ such that $n_1 + \cdots + n_m = k$. $E(k)$ is obtained by gluing top-dimensional cells such as Ψ together. When choosing internal and external basepoint, we obtain Ψ from $\tilde{\Psi}$.

Note that the interior Ψ° of the (top-dimensional) Ψ parametrize cacti without any intersection clusters. By the space $L_\Psi M$, we shall understand the space of pairs $((c, \pi), f_c)$, where (c, π) is a cactus obtained from Ψ°, and f_c are given as *cacti-like loops*; that is maps of the form $f_c \colon S(k)/\sim_c \longrightarrow M$.

We have an evaluation map $\mathrm{ev} \colon L_\Psi M \to M^{k-1}$ evaluating f_c at the $k - 1$ points of $S(k)/\sim_c$ arising from intersection tuples of c.

Furthermore, we have a map

$$\rho_\Psi \colon L_\Psi M \to \Psi \times (LM)^k$$

by $\rho((c, \pi), f_c) = (c, f_c|_{c_1}, \ldots, f_c|_{c_k})$, where $f_c|_{c_i}$ is the morphism $S^1 \to M$ given by restricting f_c to the i'th lobe of c.

Both maps are easily seen to be continuous as evaluations and restrictions along continuous maps (in the compact-open topology) are continuous maps.

Indeed (up to homeomorphism) this structure fits into the following pull-back bundle:

$$\begin{tikzcd} \Psi^\circ \times (LM)^k \arrow[d, "\mathrm{ev}_\Psi"'] & L_\Psi M \arrow[l, "\rho_\Psi"'] \arrow[d, "\mathrm{ev}"] \\ M^{2(k-1)} & M^{k-1} \arrow[l, "\Delta"] \end{tikzcd}$$

where – as usual – Δ is the diagonal embedding, and ev_Ψ is given as $((c, \pi), -) = \mathrm{ev}_c(-)$ defined in 4.5, where (c, π) is given as a point of Ψ°.

The pullback space $\Delta^*(M^{2(k-1)})$, that by definition fits into the pullback square is given as the subspace of $\Psi^\circ \times LM^k \times M^{k-1}$, specified by loops $(l_1, \ldots, l_k) \in LM^k$ evaluating at the intersection points specified by (c, π) in Ψ° such that at the i'th intersection tuple $[t_i, b_i]$, both the evaluation of t_i and b_i evaluates to the i'th entry of M^{k-1}, that is; $x_i \in M$. This is precisely the same as giving a quitent map $S(k)/\sim_c \to M$; hence the identification $\Delta^*(M^{2(k-1)}) \cong L_c M$.

Under this identification, ρ_Ψ and ev are simply the projections that fit into the pullback square, as they should be.

It follows from the proof of 4.7, that ev_Ψ is a fiber-bundle – we are just expanding the fibers to something we could call $\Psi^\circ \times \Omega^\Psi_{(x_1, \ldots, x_{2(k-1)})}$ instead, and the map ξ constructed in the proof generalizes to this situation.

By 4.4, we have that ρ_Ψ is a topological embedding. Note that the maps ρ_Ψ extend to maps by allowing points on the boundary of Ψ as well; the graph $S(k)/\sim_c$ will, under different representatives of c (i.e. different choices of intersection tuples in an intersection cluster), give rise to the same graph, as noted in 3.7. Therefore ρ_Ψ can be pasted together to a continuous map $\rho_{\text{in}}\colon L_{\mathcal{C}acti_1(k)}M \to \mathcal{C}acti_1(k)$. Similarly, we get an extension $\text{ev}\colon L_{\mathcal{C}acti_1(k)}M \to M^{k-1}$ as we are evaluating points of an intersection tuple to be the same, and hence all points of an intersection cluster evaluates to the same value.

As we have chosen M to be a smooth manifold, we get that the normal bundle $\nu_\Delta\colon N(\Delta(M^{k-1})) \to M^{k-1}$ is a rank $d(k-1)$-vector bundle. As the normal bundle is equivalent to the tangent bundle $T(M^{k-1})$, we have as M is chosen to be an orientable manifold that indeed ν_Δ is an orientable vector bundle.

By considering the pullback

$$\begin{tikzcd}
L_{\mathcal{C}acti_1(k)}M \arrow[d, "\text{ev}"] & \text{ev}^*(M^{k-1}) \arrow[l, "(\nu_\Delta)_{\text{ev}}"'] \arrow[d, "\text{ev}_{\nu_\Delta}"] \\
M^{k-1} & N(\Delta(M^{k-1})) \arrow[l, "\nu_\Delta"]
\end{tikzcd}$$

we get from 4.4 that $(\nu_\Delta)_{\text{ev}}$ indeed is an orientable rank $d(k-1)$-vector bundle.

Summarizing up, we have

Proposition 4.9 The composite

$$\text{ev}^*(M^{k-1}) \xrightarrow{(\nu_\Delta)_{\text{ev}}} L_{\mathcal{C}acti_1(k)}M \xrightarrow{\rho_{\text{in}}} \mathcal{C}acti_1(k) \times LM^{k-1}$$

defined in 4.8 has the left-most map a $d(k-1)$-vector bundle and ρ_{in} a topological embedding,

4.3 A Potential Gap in the Argument of [CJ02]

4.9 is close to what we want in order to apply a Pontrjagin-Thom collapse – cf. subsection 4.4 – to obtain an 'action' of $\mathrm{H}_*(\mathcal{C}acti_1)$ on $\mathbb{H}_*(LM)$. However, a slight subtlety still remains, concerning tubular neighborhoods.

Definition 4.10 Assume that X is some topological space, and that $p\colon V \to X$ is a vector bundle of rank k. Assume furthermore that $f\colon X \to Y$ is a topological embedding.

Let $\iota\colon X \to V$ be the inclusion of X as the zero-section in V, i.e., in locally trivial coordinates written as $\iota(x) = (x, 0)$. We say that a map $\varphi\colon V \to Y$ is a (k-)*tubular neighborhood*, if it is a topological embedding, and makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\iota"] & Y \\
V \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Given a tubular neighborhood φ, we call the vector bundle p the *associated vector bundle*

Remark 4.11 It is a standard theorem of differential geometry, that any smooth embedding $f\colon N \to M$ of a smooth manifold inside of another smooth manifold admits a tubular neighborhood, $\varphi_f\colon N(f(N)) \to M$.

It is stated in [CJ02, p.18], that the topological embedding we call ρ_{in} of 4.9 admits a tubular neighborhood.

The argument for this statement seems to be that given a diagram

$$\begin{tikzcd}
W \arrow[dr, "\tilde{\beta}"] \arrow[drr, dashed, "\tilde{\varphi}"] \arrow[ddd, "q"] & & \\
& f^*(E) \arrow[r, "f_p"] \arrow[d, "p_f"] & E \arrow[d, "p"] \\
& X \arrow[r, "f"] & B \\
Z \arrow[ur, "\beta"] \arrow[urr, "\varphi"'] & &
\end{tikzcd} \tag{20}$$

where the right-most square is the pullback of f and p, and the left-most square is obtained as the pullback of β and p_f, and φ is a tubular neighborhood to f (the triangle involving tubular neighborhoods in the diagram does not commute).

Then as indicated in the figure, there should be an induced tubular neighborhood $\tilde{\varphi}$. In [CK07, p.7] it is stated more clearly that in fact, this is the strategy to obtain a tubular neighborhood.

I have however not been able to verify the existence of an induced tubular neighborhood $\tilde{\varphi}$.

In fact, I doubt that the answer to the following question is affirmative in the full generality posed above.

Question 4.12 Is it possible – given a diagram of the form (20) (without the dotted arrow $\tilde{\varphi}$) – to produce a tubular neighborhood to the embedding f_p?

Compared to the results of 4.4, the problem is that we are not given a map $\tilde{\varphi}$ automatically. Below, we give a description of why the – in my opinion – 'obvious' way of defining $\tilde{\varphi}$ fails.

We would define $\tilde{\varphi}$ locally, i.e. fix some locally trivially neighborhood $U_x \subseteq \varphi(Z)$ of $x \in f(X)$. From the proof of 4.2, we have that $\beta^{-1}f^{-1}(U_x)$ are locally trivial neighborhoods for q.

Let $\tau\colon U_x \times F \to p^{-1}(U_x)$ be a fixed trivialization of U_x. From the proof of 4.2, we have an induced trivialization $\tilde{\tilde{\tau}}\colon \varphi^{-1}(U_x) \times F \to q^{-1}(U_x)$. Locally, we can define $\tilde{\varphi}\colon q^{-1}(\varphi^{-1}(U_x) \to p^{-1}(U_x)$ as the composition

$$q^{-1}(\beta^{-1}f^{-1}(U_x)) \xrightarrow{\tilde{\tilde{\tau}}^{-1}} \beta^{-1}f^{-1}(U_x) \times F \xrightarrow{\varphi \times \mathbb{1}} U_x \times F \xrightarrow{\tau} p^{-1}(U_x)$$

As this is a composition of homeomorphisms, this defines a homeomorphism.

However, this only defines $\tilde{\varphi}$ as a local map. In order to globalise it, we would need it to be independent of local trivializations.

Note that we via (18) have an explicit description of $\left(\tilde{\tilde{\tau}}\right)$, whose points in the domain are given through $q^{-1}(\varphi^{-1}(U_x)) \subseteq V \times_X f^*(E) = V \times_X (X \times_B E)$, so we compute:

$$\left(\tilde{\tilde{\tau}}\right)^{-1}(v, x, e) = (v, (\tilde{\tau}^{-1}(x, e))_2) = (v, (x, (\tau^{-1}(e))_2)_2) = (v, (\tau^{-1}(e))_2) \tag{21}$$

Note however that we from the above have

$$\tilde{\varphi}(v, x, e) = (\tau(\varphi(v), (\tau^{-1}(e))_2))$$

Had there instead of $\varphi(v)$ stood $p(e)$, indeed local triviality of τ would have given us that $\tilde{\varphi}$ was independent of the choice of local trivialization. However, this is not the case, and the construction fails miserably.

All my further attempts at pasting this locally defined map together to a globally defined map failed equally miserably.

Indeed, having a result as 4.12, we would – dependent on the nature of the induced tubular neighborhood only need to glue it together along the different choices of cells Φ for $\mathcal{C}acti_1(k)$, similar to what we did in 4.8.

As mentioned, I do not believe that 4.12 holds.

However, in [God07], similar concerns are made. And indeed it seems plausible that in our specific case – where we can lift constructions made in the normal bundle of the diagonal embedding $\Delta\colon M^{k-1} \to M^{2(k-1)}$ to local deformations of the loops evaluating to point in the manifolds, that the answer to the following question should be affirmative:

Question 4.13 Does ρ_{in} admit a tubular neighborhood,

$$\varphi\colon \text{ev}^*(M^{k-1}) \to \mathcal{C}acti_1(k) \times LM^{k-1}?$$

Another approach to answering 4.13, would be to redefine the notion we have given of a free loop space on a smooth manifold M, to

$$L^{H^1}M := \{f \in \text{Map}(S^1, M) \ \mid f \text{ is a } H^1\text{-curve}\}.$$

We refer to [Kli82, p. 159] for what precisely is meant by a H^1-curve. In [Kli82, 2.4.1] it is shown that $L^{H^1}M$ is a Hilbert manifold (that is, the same as a smooth manifold, but instead of requiring the chart to be going into some $\mathbb{R}^d$, we require them to go into some Hilbert-space).

Similar to existence theorems of tubular neighborhoods for differentiable manifolds, there are existence theorems of tubular neighborhoods for Hilbert-manifolds, [Lan02, p.73 Th. 9]. Using this, one could probably show that ρ_{in} transformed to this setting is a smooth embedding, and thereby obtain the tubular neighborhood automatically. The approach of [God07] seems – to me – more appealing, however.

4.4 Some Action on the Other Side of the Gap

Disclaimer 4.14 In this final subsection, we assume that the answer to 4.13 is affirmative.

Definition 4.15 Note that there is a morphism $\rho_{\text{out}}\colon L_{\mathcal{C}acti_1(k)}M \to LM$, simply given by
$$\rho_{\text{out}}((c,\pi),f) = f \circ \pi$$
That is, we via the pinching map $\pi\colon S^1 \to S(k)/\sim_c$ compose with $f\colon S(k)/\sim_c \to M$ to get a loop in M.

This allows us to define a diagram – or *correspondence* – between $\mathcal{C}acti_1(k) \times (LM)^k$ and LM, which we will call ρ:

$$\begin{tikzcd} & L_{\mathcal{C}acti_1(k)}M \arrow[dl, "\rho_{\text{in}}"'] \arrow[dr, "\rho_{\text{out}}"] & \\ \mathcal{C}acti_1(k) \times (LM)^k & & LM \end{tikzcd}$$

A true action of $\mathcal{C}acti_1$ on LM would involve a morphism $\mathcal{C}acti_1(k) \times (LM)^k \to LM$ in the category of topological spaces. The map ρ does not define an action in Top, since one of the arrows go in the wrong direction. Similar to the construction of a Verdier quotient in a triangulated category, one way of letting ρ be an action is to enlarge the set of morphisms in Top.

Definition 4.16 By Corr we will denote the category of *correspondences between topological spaces*. That is, the category given by letting the objects be topological spaces, and a morphism from X to Y is given by a diagram (or *correspondence*)

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} .$$

Composition of morphisms $X \to Y$ and $Y \to Z$ is given – via pullbacks – by the following correspondence between X and Z:

$$\begin{tikzcd} & & W \times_Y V \arrow[dl] \arrow[dr] & & \\ & W \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

Associativity of the composition is trivial, and defining the identity morphism $X \to X$, to be given by the correspondence

$$\begin{tikzcd} & X \arrow[dl, "\mathbb{1}_X"'] \arrow[dr, "\mathbb{1}_X"] & \\ X & & X \end{tikzcd}$$

makes Corr into a category, as the pullback space obtained from composition with the identity is given by the graph of $f\colon X \times_X V = \{(f(v), v) \in X \times V\} \cong V$.

Therefore – by definition of the morphisms in a pullback diagram – composition of a morphism with the identity yields the same morphism.

Construction 4.17 We show in what sense Corr is a symmetrical monoidal category. We can define a functor $F\colon \mathrm{Top} \to \mathrm{Corr}$ by being the identity on objects and map $f\colon X \to Y$ to the morphism

$$\begin{tikzcd} & X \arrow[dl, "\mathbb{1}"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} .$$

We can therefore define the symmetrical monoidal structure in Corr to be $F(X) \times F(Y) := F(X \times Y)$ – F being a functor it easily follows that the symmetric monoidal structure is preserved. In this sense, the symmetrical monoidal structure on $(\mathrm{Corr}, \times)$ is inherited from the cartesian product in $(\mathrm{Top}, \times)$.

As a first application of the $\mathcal{C}acti_1$-operad, we give an action in Corr. Corr is not that interesting a category to topologists – our usual invariants doesn't function properly. However, we shall indeed in 4.25 use its proof to construct an action of $\mathrm{H}_*(\mathcal{C}acti)$ in the category of graded vector spaces.

Proposition 4.18 The correspondence ρ of 4.15 defines an action of $\mathcal{C}acti_1$ on LM in the category of Corr.

Proof. We use the notation of 2.59. In this language, the first diagram of 2.20 translates into requiring that the following diagram in Corr commutes:

$$\begin{tikzcd} \mathcal{C}acti_1(k) \times \mathcal{C}acti_1(l) \times LM^{k+l-1} \arrow[r, "F(\circ_i \times \mathbb{1})"] \arrow[d, "\rho^i"'] & \mathcal{C}acti_1(k+l-1) \times LM^{k+l-1} \arrow[d, "\rho"] \\ \mathcal{C}acti_1(k) \times LM^k \arrow[r, "\rho"] & LM \end{tikzcd} \tag{22}$$

Here the map ρ^i is given by first shuffling the cartesian product, such that ρ^i is given by $\rho\colon \mathcal{C}acti_1(l) \times LM^l \to LM$ along the domain that lies in between the $i-1$'st and $i+l+1$'th copy of LM^{k+l-1}, and being the identity along the rest of the factors in the domain.

In order to check that this diagram is indeed commutative, we have to look at the diagrams involved in the two ways of walking 22.

Let $L_{\mathcal{C}acti_1(k)\circ_i \mathcal{C}acti_1(l)}M$ denote the space of triples $(c_k, \pi_k), (c_l, \pi_l), f)$, where $(c_k, \pi_k) \in \mathcal{C}acti_1(k)$, $(c_l, \pi_l) \in \mathcal{C}acti_1(l)$ and $f\colon S(k+l-1)/\sim_{c_k \circ_i c_l} \to M$ where $c_k \circ_i c_l$ is the result along pre-cacti of the operadic composition $(c_k, \pi_k) \circ_i (c_l, \pi_l)$.

The pullback diagram involved in the lower-left composition, is given by:

$$\begin{tikzcd} L_{\mathcal{C}acti_1(k)\circ_i \mathcal{C}acti_1(l)}M \arrow[r, "p_1"] \arrow[d, "p_2"'] & L_{\mathcal{C}acti_1(k)}M \arrow[d, "\rho_{\mathrm{in}}"] \\ \mathcal{C}acti_1(k) \times LM^i \times L_{\mathcal{C}acti_1(l)}M \times LM^{k-i-1} \arrow[r, "\mathbb{1} \times \rho_{\mathrm{out}} \times \mathbb{1}"] & \mathcal{C}acti_1(k) \times LM^k \end{tikzcd} \tag{23}$$

This is indeed similar to what we saw in 4.8. Recall namely that the composite $(c_k, \pi_k) \circ_i (c_l, \pi_l)$ is given by inserting c_l via $\pi_l \colon S^1 \to S(l)/\sim_{c_l}$ into the i'th lobe of (c_k, π_k), so via a quotient map, we get that the relation in the pullback space, is the same as giving a map $f \colon S(k+l-1)/\sim_{c_k \circ_i c_l} \to M$.

Under this identification, the projection p_1 is given by mapping $f \colon S(k+l-1)/\sim_{c_k \circ_i c_l} \to M$ to the map $f \colon S(k)/\sim_{c_k} \to M$, where c_k is given by using the domain of $\pi_l \colon S^1 \to S(l)/\sim_{c_l}$ as the i'th lobe instead of the lobes of c_l, and forgetting all intersection tuppels arising from (c_l, π_l).

The map p_2 is given as ρ_{in} along the factors where the lower horizontal arrow maps as the identity. Along the last factor, it is the restriction of the c_l-component of $f \colon S(k+l-1)/\sim_{c_k \circ c_l} \to M$.

Similarly, we have that the second composition of (22) gives the diagram

$$\begin{tikzcd}
L_{\mathcal{C}acti_1(k) \circ_i \mathcal{C}acti_1(l)} M \arrow[r, "q_1"] \arrow[d, "q_2"] & L_{\mathcal{C}acti_1(k+l-1)} M \arrow[d, "\rho_{\text{in}}"] \\
\mathcal{C}acti_1(k) \times \mathcal{C}acti_1(l) \times LM^{k+l-1} \arrow[r, "\circ_i \times \mathbb{1}"] & \mathcal{C}acti_1(k+l-1) \times LM^{k+l-1}
\end{tikzcd}$$

where q_1 is simply given as composing the cacti in $(\mathcal{C}acti_1(k) \times \mathcal{C}acti_1(l))$ together. This gives us $\rho_{\text{out}} \circ q_1 = \rho_{\text{out}} \circ p_1$, as $\rho_{\text{out}} \circ p_1$ is simply composing with pinching map in two steps.

q_2 is given in the same way as ρ_{in} of 4.8 by restricting $f \colon S(k+l-1)/\sim_{c_k \circ_i c_l}$ to each lobe on the last factor, and the identity along the two first factors. Note however that this is the same as first composing with p_2, and then mapping identically at the components different from $L_{\mathcal{C}acti_1(l)} M$ and via ρ_{in} along the $L_{\mathcal{C}acti_1(l)} M$ factor. Again, since we are simply applying ρ_{in} in various steps, we have that $q_2 = \rho_{\text{in}} \circ p_2$.

In effect we have that the two compositions of (22) indeed agree, so the diagram commutes.

To verify Σ-equivariance, we see that the diagram

$$\begin{tikzcd}
\mathcal{C}acti_1(k) \times LM^k \arrow[d, "\sigma \times \text{shuffle}_\sigma"] \arrow[dr, "\rho"] & \\
\mathcal{C}acti_1(k) \times LM^k \arrow[r, "\rho"] & LM
\end{tikzcd}$$

commutes as the action of σ on $\mathcal{C}acti_1(k)$ permutes the labellings of the lobes, which has no effect on ρ_{out}, and for ρ_{in}, is precisely the same as shuffling the factors of LM^k according to σ. $\square$

Notation 4.19 To a topological space X, we let X^∞ denote the one-point compactification of X.

Construction 4.20 Let $\beta \colon E \to B$ be a vector bundle of rank k. We want to extend the vector bundle to a bundle $\beta' \colon E' \to B$, where the fiber of β' is given as $S^k \cong F^\infty$, where $F \cong \mathbb{R}^k$ denotes the fiber of β. I.e. we want to 'add' the point ∞ to all fibers.

Dually to pullbacks, we can define pushouts in the category of topological spaces.

To define the desired fiber-bundle, choose a covering $\{U\}$ of B, that give β a local trivialization ξ_U. Consider the commutative pushout diagram

$$\begin{tikzcd}
U \times \mathbb{R}^k \arrow[rr, "\xi_U"] \arrow[dr, "\mathbb{1}\times 0"] \arrow[dd, hook] & & \beta^{-1}(U) \arrow[dl, "\beta"'] \arrow[dd, dashed, "\iota_\beta"] \\
& U & \\
U \times S^k \arrow[ur, "\mathbb{1}\times 0"] \arrow[rr, dashed, "\xi'_U"] & & (\beta')^{-1}(U) \arrow[ul, dashed, "\beta'"]
\end{tikzcd} \tag{24}$$

where β' is obtained as the unique map, we get by the universal property of the pushout of the diagram.

Note that ξ'_U is indeed a local trivialization. Obviously, ξ'_U is a homeomorphism, as ξ_U is a homeomorphism. Local triviality follows by commutativity of the diagram.

Uniqueness of β' implies that it is independent of the choice of local trivializations, so it extends to $\beta' \colon E' := \bigcup_{U \in \{U\}} p'^{-1}(U) \to B$.

In effect, $\beta' \colon E' \to B$ is in fact a fiber-bundle with fiber S^k. We call β' the *sphere-bundle.*

The construction of a sphere-bundle is natural in the sense that if we are given a morphism of vector bundles $\beta \to \hat{\beta}$

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & B \arrow[d] \\
\hat{E} \arrow[r] & \hat{B}
\end{tikzcd}$$

it gives indeed rise to a morphism of sphere-bundles, via a construction of a morphism of diagrams of the form (24).

Note that we have an inclusion $\iota \colon B \to E'$ given in local coordinates as $\iota(b) \to (b, \infty)$. We call the quotient $\mathrm{Th}(\beta) := E'/\iota(B)$ the *Thom space of β*

As a warm-up for a Serre Spectral Sequence argument, recollect from the proof of [Hat02, 4.61] that if $\beta \colon E \to B$ is a fibration it gives an action of $\pi_1(B)$ on $\mathrm{H}_*(F)$. Using the notation of the proof, this action is given by to a loop $\gamma \in \pi_1(B)$ associating a map $L_\gamma \colon \beta^{-1}(\{\gamma(0)\}) \to \beta^{-1}(\{\gamma(0)\})$.

Lemma 4.21 Let $\beta \colon V \to B$ be an orientable rank k vector bundle. For $\beta' \colon E' \to B$, with fiber $F \cong S^k$ the action of $\pi_1(B)$ on $\mathrm{H}_*(F)$ is trivial.

Proof. Choose an orientation of $\beta^{-1}(\{b\})$ for each b, this simply amounts to choosing a generator of $\mathrm{H}_k(\beta'^{-1}(\{b\}))$ for each b.

Pick a curve $\gamma \in \pi_1(B)$, and cover $\gamma([0,1]) \subseteq B$ with finitely many locally trivial neighborhoods of B, where the covering has the local compatibility of 4.3. Enumerate the locally trivial neighborhoods in some way $U_1, \ldots, U_n$, such that $\gamma([0,1]) \cap U_i \cap U_{i-1} \neq \emptyset$. Pick points $t_1, \cdots, t_n \in [0,1]$, such that $\gamma(t_i) \in U_i \cap U_{i-1}$.

Following the notation of [Hat02, 4.61], for the homotopy equivalence $\tilde{g}_t\colon F_{\gamma(0)} \to F_{\gamma(t)}$ we have by definition of local compatibility that the induced map of $\tilde{g}_{t_i}$ maps to the same sign of the generator of $\mathrm{H}_k(\beta'^{-1}(\{\gamma(t_i)\})$ as the induced map of $\tilde{g}_{t_{i-1}}$ does to $\mathrm{H}_k(\beta'^{-1}(\{\gamma(t_{i-1})\}))$. Exhausting this fact out until we reach $n = i$, we conclude that $(L_\gamma)_*\colon \mathrm{H}_*(F) \to \mathrm{H}_*(F)$ is the identity. □

We are thus ready to prove the Thom isomorphism theorem.

Theorem 4.22 Let $\mathrm{H}_*(-)$ denote homology with arbitrary coefficients.

Assume that $\beta\colon E \to B$ is an orientable rank k vector bundle. There is a natural isomorphism $T\colon \mathrm{H}_*(B) \cong \mathrm{H}_{*+k}(\mathrm{Th}(\beta))$

Proof. Note first of all that the inclusion $\iota\colon B \to E'$ has a section given by β', so in effect the long-exact sequence of the pair $(E', \iota(B))$ leads to split short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_*(B) \arrow[r, "\iota*"] & \mathrm{H}_*(E') \arrow[l, bend left, "(\beta')_*"] \arrow[r] & \mathrm{H}_*(\mathrm{Th}(\beta)) \arrow[r] & 0 \end{tikzcd} \tag{25}$$

We now turn to the Serre Spectral Sequence associated to the bundle β'. First of all, note that by 4.21, that the coefficients of the spectral sequence is trivial. Therefore – by for instance [Hat07, Th. 1.3] – we can apply the Serre Spectral Sequence to $\beta'\colon E' \to B$ of 4.20 without further ado.

The fiber of β' is S^k, and as $\mathrm{H}_*(S^k)$ is nontrivial precisely in degrees $* = 0$ or $* = k$; where it is the coefficient ring (in particular free) so we obtain that

$$E^2_{p,q} \cong \begin{cases} \mathrm{H}_p(B) & q = 0 \text{ or } q = k \\ 0 & \text{otherwise} \end{cases}$$

In effect, there is only possibility for a non-trivial differential at the page $E^{k-1}_{*,*}$.

Considering the morphism of vectors bundles:

$$\begin{tikzcd} E' \arrow[r, "\beta'"] \arrow[d, "\beta'"] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd} \tag{26}$$

Note that if we apply H_* to the diagram, we obtain by (25) a diagram as

$$\begin{tikzcd} \mathrm{H}_*(\mathrm{Th}(\beta)) \oplus \mathrm{H}_*(B) \arrow[r, "0\times\mathbb{1}"] \arrow[d, "0\times\mathbb{1}"] & \mathrm{H}_*(B) \arrow[d, "\mathbb{1}"] \\ \mathrm{H}_*(B) \arrow[r, "\mathbb{1}"] & \mathrm{H}_*(B) \end{tikzcd}$$

Let $\overline{E}$ denote the spectral sequence associated to the lower line of (26). Note that we can identify $\overline{E}^\infty_{p,0}$ with the $\mathrm{H}_*(B)$ summand of $\mathrm{H}_*(E')$; by the proof of [Hat07, Th.

1.3], the lower line $E^1_{p,0}$ is the cellular chain complex of B. In effect, we get that the associated morphism of spectral sequences, $E^{\infty}_{p,0} \to \overline{E}^{\infty}_{p,0}$ (the so-called edge-maps) must be an isomorphism.

Therefore, the lower line of $E^r_{p,0}$ must remain constant for all r. In effect, the potential differentials at page $k-1$ must be zero. Therefore $E^r_{*,*}$ collapses at $r = 2$.

Quotienting the filtration of $\mathrm{H}_*(E')$ by $\mathrm{H}_*(B)$, we can read off the $\mathrm{H}_p(\mathrm{Th}(\beta))$ summand of $\mathrm{H}_p(E')$ as the entry $E^2_{p-k,k} \cong \mathrm{H}_{p-k}(B)$.

Naturality of this isomorphism follows from naturality of the construction of the sphere-bundle, since we effectively from a morphism of vector bundles attain a morphism between short exact sequences of the form (25). □

Convention 4.23 Let M be a manifold of dimension d. We set $\mathbb{H}_*(LM) := \mathrm{H}_{*+d}(LM)$, i.e. $\mathbb{H}_*(LM)$ denotes the graded homology of LM shifted down as many degrees as the dimension of M.

Construction 4.24 Given a morphism $f\colon X \to B$ admitting a tubular neighborhood, $\varphi\colon V \to B$, where V is a rank k vector bundle $\beta\colon V \to B$, we define the corresponding *Pontrjagin-Thom collapse map* $\tau\colon B \to \mathrm{Th}(\beta)$, by letting

$$\tau(x) = \begin{cases} \varphi^{-1}(x) & x \in \varphi(V) \\ \infty & x \notin \varphi(V) \end{cases}$$

By ∞, we here mean the point arising from the quotient with $\iota(B)$ in the definition of 4.20. Since φ is a homeomorphism, this is continuous – by definition of the Thom Space.

Theorem 4.25 Let M be an orientable manifold. Then the map ρ of 4.15 gives rise to an action of $\mathrm{H}_*(\mathcal{C}acti_1)$ on $\mathbb{H}_*(LM)$

Proof. Under assumption that 4.13 holds, we have that $\rho_{\mathrm{in}}\colon L_{\mathcal{C}acti_1(k)}M \to \mathcal{C}acti_1(k) \times (LM)^k$ admits a tubular neighborhood $\varphi\colon W \to \mathcal{C}acti_1(k) \times (LM)^k$, with the associated rank $d(k-1)$ vector bundle $\beta\colon W \to L_{\mathcal{C}acti_1(k)}M$ orientable.

In effect, we have the Pontrjagin-Thom collapse map $\tau\colon\ \mathcal{C}acti_1(k) \times (LM)^k \to \mathrm{Th}(\beta)$.

We therefore have an induced map

$$\tau_* \circ \Theta\colon\ \mathrm{H}_*(\mathcal{C}acti_1(k)) \otimes \mathrm{H}_*(LM)^{\otimes k} \to \mathrm{H}_*(\mathrm{Th}(\beta))$$

where Θ is the natural morphism of 2.15. By further composing with the inverse of the Thom isomorphism of 4.22, and the map $\rho_{\mathrm{out}\,*-d(k-1)}$ induced from 4.15, we get a map

$$\rho_{\mathrm{out}\,*-d(k-1)} \circ T^{-1} \circ \tau_* \circ \Theta\colon\ \mathrm{H}_*(\mathcal{C}acti_1(k)) \otimes \mathrm{H}_*(LM)^{\otimes k} \to \mathrm{H}_{*-d(k-1)}(LM)$$

We claim that this map gives rise to an action of $\mathrm{H}_*(\mathcal{C}acti_1(k))$ on $\mathbb{H}_*(LM)$.

Note that we in the proof of 4.18 have constructed an action in the category of Corr. As the diagram (22) of the proof commutes, we apply homology to that diagram. The maps q_1, p_1 and ρ_{in} are essentially all ρ_{in}, therefore it follows that we can apply $T^{-1} \circ \tau_* \circ \Theta$ to the induced of each of the morphisms.

That the diagrams involved in the proof indeed still commutes follows from naturality of the Thom isomorphism, naturality of Θ, and assuming that the speculative construction of tubular neighborhoods in 4.13 is natural, naturality of the Pontrjagin-Thom collapse maps.

Simlilarly, Σ-equivariance of the action follows, as it does in Corr.

□

In the string topology community there has lately been some concern that the different approaches of defining Batalin-Vilkovisky algebra structures on $\mathbb{H}_*(LM)$ indeed give different structures – nevertheless, we shall call the following corollary the Chas-Sullivan loop-product:

Corollary 4.26 $\mathbb{H}_*(LM)$ carries the structure of a Batalin-Vilkovisky-algebra

Proof. Combine 4.25, 3.45 and 2.42. □

References

[Bar07] Tarje Bargheer, *Graduate project: Konfigurationsrum*, 2007, available upon request.

[Boa71] J. M. Boardman, *Homotopy structures and the language of trees*, Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970), Amer. Math. Soc., Providence, R.I., 1971, pp. 37–58. MR MR0350723 (50 #3215)

[CJ02] Ralph L. Cohen and John D. S. Jones, *A homotopy theoretic realization of string topology*, Math. Ann. **324** (2002), no. 4, 773–798. MR MR1942249 (2004c:55019)

[CK07] Ralph L. Cohen and John R. Klein, *Umkehr maps*, 2007, arXiv.org:0711.0540.

[CLM76] Frederick R. Cohen, Thomas J. Lada, and J. Peter May, *The homology of iterated loop spaces*, Springer-Verlag, Berlin, 1976, Lecture Notes in Mathematics, Vol. 533. MR MR0436146 (55 #9096)

[CS99] Moira Chas and Dennis Sullivan, *String topology*, 1999, arXiv.org:math/9911159.

[CV06] Ralph L. Cohen and Alexander A. Voronov, *Notes on string topology*, String topology and cyclic homology, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2006, pp. 1–95. MR MR2240287

[Get94] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (1994), no. 2, 265–285. MR MR1256989 (95h:81099)

[God07] Veronique Godin, *Higher string topology operations*, 2007, arXiv.org:0711.4859.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR MR1867354 (2002k:55001)

[Hat07] ———, *Spectral sequences in algebraic topology*, Preliminary internet-notes, http://www.math.cornell.edu/ hatcher/SSAT/SSATpage.html, 2007.

[Kau05] Ralph M. Kaufmann, *On several varieties of cacti and their relations*, Algebr. Geom. Topol. **5** (2005), 237–300 (electronic). MR MR2135554 (2006f:55010)

[Kli82] Wilhelm Klingenberg, *Riemannian geometry*, de Gruyter Studies in Mathematics, vol. 1, Walter de Gruyter & Co., Berlin, 1982. MR MR666697 (84j:53001)

[Lan02] Serge Lang, *Introduction to differentiable manifolds*, second ed., Universitext, Springer-Verlag, New York, 2002. MR MR1931083 (2003h:58002)

[Mar06] Martin Markl, *Operads and props*, 2006, arXiv.org:math/0601129v3.

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR MR1712872 (2001j:18001)

[Sin06] Dev Sinha, *The homology of the little disks operad*, 2006, arXiv.org:math/0610236.

[SW03] Paolo Salvatore and Nathalie Wahl, *Framed discs operads and Batalin-Vilkovisky algebras*, Q. J. Math. **54** (2003), no. 2, 213–231. MR MR1989873 (2004e:55013)

[Vor05] Alexander A. Voronov, *Notes on universal algebra*, Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, pp. 81–103. MR MR2131012 (2005k:18017)

[Wah01] Nathalie Wahl, *Ribbon braids and related operads*, 2001, Doctoral Thesis, www.math.ku.dk/˜ wahl.