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Research Proposal
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Novel String Topology Interactions

Abstract: Using diverse methods from Homotopy-, Group- and Knot-Theory, this mathematical project aims to generalise the Chas-Sullivan loop product, leading to an expansion of the foundations of String Topology

My proposed postdoctoral research project lies in the modern mathematical field of homotopy theory. The underlying physical motivation for my work is a profound theoretical problem: the theories describing our universe are divided by an unnatural and seemingly irreconcilable dichotomy between the small scale quantum world and the macro world, shaped relativistically by gravitational forces. String Theory describes the geometry of our universe in a way that incorporates both relativistic cosmology of the universe, and small scale quantum physics. It is widely hoped that String Theory will provide a testable particle description of gravity, unifying the theoretical description of the large scale, relativistic phenomenon with the so called Standard Model which predicts quantum phenomenon with stunning precision.

Homotopy theory provides theoretical tools to effectively deal with complicated geometric structures as those in String Theory. The geometry at the heart of my research will thus hopefully uncover fruitful interpretations of our models of the physical reality.

The model I will work with has its roots in the mathematical field of String Topology. This subject was initiated a decade ago with the appearance of [CS99] in which Chas and Sullivan constructed a surprising mathematical product describing the interaction of strings or particles, however far apart they are in ambient space. Ambient space is here taken to be a finite dimensional manifold, encapsulating relativistic data.

The last ten years has seen substantial mathematical effort towards understanding the complexities of String Topology. The model still falls short, however, of giving an accurate geometric account involving the quantum mysteries of our universe. One shortcoming is that in its present formulation, the Chas-Sullivan product only allows for a single mode of interaction between strings. For instance: from a quantum perspective, particles are entangled and this is not incorporated in the theory as of yet. I contend that we have only just witnessed the birth of String Topology and that Chas and Sullivan have singled out the most basic string interaction in a vast spaghetti soup of possible products.

In the first section below I give a more detailed description of String Topology, with an emphasis on the main results of my PhD-thesis. The next sections contain a specific outline of the tasks I will undertake in my postdoctoral project.

1 Mathematical Background

Taking M to be our cosmological universe, String Topology is concerned with the space of maps from the circle S^1 to M, denoted M^{S^1}. The space M^{S^1} is of an infinite dimensional nature and represents the space of closed strings within our universe. As such it constitutes a version of String Theory.

A main result in my thesis that also appear in my papers [Bar10] and [Bar11] is the following:

Theorem 1.1 $\mathbb{H}_*(M^{S^n})$ carries the structure of an $(n+1)$-Batalin-Vilkovisky algebra for M a closed, oriented, smooth manifold.

Here, $S^n \subseteq \mathbb{R}^{n+1}$ denotes the n-dimensional unit sphere and $\mathbb{H}_*$ the dimension shifted homology. The dimension shift makes it impossible to recover the algebraic structure from M^{S^n} itself, making it ever so surprising to find it in its homological shadow.

For $n = 1$, we have the Chas-Sullivan product [CS99], discussed in the introduction. What I have achieved in Theorem 1.1 is a generalisation to higher dimension. What in physics is called Brane Theory. To obtain this, I have drawn upon connections between String Topology and the theory of operads. Loosely speaking, operads are gadgets parametrizing algebraic structures. In [Vor05], Voronov introduced operads to String Topology by presenting the so-called Cacti operad. This operad was later shown, [CJ02], to give the $n = 1$ case of Theorem 1.1.

The techniques utilized in my thesis are inspired by the relationship between the Cacti operad and the Chas Sullivan product. However, new ideas were needed to formulate a geometrical controllable model in the higher dimensions. These ideas materialised in the language of coloured operads, and given any manifold $N \subset \mathbb{R}^{n+1}$ I have constructed a coloured operad, $\mathcal{C}\mathit{leav}_N$ called the N-Cleavage operad, whose geometric operations serve to cleave N into smaller parts.

One major asset of my construction is that the homotopy type of $\mathcal{C}\mathit{leav}_{S^n}$ is explicitly computable, which can be formulated thus:

Theorem 1.2 The Spherical Cleavage Operad, $\mathcal{C}\mathit{leav}_{S^n}$, is a coloured E_{n+1}-operad. The twisted $\mathcal{C}\mathit{leav}_{S^n} \rtimes \mathrm{SO}(n+1)$ parametrize $(n+1)$-Batalin-Vilkovisky algebras.

The calculation draws upon work of [SW03] and [Ber97]. My [Bar10, Ch. 5] gives the details. Such a computational model has been coveted since the inception of String Topology, see [CV06, 5.1].

2 Quantum String Topology

Knot theory arose Around 1860, when Lord Kelvin put forward a theory of atoms as strings knotted in 'aether' around us. Of course an examination of the periodic table reveals that Kelvin's ideas regarding atoms fizzled out pretty quickly. Knot theory, the study of knotted circles in 3-dimensional space, remained as an active area of research in the domain of pure mathematics. During the last twenty years, knots have begun to reemerge in quantum physics. Vaughan Jones' introduction [Jon85] of the Jones polynomial of a knot as a tool to be used in statistical mechanics, played a key role in this development. Within this framework knots are used to geometrically model physical quantum entanglements. For an exposition of these ideas, see [Kau02].

Passing such quantum ideas on to String Topology would, with a little poetic license, constitute a revival of Lord Kelvin's dreams of elementary particles endowed with an underlying knotted structure; just on a much higher energy level – where atoms are replaced by the sub-atomic building blocks of string theory.

From my thesis, something akin to this emerges. The construction of $\mathcal{Cleav}_N$ works for any embedded manifold. In particular, any knotted embedding $K\colon S^1 \to \mathbb{R}^3$ defines $\mathcal{Cleav}_K$. For every knot we get a seemingly new version of a Chas-Sullivan product similar to that of Theorem 1.1. Somewhat disappointingly, techniques of [God07] can be used to prove that all new products arising hereby are trivial and as such provide no new interactions.

One could have anticipated that the construction, in the above formulation, is too naïve. Indeed, had it yielded interesting products one undesirable ramification would be that even small wigglings of the unit-circle would give new versions of 1.1. In order to have a solid theory, we conclude, the operad needs to be a knot invariant.

Khovanov Homology [Kho00] is a type of knot invariant. A result of my doctoral work is an extension of $\mathcal{Cleav}_K$, using Khovanov Homology an operad is constructed which I call the Gordian Knot Operad and denote $\mathcal{Gord}_K$. The Gordian Knot Operad is itself a knot invariant which acts on the space of all consecutive strings, $M^{\coprod_{\mathbb{N}} S^1}$. Whilst the unit circle in $\mathbb{R}^2$, also known as the unknot, recovers 1.1, the more complicated and knotted the knot becomes, the more entangled the associated product. Happily, this is precisely the type of product sought out by theoretical quantum physicists.

This leads us to the interesting

Question 2.1 What non-trivial operations do $\mathrm{H}_*(\mathcal{Gord}_K)$ provide on $\mathrm{H}_*\left(M^{\coprod_{\mathbb{N}} S^1}\right)$, for K a given knot?

In general, explicit computations in String Topology are very hard. Khovanov Homology, however, is easy to compute, and one sees that $\mathrm{H}_*(\mathcal{Gord}_K)$ grows quickly with the complexity of the knot. Intertwining these computation with a homotopy theoretical device called Minimal Sullivan Models provide a tractable path for answering the above question. The efficiency of Khovanov Homology to discern the complexities of knots, demonstrated over the last decade in for instance [Str05], [Ras10], [PK10], makes it very likely that interesting consequences will be found in analyzing answers to Question 2.1.

The construction of the Gordian Knot Operad prompts a far reaching second question:

Question 2.2 What algebraic structure does $\mathrm{H}_*(\mathcal{Gord}_K)$ introduce to $\mathrm{H}_*(M^{\coprod_{\mathbb{N}} S^1})$?

An answer for the case of the unknot is given by Theorem 1.1; namely the Batalin-Vilkovisky Algebra. However the richness of $\mathrm{H}_*(\mathcal{Gord}_K)$ suggests that a satisfying answer for general knots will be much more elaborate, indeed the sheer number and complexity of knots makes the question impossible to answer in full.

To seep the question for answers, I will below propose three different strategies utilizing different regions of mathematics

Strategy A The work of Lauda and Pfeiffer, [LP09] gives a limit construction for patching together the Khovanov Homology of a cleaved knot. This limit construction dovetail beautifully with my construction of Gord_K. An examination of [LP09] suggests that a partial answer to Question 2.2 could be that $\mathrm{H}_*(M^{\coprod_{\mathbb{N}} S^1})$ is a so-called Knowledgeable Frobenius Algebras.

Strategy B Determining the homotopy type of Gord_K, would in analogy to Theorem 1.2 provide an answer to Question 2.2. Even though knots are too complicated for such a computation, all is not lost. In [HHRT09], an interesting construction known as the k-width of a knot has been devised. This construction distinguishes knots by examining how they can be cleaved. Their notion of cleaving a knot matches precisely the idea of cleaving developed in my thesis work. His construction could be used to create a filtration of Gord_K. This would yield spectral sequences that could compute homology classes of Gord_K and thus give partial answers to Question 2.2.

Strategy C In [Jon99], it is shown how the Jones polynomial of a knot can be thought of as arising from a larger operadic structure, within the theory of von Neumann Algebras, known as the Planar Operad. In [BN05][Ch. 5] Bar-Natan outlines how to produce actions of the planar operad from the data used to produce Khovanov Homology. This relationship hints at the existence of a morphism from Gord_K to the Planar Operad. Such morphisms would be a link back to classical quantum theory, as described by von Neumann Algebras

3 Equivariant Interactions

Groups, an abstract entity encoding symmetries are a cornerstone of many fields of mathematical endeavourIn the field of homotopy theory, great headway has been made in understanding the interplay between groups and spaces, a dynamic referred to as equivariance in the mathematical idiom. As part of my postdoctoral work, I will examine such ideas with the aim of providing new interactions in the domain of String Topology.

In [Wes08], Westerland describes a theory where actions of a group G, on an operad $\mathcal{O}$, which parametrizes products on a G-space X, produces a new operad $\mathcal{O}_G$. The resulting operad, O_G, in turn parametrizes products on the homotopy orbit space X_{hG}. This is a space capturing the symmetries of G on X. Seeing S^1 as a group, Westerland obtains an S^1-equivariant version of 1.1 by acting on the equivariant mapping space $\left(M^{S^1}\right)_{hS^1}$. Hereby he creates a more efficient model of String Theory where parametrizations of strings loose relevance. He then shows that the interactions on $\left(M^{S^1}\right)_{hS^1}$ obtained in this manner are given by the Getzler Gravity Algebra [Get95].

One of the major sticking points of classical String Topology is that the string product is not equivariant. In [Wes08] an ad-hoc resolution to this problem, depending heavily on the one dimensional nature of S^1, is employed. A fruit to be reaped of my thesis is that the action associated to the operad, Cleav_{S^1}, is indeed S^1-equivariant. This equivariance permits a clean transference of Westerland's action to higher dimensional settings, which Theorem 1.1 introduces.

The results from the above observations enables us to pose the following question:

Question 3.1 For G a Lie group describing symmetries of S^n, what is the algebraic structure of $\left(M^{S^n}\right)_{hG}$ given by the G-equivariant operad $(\mathcal{C}\mathit{leav}_{S^n})_G$?

The first case to probe the question on is the orthogonal group $G = \mathrm{SO}(n+1)$. In this case one can reasonably hope for some, yet unknown, generalisation of the Getzler gravity operad. The question becomes significantly harder if we stipulate our Lie group as being unitary or spin groups. In this context there seems to be no hope of analogies from the one dimensional case.

Letting S^n tend to the infinite dimensional sphere, S^∞, would make Theorem 1.1 yield classical intersection theory of manifolds – since $M \simeq M^{S^\infty}$. However, equivariantly we obtain the vital classifying space $BG = S^\infty_{hG}$ for many groups G. We see that in this light, prior results have bearing on interesting problems of modern homotopy theory. Namely the equivariant version of Theorem 1.1 becomes a theorem concerning the mapping spaces M^{BG}. The mapping space M^{BG} lies at the heart of homotopy theory. It is of particular relevance to the Sullivan Conjecture[1] [Mil84], which states that for any finite group G and finite complex M, M^{BG} is contractible. If we take G to be an infinite group such as S^1, and look at M^{BS^1}, the theorem no longer holds. This raises the following sweeping question:

Question 3.2 For G a Lie Group acting freely on S^∞, what does the geometry of $(\mathcal{C}\mathit{leav}_{S^\infty})_G$ tell us about the mapping spaces M^{BG}?

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