LECTURE 9: AXIOMATIC SET THEORY

The rest of the course is dedicated to a formal introduction to the most important theory of all: Set theory.

1. Basic set theory and axioms

1.1. **The iterative concept of set.** Set theory as a mathematical theory springs from the informal notion of a "collection" or "family" of objects. By a *set* we will mean a collection in the sense we will soon formalize, whereas the word "collection" will be used colloquially.

Before introducing the formal axioms, we give an informal (!) description of the set theoretic universe that the axioms intend to describe.

(a) Sets are formed in stages $0, 1, \ldots, s, \ldots$.

(b) For each stage s, there is a next stage $s + 1$.

(c) There is an "absolute infinity" of stages.

(d) V_s is the collection of all sets formed before stage s.

(e) V_0 is the empty collection.

(f) V_{s+1} is the collection of (1) all sets belonging to V_s and (2) all subcollections of V_s not previously formed into sets.

1.2. **The axioms of set theory.** The axioms for our theory will all be first order formulas in the *language of set theory*, LOST, which consists of a single, binary relation symbol $\in$, called *membership*, along with equality and the other logical symbols. As usual, we use x, y, z, u, w, $x_1, x_2, \ldots$, $y_1, y_2, \ldots$, $w_1, w_2, \ldots$, to stand for arbitrary variables, along with our formal variable symbols $v_1, v_2, \ldots$. Note that the atomic formulas of LOST all have the form $x \in y$ and $x = y$ (formally, $\in xy$ and $= xy$). If we want to express something more complicated than that, it will have to be done using formulas of higher complexity.

Abbreviations, such as $\wedge, \vee, \leftrightarrow$, and $\exists$, standing for "and", "or", "if and only if", and "there exists", will be used below. All of these can be expressed using our formal logical symbols $\rightarrow$, $\neg$ and $\forall$. For example, $\exists x \varphi$ is shorthand for $(\neg \forall x(\neg \varphi))$, while $\varphi \vee \psi$ is shorthand for $(\neg \varphi) \rightarrow \psi$. However, our *actual* axioms will be the formal formulas without abbreviations, but it is pointlessly cumbersome to write the axioms without abbreviations. We will also write $x \neq y$ and $x \notin y$ for the negation of the formulas $x = y$ and $x \in y$, respectively.

0. Axiom of Set Existence:

$$\exists v_1 v_1 = v_1.$$

(There is a set.)

1. Axiom of Extensionality:

$$\forall v_1 \forall v_2 (\forall v_3 (v_3 \in v_1 \leftrightarrow v_3 \in v_2) \rightarrow v_1 = v_2).$$

(Sets that have the same members are identical.)

2. Axiom of Foundation:

$$\forall v_1 (\exists v_2 \; v_2 \in v_1 \rightarrow \exists v_2 (v_2 \in v_1 \wedge \forall v_3 \; v_3 \notin v_1 \vee v_3 \notin v_2)).$$

(Every non-empty set has a member which has no members in common with it.)

3. Axiom Schema of Comprehension[1]: For each LOST formula $\varphi(x, z, w_1, \ldots, w_n)$ with the (distinct) free variables among those shown, the following is an axiom:

$$\forall w_1 \cdots \forall w_n \forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge \varphi)).$$

(For any sets z and any property P which can be expressed by a formula of LOST, there is a set whose members are those members of z that have property P.)

4. Axiom of Pairing:

$$\forall v_1 \forall v_2 \exists v_3 (v_1 \in v_3 \wedge v_2 \in v_3).$$

(For any two sets, there is a set to which they both belong, i.e., of which they are both members.)

5. Axiom of Union:

$$\forall v_1 \exists v_2 \forall v_3 \forall v_4 ((v_4 \in v_3 \wedge v_3 \in v_1) \rightarrow v_4 \in v_2).$$

(For any set, there is another set to which all members of members of the first set belongs.)

Remark 1.1. Comprehension, Pairing and Union give us some operations on sets. If $\varphi(x, z, w_1, \ldots, w_n)$ is a formula as in the comprehension schema, and $b, a_1, \ldots, a_n$ are sets, then we write

$$\{x \in b : \varphi(x, b, a_1, \ldots, a_n)\}$$

for the subset y of b guaranteed to exist be the comprehension schema (this notation is in line with the set-builder notation you've used since kindergarten). By the set existence axiom and Comprehension, the empty set $\varnothing$ exists since if x is some (any) set (and there must at least be one), then

$$\varnothing = \{y \in x : y \neq y\}.$$

For any sets x and y, we let $\{x, y\}$ denote the set whose members are exactly x and y; this exists by Pairing and Comprehension. For any set x, let y be the set guaranteed to exist by the Union axiom and which contains every set which is a member of a member of x, and let

$$\mathcal{U}(x) = \{z \in y : \exists w (w \in x \wedge z \in w)\};$$

[1]Comprehension is not a single axiom, but an *axiom schema*, consisting of infinitely many axioms: For each suitable formula φ, there is an axiom (sometimes called an "instance" of the axiom schema). The same is going to be the case for replacement.

this exists by Union and Comprehension. For sets x and y, we let $x \cup y$ be $\mathcal{U}(\{x, y\})$. For sets $x_1, \ldots, x_n$, we let $\{x_1, \ldots, x_n\}$ be the set containing exactly $x_1, \ldots, x_n$. (We could prove this exists by induction on n, but one then has to ask where this induction takes place. At this stage it would take place in the metatheory (which is fine). Only once the Axiom of Infinity is introduced could we endeavour to prove a corresponding internal version within set theory.)

In the statement of the next axiom, the notation $\exists! v_i$ is shorthand for the obvious way of expressing "there is exactly one v_i".

6. Axiom Schema of Replacement: For each LOST formula $\varphi(x, y, z, w_1, \ldots, w_n)$ with free variables among those shown, the following is an axiom:

$$\forall w_1 \cdots \forall w_n \forall z (\forall x (x \in z \rightarrow \exists! y \varphi) \rightarrow \exists u \forall x (x \in z \rightarrow \exists y (y \in u \wedge \varphi))).$$

(For any set z and relation R (which *must* be expressible by some first order formula φ of LOST), if each member x of z bears the relation R to exactly one set y_x, then there is a set to which all these y_x belong.)

Remark 1.2. By Comprehension, Replacement can be strengthened to give

$$\forall w_1 \cdots \forall w_n \forall z (\forall x (x \in z \rightarrow \exists! y \varphi) \rightarrow \exists u \forall x (x \in z \leftrightarrow \exists y (y \in u \wedge \varphi))).$$

For the next axiom, let $\mathcal{S}(x) = x \cup \{x\}$.

7. Axiom of Infinity:

$$\exists v_1 (\varnothing \in v_0 \wedge \forall v_1 (v_1 \in v_0 \rightarrow \mathcal{S}(v_1) \in v_0)).$$

(There is a set that has the empty set as a member and is closed under the operation $\mathcal{S}$.)

For the next axiom, let "$v_3 \subseteq v_1$" abbreviate "$\forall v_4 (v_4 \in v_3 \rightarrow v_4 \in v_1)$".

8. Axiom of Power Set:

$$\forall v_1 \exists v_2 \forall v_3 (v_3 \subseteq v_1 \rightarrow v_3 \in v_2).$$

(For any set, there is a set to which all subsets of that set belong.)

Notation: If x is a set and y is the set guaranteed to exist by Power Set (and which has all subsets of x as members), let

$$\mathcal{P}(x) = \{z \in y : z \subseteq x\}.$$

This exists by Power Set and Comprehension.

For sets x and y, we let

$$x \cap y = \{z \in x : z \in y\};$$

this exists by Comprehension. Note that the formula $z \in x \wedge z \in y$ expresses exactly that fact that z is a member of the intersection of x and y, so we will use $x \cap y$ as shorthand for this formula (with suitable variables) in the next axiom.

9. Axiom of Choice:

$$\forall v_1(\forall v_2 \forall v_3((v_2 \in v_1 \wedge v_3 \in v_1) \rightarrow (v_2 \neq \varnothing \wedge (v_2 = v_3 \vee v_2 \cap v_3 = \varnothing))) \\ \rightarrow \exists v_4 \forall v_5 (v_5 \in v_1 \rightarrow \exists! v_6 v_6 \in v_4 \cap v_5))$$

(If x is a set of pairwise disjoint non-empty sets, then there is a set that has exactly one member in common with each member of x.)

Warning: In naïve set theory (i.e., the set theory of most other math courses), comprehension is often thought of as saying that if we have a set x and a property P then we can form the subset of x consisting of all elements of x with property P. Our formal version of Comprehension is *prima facie* weaker than this, since it only allows this when P is a property expressible by a first order formula of LOST (with finitely many parameters). Likewise, in naïve set theory, Replacement, if it is acknowledged at all, is often thought of as saying that any relation which is "function-like" and has a domain which is a set, has a range which is a set. Again, this is *prima facie* stronger than the formal version, which only expresses this when the relation can be defined using a formula of LOST (with finitely many parameters).

1.3. **Justifications.** The informal iterative concept of set can be used to justify the axioms. (We leave out, for now, the justifications that are assigned for the lecture on Thursday).

0. Set existence: The empty collection is a subcollection of V_0. So $\varnothing$ is formed into a set at stage 1.

1. Extensionality: The iterative concept of set implies that all sets are collections. Since any two collections that have the same elements are identical, the same holds for sets.

2. Foundation: Let $x \neq \varnothing$ be a set. If $\varnothing \in x$ we're done. If not, let s be the first stage where some element $y \in x$ is formed. Then every element of y is formed before stage s, so y and x have no members in common.

3. Comprehension: Given sets $a, b_1, \ldots, b_n$, let s be a stage by which all these sets have been formed already; in particular, all elements of a have been formed. Let $\varphi(x, z, w_1, \ldots, w_n)$ be a formula. Then the collection of elements x in a such that $\varphi(x, b, a_1, \ldots, a_n)$ is a subcollection of V_s, so is formed into a set at stage $s + 1$.

5. Union: Given x, let s be the stage where x is formed. Every element of x is formed at some stage before s, and every element of an element of x is formed at some stage before s. So all elements of elements of x are in V_s. Since V_s is formed into a set at stage $s + 1$, V_s is a set which contains every element of an element of x.

7. Infinity: The fact that there are an "absolute infinity" of stages implies that there is an infinite stage[2], call it s. The subcollection of V_s consisting of all sets formed at finite stages contains $\varnothing$ and is closed under the operation $\mathcal{S}$. This subcollection becomes a set at stage $s + 1$, and so Infinity holds.

[2]Absolute infinity intends to express that there are "boundlessly many" stages, but it *does not* say that there is an ultimate, or last, stage. On the contrary!

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