DST HOMEWORK ASSIGNMENT 3: SMOOTH AND NON-SMOOTH EQUIVALENCE RELATIONS AND THE BECKER-KECHRIS THEOREM

This is the third and last of three mandatory homework assignments. It consists of two problems (the first with multiple parts). You must hand in your solution at the beginning of lecture on Friday, January 18, 2013. Be warned that the second problem is somewhat harder than previously assigned exercises in this class.

We start with a definition that is the basis of both problems. For any set X, we let $=_X$ denote the equality relation in X.

Definition. Let X and Y be standard Borel spaces.

(a) Let E and F be equivalence relations on X and Y, respectively. We say that E is *Borel reducible* to F, written $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that

$$(\forall x, y) xEy \iff f(x)Ff(y).$$

(b) We say that an equivalence relation E on X is *smooth* if $E \leq_B =_{2^\omega}$.

(c) On 2^ω, define the equivalence relation E_0 by

$$xE_0y \iff (\exists N)(\forall n \geq N)x_n = y_n,$$

i.e., xE_0y holds iff x and y are eventually equal.

(d) If E is an equivalence relation on X and $A \subseteq X$, we say that A is *E-invariant* if whenever $x \in A$ and yEx, then $y \in A$.

Problem 1

(1.1) Prove that $=_{2^\omega} \leq_B E_0$. (If you plan to use Mycielski's theorem from HW2, make sure you verify the hypothesis. You are *not* allowed to use Silver's dichotomy theorem.)

(1.2) Prove that any non-meagre E_0-invariant set A with the Baire property must be comeagre.

Hint: For fixed $s, t \in 2^{<\omega}$ with $\ell\mathrm{h}(s) = \ell\mathrm{h}(t)$ the function $F_{s,t} : 2^\omega \to 2^\omega$ defined by

$$F_{s,t}(x) = \begin{cases} x & \text{if } s \perp x \text{ and } t \perp x \\ t\hat{\ }y & \text{if } x = s\hat{\ }y \text{ for some } y \in 2^\omega \\ s\hat{\ }y & \text{if } x = t\hat{\ }y \text{ for some } y \in 2^\omega \end{cases}$$

is a homeomorphism. (You may use this fact without proof.) Observe that for any such s, t and any $x \in 2^\omega$ we have $xE_0F_{s,t}(x)$, and use that category notions are preserved under homeomorphisms.

(1.3) Prove that E_0 is not smooth by showing that there is no Baire measurable $f : 2^\omega \to 2^\omega$ such that

$$(\forall x, y \in 2^\omega) xE_0y \iff f(x) = f(y).$$

Hint: Suppose that f is Baire measurable satisfying

$$(\forall x, y \in 2^\omega) xE_0y \implies f(x) = f(y).$$

Use (1.2) to find $x \in 2^\omega$ such that $f^{-1}(N_{x\restriction n})$ is comeagre for all $n \in \omega$.

For the next exercise, it is immensely practical to write 2^n for the set of finite sequences in $2 = \{0, 1\}$ of length n.

Problem 2

Let X be a Polish space. Recall that a homeomorphism of X is a bijection $g : X \to X$ where both g and g^{-1} are continuous. A *group of homeomorphisms of* X is a non-empty set G of homeomorphisms of X which forms a group under composition, i.e., if $g, h \in G$ then $g \circ h \in G$, and if $g \in G$ then $g^{-1} \in G$. If G is a group of homeomorphisms of X, we define the associated *orbit equivalence relation* on X by

$$xE_G y \iff (\exists g \in G) g(x) = y.$$

Prove the following theorem due to Becker and Kechris:

Theorem. *Let X be a Polish space, let G be a group of homeomorphisms of X, and let E be an equivalence relation on X. Suppose that*

(*a*) *E is meagre as a subset of $X \times X$,*
(*b*) *$E_G \subseteq E$, i.e., for all $x \in X$ and $g \in G$ we have $xEg(x)$, and*
(*c*) *E_G has a dense equivalence class, i.e., there is some $x_0 \in X$ such that*

$$[x_0]_{E_G} = \{x \in X : xE_G x_0\}$$

is dense in X.

Then $E_0 \leq_B E$.

Hints: The idea is similar to the proof of Mycielski's theorem from HW2, but with a lot more bells and whistles. It may be helpful at first to assume that $E = E_G$, just to have less to worry about.

Fix a compatible complete metric d on X, and fix a **decreasing** sequence $U_n \subseteq X \times X$, $n \in \omega$, of open dense sets such that

$$E \cap \bigcap_{n \in \omega} U_n = \emptyset.$$

Recursively define a Cantor scheme $(V_s)_{s \in 2^{<\omega}}$ of non-empty open sets, along with group elements $g_{s,t} \in G$, for all $s, t \in 2^{<\omega}$ with $\ell\mathrm{h}(s) = \ell\mathrm{h}(t)$, such that:

(1) For all $s \in 2^{<\omega}$, $\mathrm{diam}(V_s) \leq 2^{-\ell\mathrm{h}(s)}$.
(2) If $t \supseteq s$ then $\overline{V_t} \subseteq V_s$.
(3) For all $s, t \in 2^n$ we have $V_{s^\frown 0} \times V_{t^\frown 1} \subseteq U_{n+1}$.
(4) For all $s \in 2^{<\omega}$ and $x \in X$ we have $g_{s,s}(x) = x$.
(5) For all $s, t, u \in 2^n$, $g_{s,t} \circ g_{t,u} = g_{s,u}$.
(6) For all $s, t \in 2^n$ and all $u \in 2^{<\omega}$ we have $g_{s,t} = g_{s^\frown u, t^\frown u}$.
(7) For all $s, t \in 2^n$ we have $g_{s,t} V_t = V_s$.

Argue that the map $f : 2^\omega \to X$ associated to the Cantor scheme $(V_s)_{s \in 2^{<\omega}}$ provides the desired Borel reduction. (In fact, f will be continuous.) Specifically, use (3) to prove that if $f(x)Ef(y)$ then $xE_0 y$, and use (6) and (7) to prove that if $xE_0 y$ then there is a $g \in G$ such that $g(f(x)) = f(y)$.

For the construction of the V_s and the $g_{s,t}$, it is satisfying both (3) and (7) that creates tension. Note that by (4) and (5) we must have $g_{s,t} = g_{t,s}^{-1}$, and so $g_{s,t} = g_{s,\bar{0}^n} g_{\bar{0}^n,t} = g_{s,\bar{0}^n} g_{t,\bar{0}^n}^{-1}$, where $\bar{0}^n$ denotes the constant 0 sequence of length n. So it is enough to define appropriate $g_{s,\bar{0}^n}$ and, by (7), appropriate $V_{\bar{0}^n}$. Observe that (6) then puts the onus on defining $g_{\bar{0}^{n\frown}0, \bar{0}^{n\frown}1}$.

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